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RESEARCH ARTICLE

# A two-symbol system of encoding and some of its applications

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Abstract. The article is devoted to a two-symbol system of encoding for real numbers with two bases of different signs $g_0 \in (0; \frac{1}{2}]$ and $g_1 \equiv g_0 - 1$, as well as its applications in metric number theory and the metric theory of functions. We prove that any natural number $a$ can be represented as

$$a = 2^n + \sum_{k=1}^{n} [(-1)^{1+\sigma_k} a_k 2^{n-k}] \equiv (1a_1 \ldots a_n)_G,$$

where $a_k \in \{0; 1\}$ and $\sigma_k = a_1 + \ldots + a_{k-1}$, and there exist exactly two such representations. Any number $x \in (0; g_0]$ can be represented as

$$\delta_{\alpha_1} + \sum_{k=2}^{\infty} (\delta_{\alpha_k} \prod_{j=1}^{k-1} g_{\alpha_j}) \equiv \Delta^{G_2}_{\alpha_1\alpha_2\ldots\alpha_k\ldots}, \delta_{\alpha_k} = \alpha_k g_{1-\alpha_k}.$$

Most numbers have a unique $G_2$-representation, while a countable set has exactly two representations: $\Delta^{G_2}_{c_1\ldots c_m 01(0)} = \Delta^{G_2}_{c_1\ldots c_m 11(0)}$. For $g_0 = \frac{1}{2}$, any number $x$ in the interval $[0; 1]$ has the expansion

$$x = \frac{1}{2}\alpha_0 + \sum_{k=1}^{\infty} \frac{\alpha_k(-1)^{1+\sigma_k}}{2^k} \equiv \Delta_{\alpha_0\alpha_1\ldots\alpha_n\ldots}.$$

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## Introduction

Today, exact sciences confidently operate with both finite and infinite sets and data arrays. The ideas of coordinate representation and encoding are effectively applied. As a result, powerful families of mathematical objects are described using a small number of basis (reference) objects and relations. Meaningful information about dependencies and correspondences takes a digital form, encoded by codes: sets, matrices, and sequences of elements of alphabet, which can be finite or infinite, constant or variable [11].

Traditionally, two-symbol systems of encoding of information use the alphabet $A = \{0, 1\}$. So far, they remain unmatched in applications, although three-symbol systems are more efficient in some sense. Two-symbol systems deserve a special attention, particularly due to the minimality of their alphabet. In the sequel, we will focus on the encoding of real numbers.

An encoding of real numbers from the set $D$ using the alphabet $A$ is defined as a surjective mapping (onto mapping) from the space $L = A \times A \times \ldots$ of sequences of elements of the alphabet $A$ onto the set $D$. Traditionally, the binary numeral system is the simplest two-symbol encoding of numbers. Its metric counterpart is the negabinary numeral system [6], a system with a negative base $(-2)$.

This work is devoted to an encoding system that is fundamentally different from the above-mentioned systems.

Such representation of numbers in the interval $[0; g_0]$ and its various applications were studied in the papers [4, 6–10].

Except for [1–3, 5], we are not aware of any other papers where numeral systems with two bases are considered.

## 1. Existence of a $G$-representation of a natural number

**Definition 1.** If, for a natural number $a$, there exists a sequence $(a_1, \ldots, a_n)$ of zeros and ones such that

$$a = 2^n + \sum_{k=1}^{n} [(-1)^{1+\sigma_k} a_k 2^{n-k}],$$

where $\sigma_k = a_1 + \ldots + a_{k-1}$, then we say that the number $a$ has a *$G$-representation*. Symbolically, this is written as

$$a = (1a_1 \ldots a_n)_G.$$

The number $a$ is thus an $(n + 1)$-digit number.

For example, each number within the range from 1 to 10 has a $G$-representation, and it has exactly two representations. It is easy to verify this statement:

$$\begin{aligned}
1 &= 2^0 = (1)_G = 2^1 - 2^0 = (11)_G,\\
2 &= 2^1 = (10)_G = 2^2 - 2^1 = (110)_G,\\
3 &= 2^2 - 2^1 + 2^0 = (111)_G = 2^2 - 2^0 = (101)_G,\\
4 &= 2^2 = (100)_G = 2^3 - 2^2 = (1100)_G,\\
5 &= 2^3 - 2^2 + 1 = (1101)_G = 2^3 - 2^2 + 2 - 1 = (1111)_G,\\
6 &= 2^3 - 2^2 + 2 = (1110)_G = 2^3 - 2 = (1010)_G,\\
7 &= 2^3 - 1 = (1001)_G = 2^3 - 2 + 1 = (1011)_G,\\
8 &= 2^3 = (1000)_G = 2^4 - 2^3 = (11000)_G,\\
9 &= 2^4 - 2^3 + 1 = (11001)_G = 2^4 - 2^3 + 2 - 1 = (11011)_G,\\
10 &= 2^4 - 2^3 + 2 = (11010)_G = 2^4 - 2^3 + 2 - 1 = (11110)_G.
\end{aligned}$$

**Remark 1.** As we can see, different $G$-representations of the same natural number can have a different number of digits. For example, $8 = (1000)_G = (11000)_G$, and in general

$$a = 2^n = (1\underbrace{0\ldots0}_{n})_G = (11\underbrace{0\ldots0}_{n})_G,$$

while $a = 2^n + 1 = 2^{n+1} - 2^n + 1 = (11\underbrace{0\ldots0}_{n-1}1)_G = (11\underbrace{0\ldots0}_{n-2}11)_G$.

**Theorem 1.** *Each natural number has exactly two $G$-representations, that is, for any natural number $a$, there exists a sequence of zeros and ones $(a_1, a_2, \ldots, a_n)$ such that*

$$a = 2^n + \sum_{k=1}^{n}[(-1)^{1+\sigma_k} a_k 2^{n-k}] \equiv (1a_1 \ldots a_n)_G, \tag{1}$$

*where $\sigma_1 = 0$, $\sigma_k = a_1 + \ldots + a_{k-1}$, and there are exactly two such expansions.*

*Proof.* Obviously, for any natural number $a$, there exists a natural number $n$ such that $2^{n-1} < a \leq 2^n$. We will use the method of mathematical induction on $n$. For $n = 1$, we have $1 < a \leq 2$ and the statement is evident for the numbers 1 and 2 (see the example above).

Assume that the statement holds for $n = k$, i.e., for $a \in (2^{k-1}; 2^k]$, there exists an expansion $a = 2^k + \sum_{i=1}^{k}[(-1)^{1+\sigma_i} a_i 2^{k-i}] \equiv (1a_1 \dots a_k)_G$, where $\sigma_i = a_1 + \dots + a_{i-1}$, and there are exactly two such expansions.

Consider $n = k+1$, i.e., the case of a number $a \in (2^k; 2^{k+1}]$. If $a = 2^{k+1}$, then $a = (1\underbrace{0\dots0}_{k+1})_G = (11\underbrace{0\dots0}_{k+1})_G$. Now, suppose that $2^k < a < 2^{k+1}$; then $0 < a - 2^k \equiv u < 2^k$. By the induction hypothesis, the number $u$ has a $G$-representation $u = (1c_1c_2\dots c_m)_G$, where $m < k$. Thus, we have $a = 2^k + u = 2^{k+1} - 2^k + u = (11\underbrace{0\dots0}_{k-m}c_1\dots c_m)_G$. Since the number $u \equiv a - 2^k$ satisfies the inequality $0 < u < 2^k$, it has exactly two $G$-representations by the induction hypothesis. Therefore, the number $a$ also has exactly two $G$-representations. By the principle of mathematical induction, the statement is proven for any natural number $a$. $\square$

**Definition 2.** We say that a $G$-representation of a natural number is called *canonical* if it has the minimum number of digits and the maximum number of zeros simultaneously.

For example, the canonical $G$-representation of the numbers

$$16 = (10000)_G = (110000)_G, \quad a = 2^n - 1 = (1\underbrace{0\dots0}_{n-1}1)_G = (1\underbrace{0\dots0}_{n-2}11)_G$$

is the first of the $G$-representations.

## 2. Identification and comparison of natural numbers

**Theorem 2.** *If $a = (1a_1\dots a_na_{n+1})_G$ and $b = (1b_1\dots b_n)_G$, then $a \geq b$, where equality holds only when $a = (11\underbrace{0\dots0}_{m})_G$ and $b = (1\underbrace{0\dots0}_{n})_G$. If the number of digits in the $G$-representation of $c = (1c_1\dots c_n\dots c_{n+p})_G$ exceeds the number of digits in the $G$-representation of $b$ by $p$, then $c > b$.*

*Proof.* We only need to consider the case where one number has exactly one more digit than the other. Let $a = (1a_1\dots a_na_{n+1})_G$ and $b = (1b_1\dots b_n)_G$.

Consider the difference

$$\rho \equiv a - b = (2^{n+1} - 2^n) + (\sum_{i=1}^{n+1}[(-1)^{1+\sigma_i}a_i2^{n+1-i}] - \sum_{i=1}^{n}[(-1)^{1+\sigma'_i}b_i2^{n-i}]).$$

Since $\min \sum_{i=1}^{n+1} [(-1)^{1+\sigma_i} a_i 2^{n+1-i}] = -2^n$ is attained only when $1 - a_1 = a_2 = a_3 = \ldots = a_{n+1} = 0$, and $\max \sum_{i=1}^{n} [(-1)^{1+\sigma'_i} b_i 2^{n-i}] = 0$ is attained only when $b_1 = b_2 = \ldots = b_n = 0$, it follows that $\rho \geq 2^{n+1} - 2^n - 2^n = 0$, where equality holds only when $a = (11\underbrace{0\ldots0}_{m})_G$ and $b = (1\underbrace{0\ldots0}_{n})_G$. By the theorem assumption, let $c = (1c_1 \ldots c_n \ldots c_{n+p})_G$, where $p \geq 2$, then the difference

$$\begin{aligned}\delta = c - b = (2^{n+p} - 2^n) +\\ + (\sum_{i=1}^{n+p}[(-1)^{1+\sigma_i} c_i 2^{n+1-i}] - \sum_{i=1}^{n}[(-1)^{1+\sigma'_i} b_i 2^{n-i}]) \geq \\ \geq 2^{n+p} - 2^n - 2^{n+p-1} + 0 = 2^n(2^{p-1} - 1) > 0,\end{aligned}$$

which implies $c > b$. This completes the proof. $\square$

**Remark 2.** Two $G$-representations of a natural number $a \neq 2^n$ have the same number of digits, while the $G$-representations of numbers of the form $2^n$ have different numbers of digits.

**Theorem 3.** *The numbers*

$$a = (1a_1 \ldots a_{k-1} 1 a_{k+1} \ldots a_n)_G \quad \textit{and} \quad b = (1a_1 \ldots a_{k-1} 0 b_{k+1} \ldots b_n)_G$$

*satisfy the following relation:*

*1) $a \geq b$ if $\sigma_k$ is odd;*

*2) $a \leq b$ if $\sigma_k$ is even, where equality holds only if*

$$1 - a_{k+1} = a_{k+2} = \ldots = a_n = 0 = 1 - b_{k+1} = b_{k+2} = \ldots = b_n.$$

*Proof.* Since $a_k = 1$ and $b_k = 0$, in the first case we have

$$\begin{aligned}a - b = 2^{n-k} + \sum_{i=k+1}^{n} [(-1)^{1+\sigma_i} a_i 2^{n-i}] - \sum_{i=k+1}^{n} [(-1)^{1+\sigma'_i} b_i 2^{n-i}] \geq \\ \geq 2^{n-k} - 2^{n-k-1} - 2^{n-k-1} = 0\end{aligned}$$

because $\min \sum_{i=k+1}^{n} [(-1)^{1+\sigma_i} a_i 2^{n-i}] = -2^{n-k-1}$ and

$$\max \sum_{i=k+1}^{n} [(-1)^{1+\sigma'_i} b_i 2^{n-i}] = 2^{n-k-1}.$$

In the second case, the difference is

$$a - b = -2^{n-k} + \sum_{i=k+1}^{n} [(-1)^{1+\sigma_i} a_i 2^{n-i}] - \sum_{i=k+1}^{n} [(-1)^{1+\sigma'_i} b_i 2^{n-i}] \leq$$
$$\leq -2^{n-k} + 2^{n-k-1} + 2^{n-k-1}$$

because $\min \sum_{i=k+1}^{n} [(-1)^{1+\sigma_i} a_i 2^{n-i}] = -2^{n-k-1}$ and

$$\max \sum_{i=k+1}^{n} [(-1)^{1+\sigma'_i} b_i 2^{n-i}] = 2^{n-k-1}.$$

Note that the extreme values in both cases are achieved only under the condition

$$1 - a_{k+1} = a_{k+2} = \ldots = a_n = 0 = 1 - b_{k+1} = b_{k+2} = \ldots = b_n. \qquad \square$$

## 3. $G$-representation of the fractional part of a real number

**Theorem 4** ([7])**.** *For any number $x \in [0; \frac{1}{2}]$, there exists a sequence of zeros and ones $(\alpha_n)$ such that*

$$x = \frac{\alpha_1}{2} + \sum_{k=2}^{\infty} \frac{\alpha_k (-1)^{\sigma_k}}{2^k} = \frac{\alpha_1}{2} + \sum_{k=2}^{\infty} \frac{\alpha_k}{2^{k-\sigma_k}(-2)^{\sigma_k}} \equiv \Delta^G_{\alpha_1\alpha_2\ldots\alpha_n\ldots}, \qquad (2)$$

*where $\sigma_k = \alpha_1 + \alpha_2 + \ldots + \alpha_{k-1}$.*

**Corollary 1.** *For any number $x \in [0; 1]$, there exists a sequence of zeros and ones $(\alpha_n)$ such that*

$$x = \frac{1}{2}\alpha_0 + \sum_{k=1}^{\infty} \frac{\alpha_k (-1)^{1+\sigma_k}}{2^k} \equiv \Delta_{\alpha_0\alpha_1\ldots\alpha_n\ldots}.$$

The symbolic notation $\Delta^G_{\alpha_1\alpha_2\ldots\alpha_n\ldots}$ is called the *$G$-representation* of a number $x \in [0; \frac{1}{2}]$ and of its expansion in series (2). Most numbers in the interval $[0; \frac{1}{2}]$ have a unique $G$-representation, and they are called $G$-unary numbers.

A countable set of numbers has exactly two $G$-representations:

$$\Delta^G_{c_1\ldots c_{m-1}01(0)} = \Delta^G_{c_1\ldots c_{m-1}11(0)},$$

and such numbers are called $G$-binary numbers. The numbers $0 = \Delta^G_{(0)}$ and $0{,}5 = \Delta^G_{1(0)}$ have a unique representation and are therefore not $G$-binary.

Evidently, every $G$-binary number is rational. However not every rational number is $G$-binary. This is illustrated by the example:

$$\Delta^G_{(10)} = \frac{1}{2} - \frac{1}{2^3} + \frac{1}{2^5} - \frac{1}{2^7} + \ldots = \frac{2}{3}.$$

**Definition 3.** A *$G$-cylinder* of rank $m$ with base $c_1c_2\ldots c_m$ is the set

$$\Delta^G_{c_1c_2\ldots c_m} = \{x:\ x = \Delta^G_{\alpha_1\alpha_2\ldots}, \alpha_i = c_i,\ i = \overline{1,m}\}.$$

A cylinder is a segment $[a;b]$ with endpoints

$$a = \frac{\alpha_1}{2} + \sum_{k=2}^{m} \frac{(-1)^{\alpha_1+\ldots+\alpha_{k-1}}\alpha_k}{2^k},\quad b = a + \frac{1}{2^m}.$$

Cylinders of the same rank do not overlap, and the following equality holds: $\Delta^G_{c_1\ldots c_m} = \Delta^G_{c_1\ldots c_m0} \cup \Delta^G_{c_1\ldots c_m1}$ along with the basic metric ratio: $2|\Delta^G_{c_1\ldots c_mi}| = |\Delta^G_{c_1\ldots c_m}|$.

For any sequence $(c_n)$, the following holds: $\bigcap\limits_{m=1}^{\infty} \Delta^G_{c_1\ldots c_m} = \Delta^G_{c_1\ldots c_m}$, which justifies considering a point as a cylinder of infinite rank.

It is easy to prove the following statement: if the $G$-representation of a number $x$ is periodic, then $x$ is rational.

## 4. Left shift and right shift operators

**Remark 3.** For the following functions to be well defined, we agree that among the two representations of the same $G$-binary number, we will use the representation $\Delta^G_{c_1\ldots c_{m-1}01(0)}$.

**Definition 4.** The operator $\omega^n$ of an $n$-fold left shift of the digits in the $G$-representation of numbers is the mapping defined by the equation

$$\omega^n(x = \Delta^G_{\alpha_1\alpha_2\ldots\alpha_n\ldots}) = \Delta^G_{\alpha_{n+1}\alpha_{n+2}\ldots\alpha_{n+k}\ldots}.$$

The operator $\omega^n$ has the following analytic expression:

$$\omega^n(x) = 2^n(-1)^{\sigma_n}x - \alpha_1\cdot 2^{n-1} + (-1)^{\alpha_1-1}\alpha_2 2^{n-2} + \ldots + (-1)^{\alpha_1+\ldots+\alpha_{n-1}-1}2.$$

For any $n \in N$, the operator $\omega^n$ is a piecewise-linear continuous function that attains its maximum value of 0.5 or minimum value of 0 at each $G$-binary point. This fundamentally distinguishes this representation from all other known two-symbol systems of encoding of numbers.

**Lemma 1.** *The equation $\omega^n(x) = x$ has $2^n$ solutions in the form $x = \Delta^G_{(\alpha_1\ldots\alpha_n)}$, where $\alpha_1,\ldots,\ \alpha_n$ are independent variables that take values 0 and 1.*

*Proof.* Indeed, if $x = \Delta^G_{\alpha_1\ldots\alpha_n\ldots}$ is a solution to the equation, then according to the definition of the operator $y = \omega^n(x)$ we have

$$\alpha_1 = \alpha_{n+1}, \ldots, \alpha_n = \alpha_{2n}, \alpha_{n+1} = \alpha_{2n+1} = \alpha_1, \ldots, \alpha_{2n} = \alpha_{3n} = \alpha_n.$$

So, $x = \Delta^G_{(\alpha_1\alpha_2\ldots\alpha_n)}$. And the fact that $\Delta^G_{(\alpha_1\ldots\alpha_n)}$, where $\alpha_k \in A$ are free variables, is the root of the equation is obvious.

Since the image of a $G$-binary number under the mapping $\omega^n$ is a $G$-binary number of lower rank, and $G$-binary numbers of different ranks are never equal, we see that roots of the equation $\omega^n(x) = x$ cannot be a $G$-binary number. □

**Definition 5.** The operator of the right shift of the digits in a $G$-representation with prefix $i_1\ldots i_k$ is the mapping $\tau_{i_1\ldots i_k}$, defined by the equation

$$\tau_{i_1i_2\ldots i_k}(x = \Delta^G_{\alpha_1\alpha_2\ldots\alpha_n\ldots}) = \Delta^G_{i_1i_2\ldots i_k\alpha_1\alpha_2\ldots\alpha_n\ldots}.$$

Each such function is linear and has the following analytic expression:

$$\tau_{i_1i_2\ldots i_k}(x = \Delta^G_{\alpha_1\alpha_2\ldots\alpha_n\ldots}) = \frac{(-1)^{i_1+\ldots+i_k}}{2^k}x + \sum_{m=1}^{k} \frac{i_m(-1)^{i_1+\ldots+i_{m-1}}}{2^m}.$$

The following equalities are evident:

$$\omega^n(\tau_{i_1\ldots i_n}(x)) = x, \ \ \tau_{\alpha_1(x)\ldots\alpha_n(x)}(\omega^n(x)) = x.$$

**Theorem 5.** *The equation $\omega^n(x) = \tau_{i_1\ldots i_k}(x)$ has $2^n$ solutions in the form $x = \Delta^G_{(\alpha_1\ldots\alpha_n i_1\ldots i_k)}$, where $\alpha_j$ are independent variables that take values 0 and 1.*

*Proof.* Let $x = \Delta^G_{\alpha_1\alpha_1\ldots\alpha_n\ldots}$ be a solution of the equation. Then

$$\begin{array}{ccc} \alpha_{n+1} = i_1, & \ldots & \alpha_{n+k} = i_k; \\ \alpha_{n+k+1} = \alpha_1, & \ldots & \alpha_{n+k+n} = \alpha_n; \\ \alpha_{2n+k+1} = \alpha_{n+1} = i_1, & \ldots & \alpha_{2n+k+k} = \alpha_{n+k} = i_k \text{ and so on.} \end{array}$$

Thus, $x = \Delta^G_{(\alpha_1\ldots\alpha_n i_1\ldots i_k)}$. It is evident that the number $\Delta^G_{(\alpha_1\ldots\alpha_n i_1\ldots i_k)}$ is a solution to the equation. □

**Corollary 2.** *Each solution of the equation $\omega^n(x) = \tau_{i_1\dots i_k}(x)$ is a $G$-unary number and, therefore, an irrational number.*

**Theorem 6.** *The system of equations $\omega^n(x) = x = \tau_{i_1\dots i_k}(x)$ has a unique solution $x = \Delta^G_{(i_1\dots i_k)}$ if $n \vdots k$.*

*Proof.* Let $n = k$, then the equation $\tau_{i_1\dots i_k}(x) = x$ has a unique solution $x = \Delta^G_{(i_1\dots i_k)}$, which obviously satisfies $\omega^k(x) = x$ as well.

According to Theorem 5, the equation $\omega^n(x) = \tau_{i_1\dots i_k}(x)$ has solutions in the form $x = \Delta^G_{(\alpha_1\dots\alpha_n i_1\dots i_k)}$ while the equation $\tau_{i_1\dots i_k}(x) = x$ has a unique solution $x = \Delta^G_{(i_1\dots i_k)}$. Thus, when $n$ is a multiple of $k$, the unique solution of the system is $x = \Delta^G_{(i_1\dots i_k)}$. □

**Corollary 3.** *The solution of the system of equations $\omega^k(x) = x = \tau_{i_1\dots i_k}(x)$ is an irrational number.*

**Definition 6.** The $G$-representations of the numbers $x_1 = \Delta^G_{\alpha_1\alpha_2\dots\alpha_n\dots}$ and $x_2 = \Delta^G_{c_1c_2\dots c_n\dots}$ are said *to have the same tail* if there exist indices $k$ and $m$ such that $\alpha_{k+j} = c_{m+j}$ for all $j \in N$. This is denoted symbolically as $x_1 \sim x_2$.

The binary relation "to have the same tail" is an equivalence relation. The set of all representations of a number that share the same tail, i.e., an element of the factor set, is called a tail set. Each tail set is countable, whereas the set of all tail sets has the cardinality of the continuum. All $G$-binary numbers form a single tail set.

A function $f$, defined on the interval $[0; 0,5]$, is said to preserve the tails of the $G$-representations of numbers if for any $x \in [0; 0,5]$, it holds that $f(x) \sim x$.

Recall that a bijective (i.e., one-to-one and onto) mapping of a set onto itself is called a transformation of this set. It is known [9] that the set of all continuous transformations of the interval $[0; 0,5]$ that preserve the tails of $G$-representations forms an infinite, noncommutative group under the composition of transformations. Examples of such transformations include

$$f_1(x) = \begin{cases} \tau_1(x) & \text{if } 0 \le x \le x_1 = \Delta^G_{(1)}, \\ \omega(x) & \text{if } x_1 \le x \le 0,5; \end{cases}$$

$$f_2(x) = \begin{cases} \tau_1(x) & \text{if } 0 \le x \le x_2 = \Delta^G_{(101)}, \\ \omega(x) & \text{if } x_2 \le x \le 0,5. \end{cases}$$

## 5. $G_2$-representation of numbers of interval $[0,g_0]$

$G$-representation is a special case of a more general two-symbol representation with two bases of different signs. Let us recall its definition. Suppose two bases are fixed: $g_0 \in (0; \frac{1}{2}]$, $g_1 \equiv g_0 - 1$, and the numbers $\delta_0 = 0$, $\delta_1 = g_0$.

**Theorem 7** ([8])**.** *For any number $x \in [0; g_0]$, there exists a sequence of zeros and ones $(\alpha_n)$ such that*

$$x = \delta_{\alpha_1} + \sum_{k=2}^{\infty}(\delta_{\alpha_k}\prod_{j=1}^{k-1} g_{\alpha_j}) \equiv \Delta^{G_2}_{\alpha_1\alpha_2\ldots\alpha_k\ldots}, \delta_{\alpha_k} = \alpha_k g_{1-\alpha_k}. \qquad (3)$$

**Corollary 4.** *If $g_0 = \frac{1}{2}$, then $\delta_{\alpha_k} = \frac{\alpha_k}{2^k}$, $\prod\limits_{j=1}^{k-1} g_{\alpha_j} = \frac{(-1)^{\alpha_1+\ldots+\alpha_{k-1}}}{2^{k-1}}$ and series* (3) *takes the form of* (2)*.*

The symbolic notation $\Delta^{G_2}_{\alpha_1\alpha_2\ldots\alpha_k\ldots}$ of the number $x$ and its expansion in the alternating series (3) is called the $G_2$-representation, and $\alpha_k$ is called its $k$th digit.

**Lemma 2.** *The Lebesgue measure of the set*

$$C \equiv C[G_2; \overline{s_1\ldots s_m}] = \{x : \ x = \Delta^{G_2}_{\alpha_1\ldots\alpha_n\ldots}, \overline{\alpha_k\ldots\alpha_{k+m-1}} \neq \overline{s_1\ldots s_m}, \ k \in N\}$$

*of numbers from the interval $[0, g_0]$, whose $G_2$-representation does not contain the sequence $s_1 s_2 \ldots s_m$ as consecutive digits, is equal to zero.*

*Proof.* Consider the set

$$E = \{x : x = \Delta^{G_2}_{\alpha_1\alpha_2\ldots\alpha_n\ldots}, \ \overline{\alpha_{km+1}\ldots\alpha_{km+m}} \neq \overline{s_1\ldots s_m} \ \ \forall k \in N\}.$$

Clearly, $C \subset E$. We will prove that the Lebesgue measure $\lambda(E) = 0$, which will imply that $\lambda(C) = 0$. Define $F_0 \equiv [0; g_0]$, and let $F_k$ be the union of all cylinders of rank $km$ that contain points from $E$ as interior points. Let $\overline{F}_{k+1} \equiv F_k \setminus F_{k+1}$. Then, we have $F_k = F_{k+1} \cup \overline{F}_{k+1}$, $\lambda(F_{k+1}) = \lambda(F_k) - \lambda(\overline{F}_{k+1})$, $E \supset F_k \supset F_{k+1}$ $\forall k \in N$ and $\lambda(E) = \lim\limits_{k\to\infty} \lambda(F_k)$.

Now, we express $\lambda(F_k)$ in the following form:

$$\lambda(F_k) = \frac{g_0\lambda(F_k)}{\lambda(F_{k-1})} \cdot \frac{\lambda(F_{k-1})}{\lambda(F_{k-2})} \cdot \ldots \cdot \frac{\lambda(F_1)}{\lambda(F_0)} =$$

$$= g_0 \prod_{i=1}^{k} \frac{\lambda(F_i)}{\lambda(F_{i-1})} = g_0 \prod_{i=1}^{k} \frac{\lambda(F_i) - \lambda(\overline{F}_i)}{\lambda(F_{i-1})} = g_0 \prod_{i=1}^{k}(1 - \frac{\lambda(\overline{F}_i)}{\lambda(F_{i-1})}).$$

From this, it follows that

$$\lambda(E)=\lim_{k\to\infty}\lambda(F_k)=g_0\prod_{k=1}^{\infty}\frac{\lambda(F_k)}{\lambda(F_{k-1})}=g_0\prod_{k=1}^{\infty}(1-\frac{\lambda(\overline{F}_k)}{\lambda(F_{k-1})}).$$

Since $|\Delta^{G_2}_{\alpha_1\ldots\alpha_k i}|=|g_i||\Delta^{G_2}_{\alpha_1\ldots\alpha_k}|$, we have $c_1\le\frac{|\Delta^{G_2}_{\alpha_1\ldots\alpha_k i}|}{|\Delta^{G_2}_{\alpha_1\ldots\alpha_k}|}\le c_2$, where $c_1=\min\{g_0;-g_1\}$, $c_2=\max\{g_0;-g_1\}$, and thus

$$0<c_1^m\le\frac{|\Delta^{G_2}_{\alpha_1\ldots\alpha_{km}s_1\ldots s_m}|}{|\Delta^{G_2}_{\alpha_1\ldots\alpha_{km}}|}=\prod_{i=1}^{m}|g_{s_i}|\le c_2^m<1.$$

Hence, the ratio $\frac{\lambda(\overline{F}_k)}{\lambda(F_{k-1})}$ is bounded away from zero, which means that the difference $1-\frac{\lambda(\overline{F}_k)}{\lambda(F_{k-1})}$ is bounded away from one. Therefore, the necessary condition for the convergence of the infinite product is not satisfied, implying that the product diverges to zero. □

**Corollary 5.** *The set $C$ is a Cantor-type null set with a self-similar structure.*

**Theorem 8.** *Almost every number in the interval $[0;g_0]$ contains every possible sequence of digits in its $G_2$-representation infinitely many times.*

*Proof.* Let $(s_1,\ldots,s_m)$ be an arbitrary ordered sequence of zeros and ones, let $H$ be the set of all numbers in $[0;g_0]$ whose $G_2$-representation contains the sequence of digits $s_1s_2\ldots s_m$ infinitely many times as consecutive digits, and let $D$ be the set of all numbers that contain this sequence only a finite number of times as consecutive digits in their $G_2$-representation.

We will prove that $H$ is a full Lebesgue measure set, meaning that $\lambda(H)=g_0$. To do this, it is sufficient to show that $\lambda(\overline{H})=0$, where $\overline{H}=[0;g_0]\setminus H$. Clearly, $\overline{H}=\bigcup\limits_{n=1}^{\infty}D_n$, where

$$D_n=\{x:x=\Delta^{G_2}_{\alpha_1\ldots\alpha_n\ldots},\ \alpha_{k+1}\ldots\alpha_{k+m}\ne s_1\ldots s_m,\forall k\ge n\}.$$

Now, we compute $\lambda(D_n)$. Since $D_n=\bigcup\limits_{\alpha_1\in A}\ldots\bigcup\limits_{\alpha_n\in A}[\Delta^{G_2}_{\alpha_1\ldots\alpha_n}\cap D]$,

$$\Delta^{G_2}_{\alpha_1\ldots\alpha_n}\cap D=\Delta^{G_2}_{\alpha_1\ldots\alpha_n}\cap H,$$

by the previous lemma, we have $\lambda(\Delta^{G_2}_{\alpha_1\ldots\alpha_n}\cap D)=0$, and thus, $\lambda(D_n)=0$. Hence $\lambda(\overline{H})=0$ because $\lambda(\overline{H})\le\sum\limits_{n=1}^{\infty}\lambda(D_n)=0$, which completes the proof. □

## 6. Applications of $G_2$-representation in the theory of locally complicated functions

**Theorem 9.** *If a $G_2$-representation of numbers is defined by the parameter $g_0 \in (0; \frac{1}{2}]$, and $r_0 \in (0; \frac{1}{2}]$, then the system of functional equations*

$$\begin{cases} f(g_0 x) = r_0 f(x), \\ f(g_0 + (g_0 - 1)x) = r_0 + (r_0 - 1) f(x) \end{cases}$$

*in the class of continuous functions defined on the interval $(0, g_0]$ has a unique solution. Moreover, when $g_0 = r_0$, the solution is $f(x) = x$, and when $g_0 \neq r_0$, the function $f(x)$ is singular, meaning that its derivative is equal to zero almost everywhere (with respect to the Lebesgue measure).*

*Proof.* Let $x = \Delta^{G_2}_{\alpha_1 \dots \alpha_n \dots}$ be an arbitrary number in the interval $[0, g_0]$. Then $g_0 x = g_0 \delta_{\alpha_1} + g_0 \delta_{\alpha_2} g_{\alpha_1} + g_0 \delta_{\alpha_3} g_{\alpha_1} g_{\alpha_2} + \dots = \Delta^{G_2}_{0 \alpha_1 \alpha_2 \dots}$,

$$g_0 + (g_0 - 1)x = g_0 + g_1 x = g_0 + g_1 \delta_{\alpha_1} + g_1 \delta_{\alpha_2} g_{\alpha_1} + \dots = \Delta^{G_2}_{1 \alpha_1 \alpha_2 \dots},$$

and thus, from the system of functional equations, we obtain

$$\begin{aligned} f(\Delta^{G_2}_{\alpha_1 \alpha_2 \dots}) &= \alpha_1 r_{1-\alpha_1} + r_{\alpha_1} f(\Delta^{G_2}_{\alpha_2 \alpha_3 \dots}) = \\ &= \alpha_1 r_{1-\alpha_1} + r_{\alpha_1} (\alpha_2 r_{1-\alpha_2} + r_{\alpha_2} f(\Delta^{G_2}_{\alpha_3 \dots \alpha_n \dots})) = \\ &= \alpha_1 r_{1-\alpha_1} + r_{\alpha_1} \alpha_2 r_{1-\alpha_2} + r_{\alpha_1} r_{\alpha_2} f(\Delta^{G_2}_{\alpha_3 \dots \alpha_n \dots}) = \dots = \\ &= \alpha_1 r_{1-\alpha_1} + \sum_{k=2}^{n} \alpha_k r_{1-\alpha_k} \prod_{i=1}^{k-1} r_{\alpha_i} + (\prod_{i=1}^{k} r_{\alpha_i}) f(\Delta^{G_2}_{\alpha_{n+1} \alpha_{n+2} \dots}). \end{aligned}$$

The product $\prod\limits_{i=1}^{n} r_{\alpha_i} \to 0$ as $n \to \infty$ and $f$ is continuous on $[0, g_0]$, so it attains a minimum and maximum value. Therefore, the remainder term in the expansion tends to zero, and we obtain

$$f(x) = \alpha_1 r_{1-\alpha_1} + \sum_{k=2}^{\infty} \alpha_k r_{1-\alpha_k} \prod_{i=1}^{k-1} r_{\alpha_i} \equiv \Delta^{G'_2}_{\alpha_1 \alpha_2 \dots},$$

where the last symbolic notation is the $G'_2$-representation with bases $r_0$ and $r_1 = r_0 - 1$.

As we see, $f$ acts as a projector from the $G_2$-representation to the $G'_2$-representation of numbers, which implies the continuity and strict monotonicity of $f$. When $g_0 = r_0$, the $G_2$-representation and $G'_2$-representation are identical, and thus, $f(x) = x$.

Now we prove that $f$ is singular for $g_0 \neq r_0$. Since $f$ is continuous and monotonic, by Lebesgue's theorem, it has a finite derivative almost everywhere (with respect of the Lebesgue measure). Let $C \subset [0; g_0]$ be the set of points where $f$ has a finite derivative. If $f$ has a derivative at a $G_2$-unary point $x_0 \in [0; g_0]$, then it is expressed as

$$f'(x_0) = \lim_{n\to\infty} \frac{|\Delta^{G'_2}_{\alpha_1\alpha_2\ldots\alpha_n}|}{|\Delta^{G_2}_{\alpha_1\alpha_2\ldots\alpha_n}|} = \lim_{n\to\infty} \frac{r_0 \prod\limits_{i=1}^{n} |r_{\alpha_i}|}{g_0 \prod\limits_{i=1}^{n} |g_{\alpha_i}|} =$$

$$= \lim_{n\to\infty} \frac{r_0}{g_0} \prod_{i=1}^{n} |\frac{r_{\alpha_i}}{g_{\alpha_i}}| = \frac{r_0}{g_0} \prod_{i=1}^{\infty} |\frac{r_{\alpha_i}}{g_{\alpha_i}}|.$$

If $g_0 \neq r_0$, then the infinite product diverges to zero, because the necessary condition for the convergence of an infinite product is not satisfied. Hence, $f'(x_0) = 0$. This conclusion applies to every point $x$ such that $f$ has a finite derivative at $x$ and this point belongs to the set $H$ of all numbers satisfying the conditions of the previous theorem. Therefore, $f$ is a singular function. $\square$

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