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RESEARCH ARTICLE

Classification of the almost positive posets

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Abstract. This paper introduces the notion of almost positive posets as non-negative ones that contain maximal positive subposets. Such posets include both positive posets and P-critical posets (minimal non-positive ones) which were described by the authors back in 2005. Almost positive posets also include principal posets in the sense of D. Simson. By definition, a non-negative poset $S = \{1, \cdots, n; \preceq\}$ is principal if the kernel of its Tits quadratic form $q_S(z) = q_S(z_0, z_1, \cdots, z_n)$, defined by the equality $\operatorname{Ker} q_S(z) := \{t \in \mathbb{Z}^{1+n} \mid q_S(t) = 0\}$, is an infinite cyclic subgroup of $\mathbb{Z}^{1+n}$. In 2019, the authors described all serial principal posets. This paper concludes the description of all almost positive posets.

Introduction

This paper is related to the Tits quadratic forms which play an important role in modern representation theory.

The Tits quadratic forms were first introduced by P. Gabriel [12] for finite quivers. Namely, if Q is a quiver with the set of vertices Q_0 and the set of arrows Q_1, then its *Tits quadratic form* $q_Q : \mathbb{Z}^n \to \mathbb{Z}$, $n = |Q_0|$, is given by the equality $q_Q(z) = \sum\limits_{i \in Q_0} z_i^2 - \sum\limits_{i \to j} z_i z_j$, where $i \to j$ runs through the set Q_1. For the posets, the closest structure to the quivers, the Tits quadratic form was first considered by Yu. A. Drozd [11]. By definition,

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for a poset $S \not\ni 0$, such quadratic form $q_S : \mathbb{Z}^{1+n} \to \mathbb{Z}$, $n = |S|$, is given by the equality $q_S(z) = z_0^2 + \sum_{i\in S} z_i^2 + \sum_{i<j,\, i,j\in S} z_i z_j - z_0 \sum_{i\in S} z_i$.

The main results of the mentioned papers inspired the study of posets with positive Tits quadratic form as analogues of the Dynkin diagrams (in more details, see [7, Introduction]).

In 2005, the authors described all posets with positive Tits quadratic form [4] using the analogous result on posets of width 2 obtained a little earlier in [3] (see also [7, Theorem 1] and [9, Section 4] for serial and non-serial posets, respectively). Such posets are simply called *positive*. In the same paper the authors also described all minimal posets with non-positive Tits form calling them *P-critical* (see also [6], [8] and [2][1]).

Analogously, the results of [10, 16] on quivers and [18] on posets inspired the study of posets with non-negative Tits quadratic form as the natural one (see [7, Introduction]).

In [6] the authors described all minimal posets whose Tits quadratic form is not non-negative calling them *NP-critical* (NP means Null+ Positive), Note that posets S with non-negative Tits quadratic form $q_S(z)$, which are simply called *non-negative*, are analogues of the extended Dynkin diagrams only when $\operatorname{Ker} q_S(z) := \{t \in \mathbb{Z}^{1+|S|} \mid q_S(t) = 0\}$ are infinite cyclic subgroups of $\mathbb{Z}^{1+|S|}$ (since such property holds for the quivers with non-negative Tits form that is not positive). The non-negative posets satisfying the indicated condition (and their Tits quadratic forms) are said to be *principal* [13, Definition 2(b)].

In [7] the authors described the serial principal posets. In this paper, we study non-serial principal posets; in particular, we obtain their complete classification.

Note that some classes of principal posets of order $n = 5, 6, 7$ (which in our terminology mean the non-series ones) were described by M. Gąsiorek, D. Simson and K. Zając with the help of computer programs (the paper [14] for $n = 5, 6$ and the preprint [15] for $n = 7$).

[1]The paper [4] has been often cited, but is today virtually inaccessible. The main ideas and many results of this paper are published (in a translation from Russian) in the paper [2, Sections 1–3] of the one of the previous issue of this journal. That is why we will often refer to [2] instead of (or parallel with) [4]. Sections 4 and 5 of the paper [2] present some new ideas about the minimax equivalence method. They are used in this paper.

1. Preliminary

1.1. Definitions on posets. Throughout the paper, all posets are assumed to be non-empty finite posets without the element 0. A partial order relation is denoted by $\leq$, or $\preceq$ when the elements of a poset are numbered by integer numbers. By a subposet S' of a poset S we always mean a full subposet (i.e. with the order relation induced by a given order relation on S). We identify singletons with the elements themselves, and (as rule) posets with their Hasse diagrams.

A poset T is called *dual to a poset* S and denoted by S^{op} if $T = S$ as usual sets and $x < y$ in T if and only if $x > y$ in S. The union of disjoint posets S_1, S_2 is called their *direct sum* and denoted by $S_1 \coprod S_2$.

The notation $T \cong S$ for posets means that T is isomorphic to S. When S is specific, we also say that T is of the form S; and by "T contains S" we mean that there is a subposet in T isomorphic to S.

1.2. Minimax equivalence of posets. The notion of (*min, max*)-*equivalence of posets* was introduced by the first author in [1]. In detail the properties of this equivalence were studied in [4]. Since some time we have been used the term *minimax equivalence*.

In this subsection we remember some definitions and results from [2, 4] (see footnote 1).

Let a be a minimal element of a poset S. We define by $S' = S_a^{\uparrow}$ the poset equal to S as an usual set and such that $S' \setminus a = S \setminus a$ as posets; in the same time the element a is already maximal in S', which is comparable with another element $x \in S'$ iff they are incomparable in S. Dually we define the poset $S' = S_a^{\downarrow}$ for a maximal element a of S. A poset T is called *minimax equivalent* or (min, max)-*equivalent* to a poset S if and only if T can be obtained from S in a finite number of such operations.

The notion of minimax equivalence can be naturally extended to the notion of *minimax isomorphism*: a posets T is minimax isomorphic to a poset S if there exists a poset S' which is minimax equivalent to T and isomorphic to S. In this case one writes $T \cong_{min,max} S$.

The definition of posets of the form $S' = S_a^{\uparrow}$ (respectively, $S' = S_a^{\downarrow}$) can be extended to posets of the form $\overline{S} = S_A^{\uparrow}$ (respectively, $\overline{S} = S_A^{\downarrow}$), where A is a lower (respectively, upper) subposet of S, i.e. $x \in A$ whenever $x < y$ (respectively, $x > y$) and $y \in A$. In this case the poset A becomes an upper (respectively, lower) subposet of $\overline{S}$ and comparability are already interchanged with incomparability between any $x \in A$ and $y \in S \setminus A$. By

Lemma 3 [2], $S^{\uparrow}_{A}$ (respectively, $S^{\downarrow}_{A}$) and S are minimax equivalent. We write $S^{\uparrow\uparrow}_{AB}$, $S^{\uparrow\downarrow}_{AB}$ instead of $(S^{\uparrow}_{A})^{\uparrow}_{B}$, $(S^{\uparrow}_{A})^{\downarrow}_{B}$, etc.

The following statement was first proved in [1] (see also [2, Prop. 2]).

Proposition 1. *The Tits quadratic forms of minimax equivalent posets are $\mathbb{Z}$-equivalent.*

Corollary 1. *Minimax equivalent or dual posets simultaneously are or are not positive, non-negative, etc.*

1.3. Principal posets. According [13], a poset S is called *principal* if the following conditions hold:

(1) $q_S(z)$ is non-negative;

(2) $\operatorname{Ker} q_S(z) := \{t \mid q_S(t) = 0\}$ is an infinite cyclic group, i.e. is equal to $t\mathbb{Z}$ for some $t \neq 0$ (equivalently, the rank of the symmetric matrix of $q_S(z)$ is equal to $|S|$).

Proposition 2. *For a non-negative poset S, the following conditions are equivalent:*

(a) *S is principal;*

(b) *S contains exactly one P-critical poset denoted by $\underline{S}$.*

The proposition follows from the definition of a P-critical poset and the following lemma.

Lemma 1. *Any P-critical poset is principal.*

This lemma follows from Theorem 2 [4] (according to which a P-critical poset is minimax equivalent to a Kleiner one), Corollary 1 and the fact that the Kleiner posets are principal (see, e.g., [17, Appendix 1]).

An element of a poset is called *isolated* if it is incomparable with all other its elements. From the above it follows the following statement.

Corollary 2. *Let a be an element of a poset S and $a^{<} := \{x \in S \mid x < a\}$, $a^{>} := \{x \in S \mid x > a\}$. Then the element a is isolated in the poset $\overline{S} = S^{\uparrow\downarrow}_{a^{<}a^{>}}$, and $\overline{S} \setminus a$ is positive if $a \in \underline{S}$.*

2. Main theorems

2.1. Almost positive posets. We call a non-negative poset S *almost positive* if $S \setminus x$ is positive for some $x \in S$. Obviously, positive and P-critical posets are almost critical. Any almost positive poset that is not positive is called *strictly almost positive* poset.

Proposition 3. *A poset S is strictly almost positive if and only if it is principal.*

The proposition follows from Proposition 2 and the main idea of the proof of Theorem 2 [8].

An almost positive or strictly almost positive poset is called *serial* if there is (as in the case of positive posets) an infinite increasing sequence $S \subset S^{(1)} \subset S^{(2)} \subset \ldots$ with the same terms, and *non-serial* if otherwise.

The serial strictly almost positive posets ($\equiv$ the serial principal posets) are described by the authors in [8] (see Theorem 3). Then the serial almost positive posets which are equal to the union of the serial strictly almost positive posets and the serial positive posets are described by Theorems 1 and 3 [8].

Note that a P-critical poset, which is not serial in itself (i.e. in the set of all P-critical ones), can become serial as an almost positive poset. We call them *conditionally serial*. By the classifications of serial principal posets in [7] and P-critical posets [4,6] it follows that, up to isomorphism, there are only 4 conditionally serial posets; namely, P-critical posets with numbers 1, 30, 30^{op} and 75 (which are minimax isomorphic). We call the P-critical posets (as strictly almost positive posets which are closest to positive ones) without the conditionally serial posets *near-positive*.

Thus, for a complete classification of the almost positive posets, it remains to classify the non-serial strictly positive posets without the near-positive ones. Such posets are said to be *essential almost positive*. Their classification (with theorems about minimal minimax systems of generators) is the main result of this paper. The essential almost positive posets are collected, up to isomorphism and duality, in Table 2 in the form of their Hasse diagrams. There are 247 such posets (the selfdual from which are marked on the pictures by the symbol *sd*). For the convenience uf the readers, we also collect all near-positive posets (see Table 1). In both cases, the equality of the first numbers of their indices indicates on the same class of minimax isomorphism.

Remark. We emphasize that replacing the principal posets by the strictly almost positive posets leads to a simpler combinatorics (without quadratic forms themselves and their matrices). Therefore, instead of computer programs simpler methods can be used. We traditionally use our "minimax equivalence method" suggested in [1, 2, 4].

2.2. Minimax systems of generators of essential almost positive posets. In [2, Section 5] the first author introduced the concept of the minimax system of generators. In particular, the following definitions were given. Let $\mathcal{K}$ be a class of finite posets closed under isomorphism and duality and let $U = \{U_i\}$ be a set of posets. We say that U is a *minimax system* (respectively, *d-system*) *of generators* of $\mathcal{K}$ if any $X \in \mathcal{K}$ is minimax isomorphic to a poset U_i (respectively, U_i or U_i^{op}) for some $i \in I$.

Theorem 1. *The following* 9 *posets with the isolated elements* 1 *form a minimal minimax d-system of generators for the set* $\mathcal{M}$ *of all essential almost positive posets:*

$$\begin{aligned}
M_1 &= \{1, 2 \prec 4, 3 \prec 4, 3 \prec 6, 5 \prec 6\},\\
M_2 &= \{1, 2 \prec 3 \prec 4, 5 \prec 6 \prec 7, 2 \prec 6\},\\
M_3 &= \{1, 2 \prec 4, 3 \prec 4, 3 \prec 6, 5 \prec 6 \prec 7\},\\
M_4 &= \{1, 2 \prec 5, 3 \prec 4 \prec 5, 3 \prec 6 \prec 7, 4 \prec 7\},\\
M_5 &= \{1, 2 \prec 3 \prec 4, 5 \prec 6 \prec 7, 2 \prec 8\},\\
M_6 &= \{1, 2 \prec 3 \prec 4, 5 \prec 6 \prec 7 \prec 8, 2 \prec 7\},\\
M_7 &= \{1, 2 \prec 3 \prec 4, 5 \prec 6 \prec 7 \prec 8, 2 \prec 6, 3 \prec 8\},\\
M_8 &= \{1, 2 \prec 4, 3 \prec 4, 3 \prec 6, 5 \prec 6 \prec 7 \prec 8\},\\
M_9 &= \{1, 2 \prec 5, 3 \prec 4 \prec 5, 3 \prec 6 \prec 7 \prec 8, 4 \prec 7\}.
\end{aligned}$$

This theorem will be proved in Section 3. In Section 4, for any poset M_i, the class $C(M_i)$ of all, up to isomorphism and duality, posets minimax equivalent to M_i will be determined. Thus, we will obtain a complete classification of all essential almost positive posets (see Table 2).

It will follow from the classification that the poset M_5 is, up to isomorphism and duality, the only poset of $\mathcal{M}$ minimax equivalent to a poset of the form $a \coprod X$ with X to be serial positive (although we have Corollary 2). Apparently, this fact determines the differences between the class $C(M_5)$ and the other classes in different situations.

Denote by $C_0(M_i)$ the class of all posets that are minimax isomorphic to M_i. It follows from Propositions 4–7, 9–12 that for $i \neq 5$, we have the equality $C_0(M_i) = C(M_i)$ (i.e. the class $C_0(M_i)$ is selfdual in the sense that together with a poset X it always contains the dual poset X^{op}). For $i = 5$, the situation is different: M_5 and M_5^{op} are not minimax isomorphic. Therefore, the system $M = \{M_1, \cdots, M_9\}$ (which is a d-system of $\mathcal{M}$ by Theorem 1) is not a usual one. To get such a system it is needed to additionally take M_5^{op}; then $C_0(M_5) \cup C_0(M_5^{op}) = C(M_5)$.

Let us look at three examples regarding various properties of posets. Emphasize that in the case when a classification is considered (in particular) up to duality, a property must be symmetrical (closed under duality),

For a poset X, we call the class $C(X)$ *normal* if $C(X) = C_0(X)$ and *special* if otherwise. Recall that an element of a poset is called *extreme* if it is minimal or maximal. An element of a poset is called *nodal* if it is comparable with all elements and *almost nodal* if it is incomparable to only one element.

Theorem 2. *Any normal class $C(M_i)$ contains only one poset F_i of width 2 with an almost nodal extremal element and maximal for this class of nodal elements. The special class contains two such posets. But in the second case there is only one poset F_0 with the only nodal and maximal for this class of almost nodal elements.*

Note that "one or two posets" implied up to duality. Indeed, it follows from Table 2 that $F_1 \cong AP1.2$, $F_2 \cong AP2.5$, $F_3 \cong AP3.3$, $F_4 \cong AP4.2$, $F_6 \cong AP6.7$, $F_7 \cong AP7.2$, $F_8 \cong AP8.3$, $F_9 \cong AP9.2$, $F_{51} \cong AP5.6$, $F_{52} \cong AP5.10$, $F_0 \cong AP5.13$.

Recall that $\tilde{A}_n$ (the cycles) and $\tilde{D}_n$ are the only serial extended Dynkin diagrams.

Theorem 3. *Any normal class $C(M_i)$ contains only one poset G_i with the Hasse diagram to be a cycle $\tilde{A}_n$. For the special class there are no such posets. But in the second case the class contains only one poset G_0 with the Hasse diagram to be a $\tilde{D}_n$.*

It follows from Table 2 that $G_1 \cong AP1.8$, $G_2 \cong AP2.13$, $G_3 \cong AP3.17$, $G_4 \cong AP4.7$, $G_6 \cong AP6.20$, $G_7 \cong AP7.14$, $G_8 \cong AP8.21$, $G_9 \cong AP9.15$, $G_0 \cong AP5.28$.

It is clear that in cases when we not talking about canonical representatives of the classes, there may not be any differences between the two types of classes. As an example, we give the following easily verifiable statement.

Theorem 4. *Any class $C(M_i)$ contains a poset with the Hasse diagram to be a non-serial Dynkin diagram.*

3. Proof of Theorem 1

We use a list of all non-serial positive posets (up to isomorphism and duality), first obtained in [4], in the form indicated in [9, Section 4]. By $[i, j]$ with $i = 5, 6, 7$, j from 1 to 10, 32, 66, respectively, we denote the poset $\{1\} \coprod NSPi.j$. The elements of this new poset (which is considered in the form of Hasse diagrams) are numbered by the integer numbers $1, 2, \ldots, 1 + |NSPi.j|$ in a such way that $p \prec q$ implies the element with number q stands higher or to the right of the element with number p. We also use a classification of serial positive posets in the form of Theorem 1 [7]. By $[i]_{k,s}$ with $i = 1, 2, 3$ we denote the poset $\{0\} \coprod (i)_{k,s}$, where $(i)_{k,s}$ is the poset in condition (i) of the theorem with the parameters k and s.

The posets M_i are direct sums of the one-element poset $\{1\}$ and positive posets (non-serial for $i \neq 5$ and serial for $i = 5$): $M_1 = [5.8]$, $M_2 = [6.1]$, $M_3 = [6.22]$, $M_4 = [6.27]$, $M_5 \cong [2]_{3,4}$, $M_6 = [7.1]$, $M_7 = [7.3]$, $M_8 = [7.46]$, $M_9 = [7.56]$.

These posets are non-negative by Table 4.2 [6] of all NP-critical posets NP_i. They are neither positive, nor P-critical, nor serial strictly almost positive posets by Tables 4.1–4.3 [9] of all non-serial positive posets, Table 4.1 [6] of all P-critical ones PC_i and Theorems 1, 3 [7] about the positive and serial principal posets. Hence $M_1, \ldots, M_9 \in \mathcal{M}$.

By Corollary 2, to prove the theorem it is enough to verify that any poset $T = \{*\} \coprod S$ with S being positive either does not belong to $\mathcal{M}$, or is minimax isomorphic to M_i, or is minimax isomorphic to M_j^{op} for some i, j. Obviously, T can be considered up to duality.

For the cases of non-serial posets we have the following:

(A1) The posets $[5.1], [5.3] - [5.5], [6.2] - [6.4], [6.6] - [6.7], [6.10] - [6.12], [6.14]$ are positive, because $[5.1] \cong NSP6.19$, $[5.3] \cong NSP6.26$, $[5.4] \cong NSP6.28$, $[5.5] \cong NSP6.32$, $[6.2] \cong NSP7.37$, $[6.3] \cong NSP7.42$, $[6.4] \cong NSP7.43$, $[6.6] \cong NSP7.52$, $[6.7] \cong NSP7.53$, $[6.10] \cong NSP7.54$, $[6.11] \cong NSP7.60$, $[6.12] \cong NSP7.61$, $[6.14] \cong NSP7.66$.

(A2) The posets $[5.2], [6.13], [6.15], [7.7], [7.10], [7.19], [7.21], [7.23]$, are P-critical, because $[5.2] \cong P_{32}$, $[6.13] \cong P_{36}$, $[6.15] \cong P_{37}$, $[7.7] \cong P_{47}$, $[7.10] \cong P_{67}$, $[7.19] \cong P_{68}$, $[7.21] \cong P_{43}$, $[7.23] \cong P_{44}$, $[7.25] \cong P_{45}$.

(A3) $[6.5] \cong NP_{49}, [7.22] \cong NP_{58}, [7.24] \cong NP_{59}$; $[7.2], [7.5], [7.9], [7.14], [7.16] \supset NP_{49}$; $[5.6]-[5.7], [5.9], [6.16]-[6.21], [6.23]-[6.26], [6.28]-[6.29], [6.31] - [6.32], [7.26] - [7.45], [7.47] - [7.55], [7.57] - [7.58], [7.60] - [7.62], [7.64] - [7.66] \supset NP_{112}$.

(A4) $[5.10]_3^{\uparrow} \cong [5.8] = M_1$; $[6.8]_{27}^{\uparrow\downarrow} \cong [6.1]^{op} = M_2^{op}$, $[6.9]_{76}^{\downarrow\downarrow} \cong [6.1] = M_2$; $[6.30]_3^{\uparrow} \cong [6.22] = M_3$; $[6.27] = M_4$; $[7.8]_{387}^{\downarrow\downarrow\downarrow} \cong M_5^{op}$, $[7.11]_{872}^{\downarrow\downarrow\uparrow} \cong M_5$, $[7.12]_{28}^{\uparrow\downarrow} \cong M_5^{op}$, $[7.15]_{876}^{\downarrow\downarrow\downarrow} \cong M_5^{op}$, $[7.18]_{87}^{\downarrow\downarrow} \cong M_5$; $[7.20]_{238}^{\uparrow\uparrow\downarrow} \cong M_5$, $[7.13]_{28}^{\uparrow\downarrow} \cong [7.1]^{op} = M_6^{op}$, $[7.17]_{87}^{\downarrow\downarrow} \cong [7.1] = M_6$; $[7.4]_{148}^{\downarrow\downarrow\downarrow} \cong [7.3] = M_7$,

$[7.6]^{\uparrow\downarrow\downarrow\downarrow}_{2874} \cong [7.3] = M_7$; $[7.63]^{\uparrow}_{3} \cong [7.46] = M_8$; $[7.59]^{\uparrow\uparrow\downarrow}_{348} \cong [7.56] = M_9$.

For the cases of serial posets we have the following:

(B1) $[1]_{0,s} \cong (1)_{1,s}$, $[1]_{2,2} \cong NSP5.6$, $[1]_{2,3} \cong NSP6.18$, $[1]_{2,4} \cong NSP7.41$, $[1]_{2,5} \cong PC_{43}$, $[1]_{2,s} \supset NPC_{70}$ for $s \geq 6$, $[1]_{3,3} \cong PC_{35}$, $[1]_{k,s} \supset NPC_{55}$ for $k \geq 3$, $s \geq 4$;

(B2) $[2]_{1,3} \cong NSP5.9$, $[2]_{1,4} \cong NSP6.29$, $[2]_{1,5} \cong NSP7.62$, $[2]_{1,6} \cong PC_{46}$, $[2]_{1,s} \supset NPC_{74}$ for $s \geq 7$, $[2]_{2,2} \cong NSP5.7$, $[2]_{2,3} \cong NSP6.20$, $[2]_{2,4} \cong NSP7.44$, $[2]_{2,5} \cong PC_{48}$, $[2]_{2,s} \supset NPC_{76}$ for $s \geq 6$, $[2]_{3,3} \cong NSP7.26$, $[2]_{3,4} \cong M_5$, $[2]_{k,s} \supset NPC_{55}$ for $k \geq 3$, $s \geq 5$;

(B3) $[3]_{k,0} \cong (3)_{k,1}$, $[3]_{k,1}$ is a serial strictly almost positive poset (see Theorem 2 [7]), $[3]_{k,s} \supset NPC_{112}$ for $s \geq 2$.

So we have proved that the posets $M_1, \ldots, M_9$ form a minimax d-system of generators for the set $\mathcal{M}$.

Now we prove that this d-system is minimal.

Lemma 2. *Let S be a poset and, for $b \in S$, $S_b^< := \{x \in S \,|\, x < b\}$, $b^< := \{x \in S \,|\, x < b\}$, $b^> := \{x \in S \,|\, x > b\}$, $S_b^\circ = S^{\uparrow\downarrow}_{\{b^<\}\{b^>\}}$. Let T be a poset minimax equivalent to the poset S, Then the element b is isolated in T if and only if $T = S_b^\circ$.*

The lemma follows directly from the definitions.

Lemma 3. *The posets that are minimax equivalent to the poset $T_1 = \{1, 2, 3, 4\}$ with incomparable elements are exhausted up to isomorphism by the posets T_1, $T_2 = \{1 \prec 2, 3, 4\}$, $T_3 = \{1, 2, 3 \prec 4\}$ and $T_4 = \{1, 2 \prec 3, 4\}$.*

The lemma can be proved by simple calculations.

Consider first the posets M_2, M_3, M_4 of order 7.

($a1$) Obviously, $\underline{M_3} = T_1$. By Lemma 3, for any poset X minimax equivalent to M_3, $\underline{X}$ is isomorphic to some T_i. Then M_2 can not be minimax isomorphic to M_3, otherwise M_2 will contain two different P-critical subposets — $\underline{M_2} = \{1, 2 \prec 3 \prec 4, 2 \prec 6 \prec 7\}$ and $K \cong T_i$ for some $i = 1, 2, 3, 4$ (what is impossible according to Proposition 2). Analogously, $M_2 \ncong_{min,max} M_3^{op}$, $M_2 \ncong_{min,max} M_4$ and $M_2 \ncong_{min,max} M_4^{op}$.

($a2$) Suppose that a poset T is minimax equivalent to the poset $S := M_3$ and isomorphic to the poset M_4. Then by Lemma 2, $T = S_i^\circ$ for some $1 \leq i \leq 7$. The poset M_4 is of width 4 and its Hasse diagram has a cycle. Since S_i° is of width less than 4 for $i = 4, 6, 7$ and its Hasse diagram is a tree for $i = 1, 2, 3$, we came to a contradiction. So $M_3 \ncong_{min,max} M_4$. Analogously, $M_3 \ncong_{min,max} M_4^{op}$.

Consider now the posets $M_5, \ldots, M_9$ of order 8.

($b1$) Similarly as in ($a1$) it is can be proved that $X \not\cong_{min,max} Y$ for $X = M_5, M_6, M_7$ and $Y = M_8, M_9, M_8^{op}, M_9^{op}$. Also similarly when $X = M_5$, $Y = M_6, M_7, M_6^{op}, M_7^{op}$ if Lemma 3 is replaced by the fact that the poset $\{1, 2 \prec 3 \prec 4, 2 \prec 5 \prec 6\} \cong \underline{M_6} \cong \underline{M_7}$ is not minimax isomorphic to the poset $\{1, 2 \prec 3 \prec 4, 5 \prec 5 \prec 7\} = \underline{M_5}$ (see Table 1).

($b2$) Similarly as in ($a2$) it is proved that $M_8 \not\cong_{min,max} M_9, M_9^{op}$.

(c) Let us write out, up to isomorphism, all posets with isolated elements minimax equivalent to the poset $S := M_6$ (see Lemma 2):

$$
\begin{aligned}
&S_1^\circ = M_6;\\
&S_2^\circ \cong \{1 \prec 6, 2 \prec 3 \prec 4 \prec 5 \prec 6 \prec 7, 3 \prec 8\};\\
&S_3^\circ \cong \{1, 2 \prec 3 \prec 8, 2 \prec 6, 4 \prec 5 \prec 6 \prec 7 \prec 8\}^{op};\\
&S_4^\circ \cong \{1, 2 \prec 3 \prec 4 \prec 5 \prec 8, 3 \prec 7, 6 \prec 7 \prec 8\};\\
&S_5^\circ \cong \{1 \prec 2 \prec 3 \prec 7, 4 \prec 5 \prec 6 \prec 7 \prec 8, 2 \prec 5\}^{op};\\
&S_6^\circ \cong \{1, 2 \prec 3 \prec 4 \prec 8, 3 \prec 6, 5 \prec 6 \prec 7 \prec 8\};\\
&S_7^\circ \cong \{1 \prec 6, 2 \prec 3 \prec 4 \prec 5 \prec 6, 5 \prec 8, 3 \prec 7 \prec 8\}^{op};\\
&S_8^\circ \cong \{1, 2 \prec 3 \prec 8, 2 \prec 6, 4 \prec 5 \prec 6 \prec 7 \prec 8\}.
\end{aligned}
$$

It is easy to check that none of these posets is isomorphic to either M_7 or M_7^{op}.

Conditions ($a1$)–(c) prove that our d-system is minimal.

The proof is complete.

4. The classification of the essential almost positive posets

From Theorem 1 it follows that all essential almost positive posets can be divided into 9 classes with respect to minimax isomorphism and duality. The posets $M_1, \ldots M_9$ will be taken as their representatives. To classify all the essential almost positive posets (up to isomorphism and duality), we apply the algorithm [2, Subsection 2.2][2] to each class separately.

Proposition 4. *The posets minimax equivalent to M_1 are exhausted, up to isomorphism and duality, by the posets* 1.1, …, 1.12 *(see Table 2).*

Proof. **Step I.** Describe, up to strong isomorphism, all lower subposets of M_1: $X_{1,0} = \varnothing$, $X_{1,1} = \{1\}$, $X_{1,2} = \{2\}$, $X_{1,3} = \{3\}$, $X_{1,4} = \{1, 2\}$, $X_{1,5} = \{1, 3\}$, $X_{1,6} = \{2, 3\}$, $X_{1,7} = \{2, 5\}$, $X_{1,8} = \{1, 2, 3\}$,

[2]See also Section 4 [2].

$X_{1,9} = \{1,2,5\}$, $X_{1,10} = \{2,3,4\}$, $X_{1,11} = \{2,3,5\}$, $X_{1,12} = \{1,2,3,4\}$, $X_{1,13} = \{1,2,3,5\}$, $X_{1,14} = \{2,3,4,5\}$,$X_{1,15} = \{1,2,3,4,5\}$, $X_{1,16} = \{2,3,4,5,6\}$.

Denote by $K_{1,j}$ the poset $S_X^{\uparrow}$ for $S = M_1$ and $X = X_{1,j}$. Then $K_{1,0} \cong AP_{1.11}$, $K_{1,1} \cong AP_{1.6}$, $K_{1,2} \cong AP_{1.5}$, $K_{1,3} \cong AP_{1.3}$, $K_{1,4} \cong AP_{1.1}$, $K_{1,5} \cong AP_{1.8}$, $K_{1,6} \cong AP_{1.4}$, $K_{1,7} \cong AP_{1.8}^{op}$, $K_{1,8} \cong AP_{1.5}^{op}$, $K_{1,9} \cong AP_{1.6}^{op}$, $K_{1,10} \cong AP_{1.2}$, $K_{1,11} \cong AP_{1.3}^{op}$, $K_{1,12} \cong AP_{1.10}^{op}$, $K_{1,13} \cong AP_{1.11}^{op}$, $K_{1,14} \cong AP_{1.7}^{op}$, $K_{1,15} \cong AP_{1.12}$, $K_{1,16} \cong AP_{1.9}^{op}$.

Step II. Describe, up to strong isomorphism, all pairs (X, V) of proper lower subposets in M_1 such that $V \subseteq X$ and $V < S \setminus X$: $Y_{1,1} = (X_{1,13}, \{3\})$, $Y_{1,2} = (X_{1,15}, \{3\})$, $Y_{1,3} = (X_{1,22}, \{5\})$, $Y_{1,4} = (X_{1,22}, \{3,5\})$.

Denote by $K'_{1,j}$ the poset $(S_X^{\uparrow})_V^{\uparrow}$ for $S = M_1$ and $(X, V) = Y_{1,j}$. Then $K'_{1,1} \cong AP_{1.9}$, $K'_{1,2} \cong AP_{1.7}$, $K'_{1,3} \cong AP_{1.10}$, $K'_{1,4} \cong AP_{1.2}^{op}$.

Step III. As a result we have the posets indicated in the proposition and posets that are dual to non-selfdual of them (a normal class). $\square$

Proposition 5. *The posets minimax equivalent to M_2 are exhausted, up to isomorphism and duality, by the posets* $2.1, \ldots, 2.22$ *(see Table 2).*

Proof. **Step I.** Describe, up to strong isomorphism, all lower subposets of M_2: $X_{2,0} = \varnothing$, $X_{2,1} = \{1\}$, $X_{2,2} = \{2\}$, $X_{2,3} = \{5\}$, $X_{2,4} = \{1,2\}$, $X_{2,5} = \{1,5\}$, $X_{2,6} = \{2,3\}$, $X_{2,7} = \{2,5\}$, $X_{2,8} = \{1,2,3\}$, $X_{2,9} = \{1,2,5\}$, $X_{2,10} = \{2,3,4\}$, $X_{2,11} = \{2,3,5\}$, $X_{2,12} = \{2,5,6\}$, $X_{2,13} = \{1,2,3,4\}$, $X_{2,14} = \{1,2,3,5\}$, $X_{2,15} = \{1,2,5,6\}$, $X_{2,16} = \{2,3,4,5\}$, $X_{2,17} = \{2,3,5,6\}$, $X_{2,18} = \{2,5,6,7\}$, $X_{2,19} = \{1,2,3,4,5\}$, $X_{2,20} = \{1,2,3,5,6\}$, $X_{2,21} = \{1,2,5,6,7\}$, $X_{2,22} = \{2,3,4,5,6\}$, $X_{2,23} = \{2,3,5,6,7\}$, $X_{2,24} = \{1,2,3,4,5,6\}$, $X_{2,25} = \{1,2,3,5,6,7\}$, $X_{2,26} = \{2,3,4,5,6,7\}$.

Denote by $K_{2,j}$ the poset $S_X^{\uparrow}$ for $S = M_2$ and $X = X_{2,j}$. Then $K_{2,0} \cong AP_{2.8}$, $K_{2,1} \cong AP_{2.2}$, $K_{2,2} \cong AP_{2.9}^{op}$, $K_{2,3} \cong AP_{2.17}$, $K_{2,4} \cong AP_{2.17}^{op}$, $K_{2,5} \cong AP_{2.2}^{op}$, $K_{2,6} \cong AP_{2.16}$, $K_{2,7} \cong AP_{2.9}$, $K_{2,8} \cong AP_{2.21}$, $K_{2,9} \cong AP_{2.8}^{op}$, $K_{2,10} \cong AP_{2.5}$, $K_{2,11} \cong AP_{2.11}^{op}$, $K_{2,12} \cong AP_{2.15}$, $K_{2,13} \cong AP_{2.6}^{op}$, $K_{2,14} \cong AP_{2.10}^{op}$, $K_{2,15} \cong AP_{2.19}$, $K_{2,16} \cong AP_{2.18}^{op}$; $K_{2,17} \cong AP_{2.12}$, $K_{2,18} \cong AP_{2.4}$, $K_{2,19} \cong AP_{2.14}$, $K_{2,20} \cong AP_{2.13}$, $K_{2,21} \cong AP_{2.22}$, $K_{2,22} \cong AP_{2.20}$, $K_{2,23} \cong AP_{2.1}$, $K_{2,24} \cong AP_{2.10}$, $K_{2,25} \cong AP_{2.19}$, $K_{2,26} \cong AP_{2.3}^{op}$.

Step II. Describe, up to strong isomorphism, all pairs (X, V) of proper lower subposets in M_2 such that $V \subseteq X$ and $V < S \setminus X$: $Y_{2,1} = (X_{2,9}, \{2\})$, $Y_{2,2} = (X_{2,14}, \{2\})$, $Y_{2,3} = (X_{2,15}, \{2\})$, $Y_{2,4} = (X_{2,19}, \{2\})$,

$Y_{2,5} = (X_{2,19}, \{5\})$, $Y_{2,6} = (X_{2,19}, \{2,5\})$, $Y_{2,7} = (X_{2,20}, \{2\})$, $Y_{2,8} = (X_{2,21}, \{2\})$, $Y_{2,9} = (X_{2,24}, \{2\})$, $Y_{2,10} = (X_{2,24}, \{5\})$, $Y_{2,11} = (X_{2,24}, \{2,5\})$, $Y_{2,12} = (X_{2,24}, \{2,5,6\})$, $Y_{2,13} = (X_{2,25}, \{2\})$, $Y_{2,14} = (X_{2,25}, \{2,3\})$.

Denote by $K'_{2,j}$ the poset $(S^{\uparrow}_X)^{\uparrow}_V$ for $S = M_2$ and $(X,V) = Y_{2,j}$. Then $K'_{2,1} \cong AP_{2.3}$, $K'_{2,2} \cong AP^{op}_{2.20}$, $K'_{2,3} \cong AP^{op}_{2.1}$, $K'_{2,4} \cong AP_{2.18}$, $K'_{2,5} \cong AP_{2.6}$, $K'_{2,6} \cong AP^{op}_{2.5}$, $K'_{2,7} \cong AP^{op}_{2.12}$, $K'_{2,8} \cong AP^{op}_{2.4}$, $K'_{2,9} \cong AP_{2.11}$, $K'_{2,10} \cong AP^{op}_{2.21}$, $K'_{2,11} \cong AP^{op}_{2.16}$, $K'_{2,12} \cong AP_{2.7}$, $K'_{2,13} \cong AP^{op}_{2.15}$, $K'_{2,14} \cong AP_{2.7}$.

Step III. We have the same result as in case M_1. □

Proposition 6. *The posets minimax equivalent to M_3 are exhausted, up to isomorphism and duality, by the posets* 3.1, …, 3.24 *(see Table 2).*

Proof. **Step I.** Describe, up to strong isomorphism, all lower subposets of M_3: $X_{3,0} = \varnothing$, $X_{3,1} = \{1\}$, $X_{3,2} = \{2\}$, $X_{3,3} = \{3\}$, $X_{3,4} = \{5\}$, $X_{3,5} = \{1,2\}$, $X_{3,6} = \{1,3\}$, $X_{3,7} = \{1,5\}$, $X_{3,8} = \{2,3\}$, $X_{3,9} = \{2,5\}$, $X_{3,10} = \{3,5\}$, $X_{3,11} = \{1,2,3\}$, $X_{3,12} = \{1,2,5\}$, $X_{3,13} = \{1,3,5\}$, $X_{3,14} = \{2,3,4\}$, $X_{3,15} = \{2,3,5\}$, $X_{3,16} = \{3,5,6\}$, $X_{3,17} = \{1,2,3,4\}$, $X_{3,18} = \{1,2,3,5\}$, $X_{3,19} = \{1,3,5,6\}$, $X_{3,20} = \{2,3,4,5\}$, $X_{3,21} = \{2,3,5,6\}$, $X_{3,22} = \{3,5,6,7\}$, $X_{3,23} = \{1,2,3,4,5\}$, $X_{3,24} = \{1,2,3,5,6\}$, $X_{3,25} = \{1,3,5,6,7\}$, $X_{3,26} = \{2,3,4,5,6\}$, $X_{3,27} = \{2,3,5,6,7\}$, $X_{3,28} = \{1,2,3,4,5,6\}$, $X_{3,29} = \{1,2,3,5,6,7\}$, $X_{3,30} = \{2,3,4,5,6,7\}$.

Denote by $K_{3,j}$ the poset $S^{\uparrow}_X$ for $S = M_3$ and $X = X_{3,j}$. Then $K_{3,0} \cong AP_{3.22}$, $K_{3,1} \cong AP_{3.14}$, $K_{3,2} \cong AP_{3.12}$, $K_{3,3} \cong AP_{3.8}$, $K_{3,4} \cong AP_{3.7}$, $K_{3,5} \cong AP_{3.1}$, $K_{3,6} \cong AP_{3.17}$, $K_{3,7} \cong AP_{3.2}$, $K_{3,8} \cong AP_{3.11}$, $K_{3,9} \cong AP^{op}_{3.17}$, $K_{3,10} \cong AP_{3.6}$, $K_{3,11} \cong AP^{op}_{3.12}$, $K_{3,12} \cong AP^{op}_{3.14}$, $K_{3,13} \cong AP^{op}_{3.7}$, $K_{3,14} \cong AP_{3.4}$, $K_{3,15} \cong AP^{op}_{3.8}$, $K_{3,16} \cong AP_{3.13}$, $K_{3,17} \cong AP^{op}_{3.20}$, $K_{3,18} \cong AP^{op}_{3.22}$, $K_{3,19} \cong AP^{op}_{3.18}$, $K_{3,20} \cong AP^{op}_{3.15}$, $K_{3,21} \cong AP^{op}_{3.9}$, $K_{3,22} \cong AP_{3.3}$, $K_{3,23} \cong AP^{op}_{3.24}$, $K_{3,24} \cong AP^{op}_{3.23}$, $K_{3,25} \cong AP^{op}_{3.21}$, $K_{3,26} \cong AP^{op}_{3.10}$, $K_{3,27} \cong AP^{op}_{3.16}$, $K_{3,28} \cong AP_{3.23}$, $K_{3,29} \cong AP_{3.24}$, $K_{3,30} \cong AP^{op}_{3.19}$.

Step II. Describe, up to strong isomorphism, all pairs (X,V) of proper lower subposets in M_3 such that $V \subseteq X$ and $V < S \setminus X$: $Y_{3,1} = (X_{3,18}, \{3\})$, $Y_{3,2} = (X_{3,23}, \{3\})$, $Y_{3,3} = (X_{3,23}, \{5\})$, $Y_{3,4} = (X_{3,23}, \{3,5\})$, $Y_{3,5} = (X_{3,24}, \{3\})$, $Y_{3,6} = (X_{3,28}, \{3\})$, $Y_{3,7} = (X_{3,28}, \{5\})$, $Y_{3,8} = (X_{3,28}, \{3,5\})$, $Y_{3,9} = (X_{3,28}, \{3,5,6\})$, $Y_{3,10} = (X_{3,29}, \{2\})$, $Y_{3,11} = (X_{3,29}, \{3\})$, $Y_{3,12} = (X_{3,29}, \{2,3\})$.

Denote by $K'_{3,j}$ the poset $(S^{\uparrow}_X)^{\uparrow}_V$ for $S = M_3$ and $(X,V) = Y_{3,j}$. Then $K'_{3,1} \cong AP_{3.19}$, $K'_{3,2} \cong AP_{3.16}$, $K'_{3,3} \cong AP_{3.21}$, $K'_{3,4} \cong AP^{op}_{3.3}$, $K'_{3,5} \cong AP_{3.10}$, $K'_{3,6} \cong AP_{3.9}$, $K'_{3,7} \cong AP_{3.18}$, $K'_{3,8} \cong AP^{op}_{3.13}$, $K'_{3,9} \cong AP_{3.5}$, $K'_{3,10} \cong AP_{3.20}$, $K'_{3,11} \cong AP_{3.15}$, $K'_{3,12} \cong AP^{op}_{3.4}$.

Step III. We have the same result as in case M_1. $\square$

Proposition 7. *The posets minimax equivalent to M_4 are exhausted, up to isomorphism and duality, by the posets $4.1, \dots, 4.15$ (see Table 2).*

Proof. **Step I.** Describe, up to strong isomorphism, all lower subposets of M_4: $X_{4,0} = \varnothing$, $X_{4,1} = \{1\}$, $X_{4,2} = \{2\}$, $X_{4,3} = \{3\}$, $X_{4,4} = \{1,2\}$, $X_{4,5} = \{1,3\}$, $X_{4,6} = \{2,3\}$, $X_{4,7} = \{3,4\}$, $X_{4,8} = \{3,6\}$, $X_{4,9} = \{1,2,3\}$, $X_{4,10} = \{1,3,4\}$, $X_{4,11} = \{1,3,6\}$, $X_{4,12} = \{2,3,4\}$, $X_{4,13} = \{2,3,6\}$, $X_{4,14} = \{3,4,6\}$, $X_{4,15} = \{1,2,3,4\}$, $X_{4,16} = \{1,2,3,6\}$, $X_{4,17} = \{1,3,4,6\}$, $X_{4,18} = \{2,3,4,5\}$, $X_{4,19} = \{2,3,4,6\}$, $X_{4,20} = \{3,4,6,7\}$, $X_{4,21} = \{1,2,3,4,5\}$, $X_{4,22} = \{1,2,3,4,6\}$, $X_{4,23} = \{1,3,4,6,7\}$, $X_{4,24} = \{2,3,4,5,6\}$, $X_{4,25} = \{2,3,4,6,7\}$, $X_{4,26} = \{1,2,3,4,5,6\}$, $X_{4,27} = \{1,2,3,4,6,7\}$, $X_{4,28} = \{2,3,4,5,6,7\}$.

Denote by $K_{4,j}$ the poset $S^{\uparrow}_{X}$ for $S = M_4$ and $X = X_{4,j}$. Then $K_{4,0} \cong AP_{4.14}$, $K_{4,1} \cong AP_{4.11}$, $K_{4,2} \cong AP_{4.10}$, $K_{4,3} \cong AP_{4.13}$, $K_{4,4} \cong AP^{op}_{4.2}$, $K_{4,5} \cong AP_{4.6}$, $K_{4,6} \cong AP_{4.5}$, $K_{4,7} \cong AP_{4.5}$, $K_{4,8} \cong AP_{4.6}$, $K_{4,9} \cong AP_{4.3}$, $K_{4,10} \cong AP_{4.7}$, $K_{4,11} \cong AP_{4.8}$, $K_{4,12} \cong AP_{4.4}$, $K_{4,13} \cong AP_{4.7}$, $K_{4,14} \cong AP_{4.3}$, $K_{4,15} \cong AP^{op}_{4.5}$, $K_{4,16} \cong AP^{op}_{4.6}$, $K_{4,17} \cong AP^{op}_{4.6}$, $K_{4,18} \cong AP^{op}_{4.1}$, $K_{4,19} \cong AP^{op}_{4.5}$, $K_{4,20} \cong AP_{4.2}$, $K_{4,21} \cong AP^{op}_{4.9}$, $K_{4,22} \cong AP^{op}_{4.13}$, $K_{4,23} \cong AP^{op}_{4.11}$, $K_{4,24} \cong AP^{op}_{4.9}$, $K_{4,25} \cong AP^{op}_{4.10}$, $K_{4,26} \cong AP^{op}_{4.15}$, $K_{4,27} \cong AP^{op}_{4.14}$, $K_{4,28} \cong AP^{op}_{4.12}$.

Step II. Describe, up to strong isomorphism, all pairs (X, V) of proper lower subposets in M_4 such that $V \subseteq X$ and $V < S \setminus X$: $Y_{4,1} = (X_{4,9}, \{3\})$, $Y_{4,2} = (X_{4,15}, \{3\})$, $Y_{4,3} = (X_{4,16}, \{3\})$, $Y_{4,4} = (X_{4,21}, \{3\})$, $Y_{4,5} = (X_{4,22}, \{3\})$, $Y_{4,6} = (X_{4,22}, \{3,4\})$, $Y_{4,7} = (X_{4,26}, \{3\})$, $Y_{4,8} = (X_{4,26}, \{3,4\})$, $Y_{4,9} = (X_{4,26}, \{3,6\})$, $Y_{4,10} = (X_{4,26}, \{3,4,6\})$, $Y_{4,11} = (X_{4,27}, \{2\})$, $Y_{4,12} = (X_{4,27}, \{3\})$, $Y_{4,13} = (X_{4,27}, \{2,3\})$, $Y_{4,14} = (X_{4,27}, \{3,4\})$, $Y_{4,15} = (X_{4,27}, \{2,3,4\})$.

Denote by $K'_{4,j}$ the poset $(S^{\uparrow}_{X})^{\uparrow}_{V}$ for $S = M_4$ and $(X, V) = Y_{4,j}$. Then $K'_{4,1} \cong AP_{4.2}$, $K'_{4,2} \cong AP^{op}_{4.10}$, $K'_{4,3} \cong AP^{op}_{4.11}$, $K'_{4,4} \cong AP^{op}_{4.12}$, $K'_{4,5} \cong AP^{op}_{4.14}$, $K'_{4,6} \cong AP_{4.12}$, $K'_{4,7} \cong AP_{4.14}$, $K'_{4,8} \cong AP_{4.10}$, $K'_{4,9} \cong AP_{4.11}$, $K'_{4,10} \cong AP^{op}_{4.2}$, $K'_{4,11} \cong AP_{4.12}$, $K'_{4,12} \cong AP_{4.15}$, $K'_{4,13} \cong AP_{4.9}$, $K'_{4,14} \cong AP_{4.9}$, $K'_{4,15} \cong AP^{op}_{4.1}$.

Step III. We have the same result as in case M_1. $\square$

Proposition 8. *The posets minimax equivalent to M_5 are exhausted, up to isomorphism and duality, by the posets $5.1, \dots, 5.52$ (see Table 2).*

Proof. **Step I.** Describe, up to strong isomorphism, all lower subposets of M_5: $X_{5,0} = \varnothing$, $X_{5,1} = \{1\}$, $X_{5,2} = \{2\}$, $X_{5,3} = \{5\}$, $X_{5,4} =$

$\{1,2\}$, $X_{5,5}=\{1,5\}$, $X_{5,6}=\{2,3\}$, $X_{5,7}=\{2,5\}$, $X_{5,8}=\{5,6\}$, $X_{5,9}=\{1,2,3\}$, $X_{5,10}=\{1,2,5\}$, $X_{5,11}=\{1,5,6\}$, $X_{5,12}=\{2,3,4\}$, $X_{5,13}=\{2,3,5\}$, $X_{5,14}=\{2,5,6\}$, $X_{5,15}=\{5,6,7\}$, $X_{5,16}=\{1,2,3,4\}$, $X_{5,17}=\{1,2,3,5\}$, $X_{5,18}=\{1,2,5,6\}$, $X_{5,19}=\{1,5,6,7\}$, $X_{5,20}=\{2,3,4,5\}$, $X_{5,21}=\{2,3,5,6\}$, $X_{5,22}=\{2,5,6,7\}$, $X_{5,23}=\{1,2,3,4,5\}$, $X_{5,24}=\{1,2,3,5,6\}$, $X_{5,25}=\{1,2,5,6,7\}$, $X_{5,26}=\{2,3,4,5,6\}$, $X_{5,27}=\{2,3,5,6,7\}$, $X_{5,28}=\{2,5,6,7,8\}$, $X_{5,29}=\{1,2,3,4,5,6\}$, $X_{5,30}=\{1,2,3,5,6,7\}$, $X_{5,31}=\{1,2,5,6,7,8\}$, $X_{5,32}=\{2,3,4,5,6,7\}$, $X_{5,33}=\{2,3,5,6,7,8\}$, $X_{5,34}=\{1,2,3,4,5,6,7\}$, $X_{5,35}=\{1,2,3,5,6,7,8\}$, $X_{5,36}=\{2,3,4,5,6,7,8\}$.

Denote by $K_{5,j}$ the poset $S_X^{\uparrow}$ for $S=M_5$ and $X=X_{5,j}$. Then $K_{5,0}\cong AP_{5.21}$, $K_{5,1}\cong AP_{5.4}$, $K_{5,2}\cong AP_{5.22}^{op}$, $K_{5,3}\cong AP_{5.29}$, $K_{5,4}\cong AP_{5.43}^{op}$, $K_{5,5}\cong AP_{5.1}^{op}$, $K_{5,6}\cong AP_{5.42}$, $K_{5,7}\cong AP_{5.18}^{op}$, $K_{5,8}\cong AP_{5.44}$, $K_{5,9}\cong AP_{5.50}$, $K_{5,10}\cong AP_{5.28}^{op}$, $K_{5,11}\cong AP_{5.3}^{op}$, $K_{5,12}\cong AP_{5.11}$, $K_{5,13}\cong AP_{5.36}$, $K_{5,14}\cong AP_{5.30}$, $K_{5,15}\cong AP_{5.51}$, $K_{5,16}\cong AP_{5.10}^{op}$, $K_{5,17}\cong AP_{5.35}^{op}$, $K_{5,18}\cong AP_{5.26}^{op}$, $K_{5,19}\cong AP_{5.9}^{op}$, $K_{5,20}\cong AP_{5.49}^{op}$, $K_{5,21}\cong AP_{5.20}^{op}$, $K_{5,22}\cong AP_{5.37}$, $K_{5,23}\cong AP_{5.41}^{op}$, $K_{5,24}\cong AP_{5.19}$, $K_{5,25}\cong AP_{5.38}$, $K_{5,26}\cong AP_{5.34}^{op}$, $K_{5,27}\cong AP_{5.25}$, $K_{5,28}\cong AP_{5.8}$, $K_{5,29}\cong AP_{5.24}^{op}$, $K_{5,30}\cong AP_{5.31}^{op}$, $K_{5,31}\cong AP_{5.52}^{op}$, $K_{5,32}\cong AP_{5.27}$, $K_{5,33}\cong AP_{5.2}$, $K_{5,34}\cong AP_{5.17}$, $K_{5,35}\cong AP_{5.45}^{op}$, $K_{5,36}\cong AP_{5.5}^{op}$.

Step II. Describe, up to strong isomorphism, all pairs (X,V) of proper lower subposets in M_5 such that $V\subseteq X$ and $V<S\setminus X$: $Y_{5,1}=(X_{5,23},\{5\})$, $Y_{5,2}=(X_{5,25},\{2\})$, $Y_{5,3}=(X_{5,29},\{5\})$, $Y_{5,4}=(X_{5,29},\{5,6\})$, $Y_{5,5}=(X_{5,30},\{2\})$, $Y_{5,6}=(X_{5,31},\{2\})$, $Y_{5,7}=(X_{5,34},\{2\})$, $Y_{5,8}=(X_{5,34},\{5\})$, $Y_{5,9}=(X_{5,34},\{2,5\})$, $Y_{5,10}=(X_{5,34},\{5,6\})$, $Y_{5,11}=(X_{5,34},\{2,5,6\})$, $Y_{5,12}=(X_{5,34},\{5,6,7\})$, $Y_{5,13}=(X_{5,34},\{2,5,6,7\})$, $Y_{5,14}=(X_{5,35},\{2\})$, $Y_{5,15}=(X_{5,35},\{2,3\})$.

Denote by $K'_{5,j}$ the poset $(S_X^{\uparrow})_V^{\uparrow}$ for $S=M_5$ and $(X,V)=Y_{5,j}$. Then $K'_{5,1}\cong AP_{5.16}^{op}$, $K'_{5,2}\cong AP_{5.12}$, $K'_{5,3}\cong AP_{5.46}$, $K'_{5,4}\cong AP_{5.14}$, $K'_{5,5}\cong AP_{5.48}$, $K'_{5,6}\cong AP_{5.7}^{op}$, $K'_{5,7}\cong AP_{5.32}^{op}$, $K'_{5,8}\cong AP_{5.23}$, $K'_{5,9}\cong AP_{5.33}$, $K'_{5,10}\cong AP_{5.39}$, $K'_{5,11}\cong AP_{5.47}$, $K'_{5,12}\cong AP_{5.6}$, $K'_{5,13}\cong AP_{5.13}^{op}$, $K'_{5,14}\cong AP_{5.40}^{op}$, $K'_{5,15}\cong AP_{5.15}^{op}$.

Step III. As a result we have up to duality every poset indicated in the proposition, but not have dual to them (a special class). $\square$

Proposition 9. *The posets minimax equivalent to M_6 are exhausted, up to isomorphism and duality, by the posets* 6.1, …, 6.30 *(see Table 2).*

Proof. **Step I.** Describe, up to strong isomorphism, all lower of M_6: $X_{6,0}=\varnothing$, $X_{6,1}=\{1\}$, $X_{6,2}=\{2\}$, $X_{6,3}=\{5\}$, $X_{6,4}=\{1,2\}$, $X_{6,5}=$

$\{1,5\}$, $X_{6,6}=\{2,3\}$, $X_{6,7}=\{2,5\}$, $X_{6,8}=\{5,6\}$, $X_{6,9}=\{1,2,3\}$, $X_{6,10}=\{1,2,5\}$, $X_{6,11}=\{1,5,6\}$, $X_{6,12}=\{2,3,4\}$, $X_{6,13}=\{2,3,5\}$, $X_{6,14}=\{2,5,6\}$, $X_{6,15}=\{1,2,3,4\}$, $X_{6,16}=\{1,2,3,5\}$, $X_{6,17}=\{1,2,5,6\}$, $X_{6,18}=\{2,3,4,5\}$, $X_{6,19}=\{2,3,5,6\}$, $X_{6,20}=\{2,5,6,7\}$, $X_{6,21}=\{1,2,3,4,5\}$, $X_{6,22}=\{1,2,3,5,6\}$, $X_{6,23}=\{1,2,5,6,7\}$, $X_{6,24}=\{2,3,4,5,6\}$, $X_{6,25}=\{2,3,5,6,7\}$, $X_{6,26}=\{2,5,6,7,8\}$, $X_{6,27}=\{1,2,3,4,5,6\}$, $X_{6,28}=\{1,2,3,5,6,7\}$, $X_{6,29}=\{1,2,5,6,7,8\}$, $X_{6,30}=\{2,3,4,5,6,7\}$, $X_{6,31}=\{2,3,5,6,7,8\}$, $X_{6,32}=\{1,2,3,4,5,6,7\}$, $X_{6,33}=\{1,2,3,5,6,7,8\}$, $X_{6,34}=\{2,3,4,5,6,7,8\}$.

Denote by $K_{6,j}$ the poset $S^{\uparrow}_{X}$ for $S=M_6$ and $X=X_{6,j}$. Then $K_{6,0}\cong AP_{6.13}$, $K_{6,1}\cong AP_{6.3}$, $K_{6,2}\cong AP^{op}_{6.14}$, $K_{6,3}\cong AP_{6.16}$, $K_{6,4}\cong AP^{op}_{6.26}$, $K_{6,5}\cong AP_{6.1}$, $K_{6,6}\cong AP_{6.25}$, $K_{6,7}\cong AP_{6.12}$, $K_{6,8}\cong AP_{6.26}$, $K_{6,9}\cong AP_{6.29}$, $K_{6,10}\cong AP^{op}_{6.16}$, $K_{6,11}\cong AP^{op}_{6.3}$, $K_{6,12}\cong AP_{6.7}$, $K_{6,13}\cong AP_{6.22}$, $K_{6,14}\cong AP_{6.14}$, $K_{6,15}\cong AP^{op}_{6.6}$, $K_{6,16}\cong AP^{op}_{6.21}$, $K_{6,17}\cong AP^{op}_{6.13}$, $K_{6,18}\cong AP^{op}_{6.28}$, $K_{6,19}\cong AP_{6.11}$, $K_{6,20}\cong AP_{6.23}$, $K_{6,21}\cong AP^{op}_{6.24}$, $K_{6,22}\cong AP^{op}_{6.10}$, $K_{6,23}\cong AP_{6.27}$, $K_{6,24}\cong AP_{6.17}$, $K_{6,25}\cong AP_{6.18}$, $K_{6,26}\cong AP_{6.5}$, $K_{6,27}\cong AP_{6.15}$, $K_{6,28}\cong AP_{6.20}$, $K_{6,29}\cong AP_{6.30}$, $K_{6,30}\cong AP_{6.19}$, $K_{6,31}\cong AP_{6.2}$, $K_{6,32}\cong AP_{6.10}$, $K_{6,33}\cong AP^{op}_{6.27}$, $K_{6,34}\cong AP^{op}_{6.4}$.

Step II. Describe, up to strong isomorphism, all pairs (X,V) of proper lower subposets in M_6 such that $V\subseteq X$ and $V<S\setminus X$: $Y_{6,1}=(X_{6,17},\{2\})$, $Y_{6,2}=(X_{6,21},\{5\})$, $Y_{6,3}=(X_{6,22},\{2\})$, $Y_{6,4}=(X_{6,23},\{2\})$, $Y_{6,5}=(X_{6,27},\{2\})$, $Y_{6,6}=(X_{6,27},\{5\})$, $Y_{6,7}=(X_{6,27},\{2,5\})$, $Y_{6,8}=(X_{6,27},\{5,6\})$, $Y_{6,9}=(X_{6,27},\{2,5,6\})$, $Y_{6,10}=(X_{6,28},\{2\})$, $Y_{6,11}=(X_{6,29},\{2\})$, $Y_{6,12}=(X_{6,32},\{2\})$, $Y_{6,13}=(X_{6,32},\{5\})$, $Y_{6,14}=(X_{6,32},\{2,5\})$, $Y_{6,15}=(X_{6,32},\{5,6\})$, $Y_{6,16}=(X_{6,32},\{2,5,6\})$, $Y_{6,17}=(X_{6,32},\{2,5,6,7\})$, $Y_{6,18}=(X_{6,33},\{2\})$, $Y_{6,19}=(X_{6,33},\{2,3\})$.

Denote by $K'_{6,j}$ the poset $(S^{\uparrow}_{X})^{\uparrow}_{V}$ for $S=M_6$ and $(X,V)=Y_{6,j}$. Then $K'_{6,1}\cong AP_{6.4}$, $K'_{6,2}\cong AP_{6.9}$, $K'_{6,3}\cong AP^{op}_{6.19}$, $K'_{6,4}\cong AP^{op}_{6.2}$, $K'_{6,5}\cong AP^{op}_{6.17}$, $K'_{6,6}\cong AP_{6.24}$, $K'_{6,7}\cong AP_{6.28}$, $K'_{6,8}\cong AP_{6.6}$, $K'_{6,9}\cong AP^{op}_{6.7}$, $K'_{6,10}\cong AP^{op}_{6.18}$, $K'_{6,11}\cong AP^{op}_{6.5}$, $K'_{6,12}\cong AP^{op}_{6.11}$, $K'_{6,13}\cong AP_{6.21}$, $K'_{6,14}\cong AP^{op}_{6.22}$, $K'_{6,15}\cong AP^{op}_{6.29}$, $K'_{6,16}\cong AP^{op}_{6.25}$, $K'_{6,17}\cong AP_{6.8}$, $K'_{6,18}\cong AP^{op}_{6.23}$, $K'_{6,19}\cong AP^{op}_{6.8}$.

Step III. We have the same result as in case M_1. □

Proposition 10. *The posets minimax equivalent to M_7 are exhausted, up to isomorphism and duality, by the posets* 7.1, …, 7.31 *(see Table 2).*

Proof. **Step I.** Describe, up to strong isomorphism, all lower subposets of M_7: $X_{7,0}=\varnothing$, $X_{7,1}=\{1\}$, $X_{7,2}=\{2\}$, $X_{7,3}=\{5\}$, $X_{7,4}=\{1,2\}$,

$X_{7,5} = \{1,5\}$, $X_{7,6} = \{2,3\}$, $X_{7,7} = \{2,5\}$, $X_{7,8} = \{1,2,3\}$, $X_{7,9} = \{1,2,5\}$, $X_{7,10} = \{2,3,4\}$, $X_{7,11} = \{2,3,5\}$, $X_{7,12} = \{2,5,6\}$, $X_{7,13} = \{1,2,3,4\}$, $X_{7,14} = \{1,2,3,5\}$, $X_{7,15} = \{1,2,5,6\}$, $X_{7,16} = \{2,3,4,5\}$, $X_{7,17} = \{2,3,5,6\}$, $X_{7,18} = \{2,5,6,7\}$, $X_{7,19} = \{1,2,3,4,5\}$, $X_{7,20} = \{1,2,3,5,6\}$, $X_{7,21} = \{1,2,5,6,7\}$, $X_{7,22} = \{2,3,4,5,6\}$, $X_{7,23} = \{2,3,5,6,7\}$, $X_{7,24} = \{1,2,3,4,5,6\}$, $X_{7,25} = \{1,2,3,5,6,7\}$, $X_{7,26} = \{2,3,4,5,6,7\}$, $X_{7,27} = \{2,3,5,6,7,8\}$, $X_{7,28} = \{1,2,3,4,5,6,7\}$, $X_{7,29} = \{1,2,3,5,6,7,8\}$, $X_{7,30} = \{2,3,4,5,6,7,8\}$.

Denote by $K_{7,j}$ the poset $S_X^{\uparrow}$ for $S = M_7$ and $X = X_{7,j}$. Then $K_{7,0} \cong AP_{7.16}$, $K_{7,1} \cong AP_{7.4}$, $K_{7,2} \cong AP_{7.19}$, $K_{7,3} \cong AP_{7.22}$, $K_{7,4} \cong AP^{op}_{7.29}$, $K_{7,5} \cong AP_{7.6}$, $K_{7,6} \cong AP^{op}_{7.16}$, $K_{7,7} \cong AP_{7.12}$, $K_{7,8} \cong AP_{7.30}$, $K_{7,9} \cong AP_{7.17}$, $K_{7,10} \cong AP_{7.7}$, $K_{7,11} \cong AP^{op}_{7.11}$, $K_{7,12} \cong AP_{7.25}$, $K_{7,13} \cong AP^{op}_{7.3}$, $K_{7,14} \cong AP^{op}_{7.18}$, $K_{7,15} \cong AP_{7.23}$, $K_{7,16} \cong AP^{op}_{7.21}$, $K_{7,17} \cong AP_{7.13}$, $K_{7,18} \cong AP^{op}_{7.31}$, $K_{7,19} \cong AP^{op}_{7.15}$, $K_{7,20} \cong AP_{7.14}$, $K_{7,21} \cong AP_{7.28}$, $K_{7,22} \cong AP_{7.27}$, $K_{7,23} \cong AP_{7.15}$, $K_{7,24} \cong AP^{op}_{7.13}$, $K_{7,25} \cong AP_{7.18}$, $K_{7,26} \cong AP_{7.21}$, $K_{7,27} \cong AP_{7.3}$, $K_{7,28} \cong AP_{7.11}$, $K_{7,29} \cong AP^{op}_{7.30}$, $K_{7,30} \cong AP^{op}_{7.7}$.

Step II. Describe, up to strong isomorphism, all pairs (X, V) of proper lower subposets in M_2 such that $V \subseteq X$ and $V < S \setminus X$: $Y_{7,1} = (X_{7,9}, \{2\})$, $Y_{7,2} = (X_{7,18}, \{2\})$, $Y_{7,3} = (X_{7,19}, \{2\})$, $Y_{7,4} = (X_{7,19}, \{5\})$, $Y_{7,5} = (X_{7,19}, \{2,5\})$, $Y_{7,6} = (X_{7,20}, \{2\})$, $Y_{7,7} = (X_{7,21}, \{2\})$, $Y_{7,8} = (X_{7,24}, \{2\})$, $Y_{7,9} = (X_{7,24}, \{5\})$, $Y_{7,10} = (X_{7,24}, \{2,5\})$, $Y_{7,11} = (X_{7,24}, \{2,5,6\})$, $Y_{7,12} = (X_{7,25}, \{2\})$, $Y_{7,13} = (X_{7,25}, \{2,3\})$, $Y_{7,14} = (X_{7,28}, \{2\})$, $Y_{7,15} = (X_{7,28}, \{5\})$, $Y_{7,16} = (X_{7,28}, \{2,3\})$, $Y_{7,17} = (X_{7,28}, \{2,5\})$, $Y_{7,18} = (X_{7,28}, \{2,3,5\})$, $Y_{7,19} = (X_{7,28}, \{2,5,6\})$, $Y_{7,20} = (X_{7,28}, \{2,3,5,6\})$, $Y_{7,21} = (X_{7,28}, \{2,5,6,7\})$, $Y_{7,22} = (X_{7,28}, \{2,3,5,6,7\})$, $Y_{7,23} = (X_{7,29}, \{2\})$, $Y_{7,24} = (X_{7,29}, \{2,3\})$.

Denote by $K'_{7,j}$ the $(S_X^{\uparrow})_V^{\uparrow}$ for $S = M_7$ and $(X, V) = Y_{7,j}$. Then $K'_{7,1} \cong AP_{7.5}$, $K'_{7,2} \cong AP^{op}_{7.28}$, $K'_{7,3} \cong AP_{7.31}$, $K'_{7,4} \cong AP_{7.8}$, $K'_{7,5} \cong AP^{op}_{7.2}$, $K'_{7,6} \cong AP^{op}_{7.23}$, $K'_{7,7} \cong AP^{op}_{7.5}$, $K'_{7,8} \cong AP^{op}_{7.25}$, $K'_{7,9} \cong AP^{op}_{7.26}$, $K'_{7,10} \cong AP^{op}_{7.20}$, $K'_{7,11} \cong AP_{7.9}$, $K'_{7,12} \cong AP^{op}_{7.17}$, $K'_{7,13} \cong AP^{op}_{7.6}$, $K'_{7,14} \cong AP^{op}_{7.12}$, $K'_{7,15} \cong AP^{op}_{7.24}$, $K'_{7,16} \cong AP^{op}_{7.22}$, $K'_{7,17} \cong AP_{7.10}$, $K'_{7,18} \cong AP_{7.24}$, $K'_{7,19} \cong AP_{7.20}$, $K'_{7,20} \cong AP_{7.26}$, $K'_{7,21} \cong AP_{7.2}$, $K'_{7,22} \cong AP^{op}_{7.8}$, $K'_{7,23} \cong AP_{7.29}$, $K'_{7,24} \cong AP^{op}_{7.4}$.

Step III. We have the same result as in case M_1. $\square$

Proposition 11. *The posets minimax equivalent to M_8 are exhausted, up to isomorphism and duality, by the posets* 8.1, ..., 8.30 *(see Table 2).*

Proof. **Step I.** Describe, up to strong isomorphism, all lower of M_8: $X_{8,0} = \varnothing$, $X_{8,1} = \{1\}$, $X_{8,2} = \{2\}$, $X_{8,3} = \{3\}$, $X_{8,4} = \{5\}$, $X_{8,5} = \{1,2\}$,

$X_{8,6} = \{1,3\}$, $X_{8,7} = \{1,5\}$, $X_{8,8} = \{2,3\}$, $X_{8,9} = \{2,5\}$, $X_{8,10} = \{3,5\}$, $X_{8,11} = \{1,2,3\}$, $X_{8,12} = \{1,2,5\}$, $X_{8,13} = \{1,3,5\}$, $X_{8,14} = \{2,3,4\}$, $X_{8,15} = \{2,3,5\}$, $X_{8,16} = \{3,5,6\}$, $X_{8,17} = \{1,2,3,4\}$, $X_{8,18} = \{1,2,3,5\}$, $X_{8,19} = \{1,3,5,6\}$, $X_{8,20} = \{2,3,4,5\}$, $X_{8,21} = \{2,3,5,6\}$, $X_{8,22} = \{3,5,6,7\}$, $X_{8,23} = \{1,2,3,4,5\}$, $X_{8,24} = \{1,2,3,5,6\}$, $X_{8,25} = \{1,3,5,6,7\}$, $X_{8,26} = \{2,3,4,5,6\}$, $X_{8,27} = \{2,3,5,6,7\}$, $X_{8,28} = \{3,5,6,7,8\}$, $X_{8,29} = \{1,2,3,4,5,6\}$, $X_{8,30} = \{1,2,3,5,6,7\}$, $X_{8,31} = \{1,3,5,6,7,8\}$, $X_{8,32} = \{2,3,4,5,6,7\}$, $X_{8,33} = \{2,3,5,6,7,8\}$, $X_{8,34} = \{1,2,3,4,5,6,7\}$, $X_{8,35} = \{1,2,3,5,6,7,8\}$, $X_{8,36} = \{2,3,4,5,6,7,8\}$.

Denote by $K_{8,j}$ the poset $S^{\uparrow}_{X}$ for $S = M_8$ and $X = X_{8,j}$. Then $K_{8,0} \cong AP_{8.28}$, $K_{8,1} \cong AP_{8.18}$, $K_{8,2} \cong AP_{8.16}$, $K_{8,3} \cong AP_{8.9}$, $K_{8,4} \cong AP_{8.8}$, $K_{8,5} \cong AP_{8.1}$, $K_{8,6} \cong AP_{8.21}$, $K_{8,7} \cong AP_{8.2}$, $K_{8,8} \cong AP_{8.15}$, $K_{8,9} \cong AP^{op}_{8.21}$, $K_{8,10} \cong AP_{8.6}$, $K_{8,11} \cong AP^{op}_{8.16}$, $K_{8,12} \cong AP^{op}_{8.18}$, $K_{8,13} \cong AP^{op}_{8.8}$, $K_{8,14} \cong AP_{8.4}$, $K_{8,15} \cong AP^{op}_{8.9}$, $K_{8,16} \cong AP_{8.10}$, $K_{8,17} \cong AP^{op}_{8.25}$, $K_{8,18} \cong AP^{op}_{8.28}$, $K_{8,19} \cong AP^{op}_{8.12}$, $K_{8,20} \cong AP^{op}_{8.19}$, $K_{8,21} \cong AP_{8.7}$, $K_{8,22} \cong AP_{8.17}$, $K_{8,23} \cong AP^{op}_{8.30}$, $K_{8,24} \cong AP^{op}_{8.27}$, $K_{8,25} \cong AP^{op}_{8.22}$, $K_{8,26} \cong AP^{op}_{8.13}$, $K_{8,27} \cong AP_{8.11}$, $K_{8,28} \cong AP_{8.3}$, $K_{8,29} \cong AP_{8.29}$, $K_{8,30} \cong AP^{op}_{8.29}$, $K_{8,31} \cong AP^{op}_{8.26}$, $K_{8,32} \cong AP^{op}_{8.14}$, $K_{8,33} \cong AP^{op}_{8.20}$, $K_{8,34} \cong AP_{8.27}$, $K_{8,35} \cong AP_{8.30}$, $K_{8,36} \cong AP^{op}_{8.24}$.

Step II. Describe, up to strong isomorphism, all pairs (X, V) of proper lower subposets in M_8 such that $V \subseteq X$ and $V < S \setminus X$: $Y_{8,1} = (X_{8,18}, \{3\})$, $Y_{8,2} = (X_{8,23}, \{3\})$, $Y_{8,3} = (X_{8,23}, \{5\})$, $Y_{8,4} = (X_{8,23}, \{3,5\})$, $Y_{8,5} = (X_{8,24}, \{3\})$, $Y_{8,6} = (X_{8,29}, \{3\})$, $Y_{8,7} = (X_{8,29}, \{5\})$, $Y_{8,8} = (X_{8,29}, \{3,5\})$, $Y_{8,9} = (X_{8,29}, \{3,5,6\})$, $Y_{8,10} = (X_{8,30}, \{3\})$, $Y_{8,11} = (X_{8,34}, \{3\})$, $Y_{8,12} = (X_{8,34}, \{5\})$, $Y_{8,13} = (X_{8,34}, \{3,5\})$, $Y_{8,14} = (X_{8,34}, \{3,5,6\})$, $Y_{8,15} = (X_{8,34}, \{3,5,6,7\})$, $Y_{8,16} = (X_{8,35}, \{2\})$, $Y_{8,17} = (X_{8,35}, \{3\})$, $Y_{8,18} = (X_{8,35}, \{2,3\})$.

Denote by $K'_{8,j}$ the $(S^{\uparrow}_{X})^{\uparrow}_{V}$ for $S = M_8$ and $(X, V) = Y_{8,j}$. Then $K'_{8,1} \cong AP_{8.24}$, $K'_{8,2} \cong AP_{8.20}$, $K'_{8,3} \cong AP_{8.26}$, $K'_{8,4} \cong AP^{op}_{8.3}$, $K'_{8,5} \cong AP_{8.14}$, $K'_{8,6} \cong AP^{op}_{8.11}$, $K'_{8,7} \cong AP_{8.22}$, $K'_{8,8} \cong AP^{op}_{8.17}$, $K'_{8,9} \cong AP^{op}_{8.5}$, $K'_{8,10} \cong AP_{8.13}$, $K'_{8,11} \cong AP^{op}_{8.7}$, $K'_{8,12} \cong AP_{8.12}$, $K'_{8,13} \cong AP^{op}_{8.10}$, $K'_{8,14} \cong AP_{8.23}$, $K'_{8,15} \cong AP_{8.5}$, $K'_{8,16} \cong AP_{8.25}$, $K'_{8,17} \cong AP_{8.19}$, $K'_{8,18} \cong AP^{op}_{8.4}$.

Step III. We have the same result as in case M_1. □

Proposition 12. *The posets minimax equivalent to M_9 are exhausted, up to isomorphism and duality, by the posets* 9.1, …, 9.31 *(see Table 2).*

Proof. **Step I.** Describe, up to strong isomorphism, all lower subposets of M_9: $X_{9,0} = \varnothing$, $X_{9,1} = \{1\}$, $X_{9,2} = \{2\}$, $X_{9,3} = \{3\}$, $X_{9,4} = \{1,2\}$, $X_{9,5} =$

$\{1,3\}$, $X_{9,6} = \{2,3\}$, $X_{9,7} = \{3,4\}$, $X_{9,8} = \{3,6\}$, $X_{9,9} = \{1,2,3\}$, $X_{9,10} = \{1,3,4\}$, $X_{9,11} = \{1,3,6\}$, $X_{9,12} = \{2,3,4\}$, $X_{9,13} = \{2,3,6\}$, $X_{9,14} = \{3,4,6\}$, $X_{9,15} = \{1,2,3,4\}$, $X_{9,16} = \{1,2,3,6\}$, $X_{9,17} = \{1,3,4,6\}$, $X_{9,18} = \{2,3,4,5\}$, $X_{9,19} = \{2,3,4,6\}$, $X_{9,20} = \{3,4,6,7\}$, $X_{9,21} = \{1,2,3,4,5\}$, $X_{9,22} = \{1,2,3,4,6\}$, $X_{9,23} = \{1,3,4,6,7\}$, $X_{9,24} = \{2,3,4,5,6\}$, $X_{9,25} = \{2,3,4,6,7\}$, $X_{9,26} = \{3,4,6,7,8\}$, $X_{9,27} = \{1,2,3,4,5,6\}$, $X_{9,28} = \{1,2,3,4,6,7\}$, $X_{9,29} = \{1,3,4,6,7,8\}$, $X_{9,30} = \{2,3,4,5,6,7\}$, $X_{9,31} = \{2,3,4,6,7,8\}$, $X_{9,32} = \{1,2,3,4,5,6,7\}$, $X_{9,33} = \{1,2,3,4,6,7,8\}$, $X_{9,34} = \{2,3,4,5,6,7,8\}$.

Denote by $K_{9,j}$ the poset $S_X^{\uparrow}$ for $S = M_9$ and $X = X_{9,j}$. Then $K_{9,0} \cong AP_{9.29}$, $K_{9,1} \cong AP_{9.22}$, $K_{9,2} \cong AP_{9.19}$, $K_{9,3} \cong AP_{9.26}$, $K_{9,4} \cong AP_{9.3}^{op}$, $K_{9,5} \cong AP_{9.13}$, $K_{9,6} \cong AP_{9.10}$, $K_{9,7} \cong AP_{9.6}$, $K_{9,8} \cong AP_{9.7}$, $K_{9,9} \cong AP_{9.8}$, $K_{9,10} \cong AP_{9.15}$, $K_{9,11} \cong AP_{9.17}$, $K_{9,12} \cong AP_{9.9}$, $K_{9,13} \cong AP_{9.15}^{op}$, $K_{9,14} \cong AP_{9.5}$, $K_{9,15} \cong AP_{9.10}^{op}$, $K_{9,16} \cong AP_{9.13}^{op}$, $K_{9,17} \cong AP_{9.7}^{op}$, $K_{9,18} \cong AP_{9.1}$, $K_{9,19} \cong AP_{9.6}^{op}$, $K_{9,20} \cong AP_{9.11}$, $K_{9,21} \cong AP_{9.20}^{op}$, $K_{9,22} \cong AP_{9.26}^{op}$, $K_{9,23} \cong AP_{9.16}^{op}$, $K_{9,24} \cong AP_{9.12}^{op}$, $K_{9,25} \cong AP_{9.14}$, $K_{9,26} \cong AP_{9.2}$, $K_{9,27} \cong AP_{9.31}^{op}$, $K_{9,28} \cong AP_{9.27}^{op}$, $K_{9,29} \cong AP_{9.23}^{op}$, $K_{9,30} \cong AP_{9.18}^{op}$, $K_{9,31} \cong AP_{9.21}^{op}$, $K_{9,32} \cong AP_{9.28}^{op}$, $K_{9,33} \cong AP_{9.30}^{op}$, $K_{9,34} \cong AP_{9.25}$.

Step II. Describe, up to strong isomorphism, all pairs (X,V) of proper lower subposets in M_2 such that $V \subseteq X$ and $V < S \setminus X$: $Y_{9,1} = (X_{9,9},\{3\})$, $Y_{9,2} = (X_{9,15},\{3\})$, $Y_{9,3} = (X_{9,16},\{3\})$, $Y_{9,4} = (X_{9,21},\{3\})$, $Y_{9,5} = (X_{9,22},\{3\})$, $Y_{9,6} = (X_{9,22},\{3,4\})$, $Y_{9,7} = (X_{9,27},\{3\})$, $Y_{9,8} = (X_{9,27},\{3,4\})$, $Y_{9,9} = (X_{9,27},\{3,6\})$, $Y_{9,10} = (X_{9,27},\{3,4,6\})$, $Y_{9,11} = (X_{9,28},\{3\})$, $Y_{9,12} = (X_{9,28},\{3,4\})$, $Y_{9,13} = (X_{9,32},\{3\})$, $Y_{9,14} = (X_{9,32},\{3,4\})$, $Y_{9,15} = (X_{9,32},\{3,6\})$, $Y_{9,16} = (X_{9,32},\{3,4,6\})$, $Y_{9,17} = (X_{9,32},\{3,4,6,7\})$, $Y_{9,18} = (X_{9,33},\{2\})$, $Y_{9,19} = (X_{9,33},\{3\})$, $Y_{9,20} = (X_{9,33},\{2,3\})$, $Y_{9,21} = (X_{9,33},\{3,4\})$, $Y_{9,22} = (X_{9,33},\{2,3,4\})$.

Denote by $K'_{9,j}$ the poset $(S_X^{\uparrow})_V^{\uparrow}$ for $S = M_9$ and $(X,V) = Y_{9,j}$. Then $K'_{9,1} \cong AP_{9.3}$, $K'_{9,2} \cong AP_{9.19}^{op}$, $K'_{9,3} \cong AP_{9.22}^{op}$, $K'_{9,4} \cong AP_{9.21}^{op}$, $K'_{9,5} \cong AP_{9.29}^{op}$, $K'_{9,6} \cong AP_{9.25}$, $K'_{9,7} \cong AP_{9.30}$, $K'_{9,8} \cong AP_{9.21}$, $K'_{9,9} \cong AP_{9.23}$, $K'_{9,10} \cong AP_{9.2}^{op}$, $K'_{9,11} \cong AP_{9.28}$, $K'_{9,12} \cong AP_{9.18}$, $K'_{9,13} \cong AP_{9.27}$, $K'_{9,14} \cong AP_{9.14}^{op}$, $K'_{9,15} \cong AP_{9.16}$, $K'_{9,16} \cong AP_{9.11}^{op}$, $K'_{9,17} \cong AP_{9.4}$, $K'_{9,18} \cong AP_{9.21}$, $K'_{9,19} \cong AP_{9.31}$, $K'_{9,20} \cong AP_{9.20}$, $K'_{9,21} \cong AP_{9.12}$, $K'_{9,22} \cong AP_{9.1}^{op}$.

Step III. We have the same result as in case M_1. □

Propositions 4–12 provide a complete classification up to isomorphism and duality of essential almost positive posets (see Table 2).

Table 1. The near-positive posets $aP_{i.j}$.

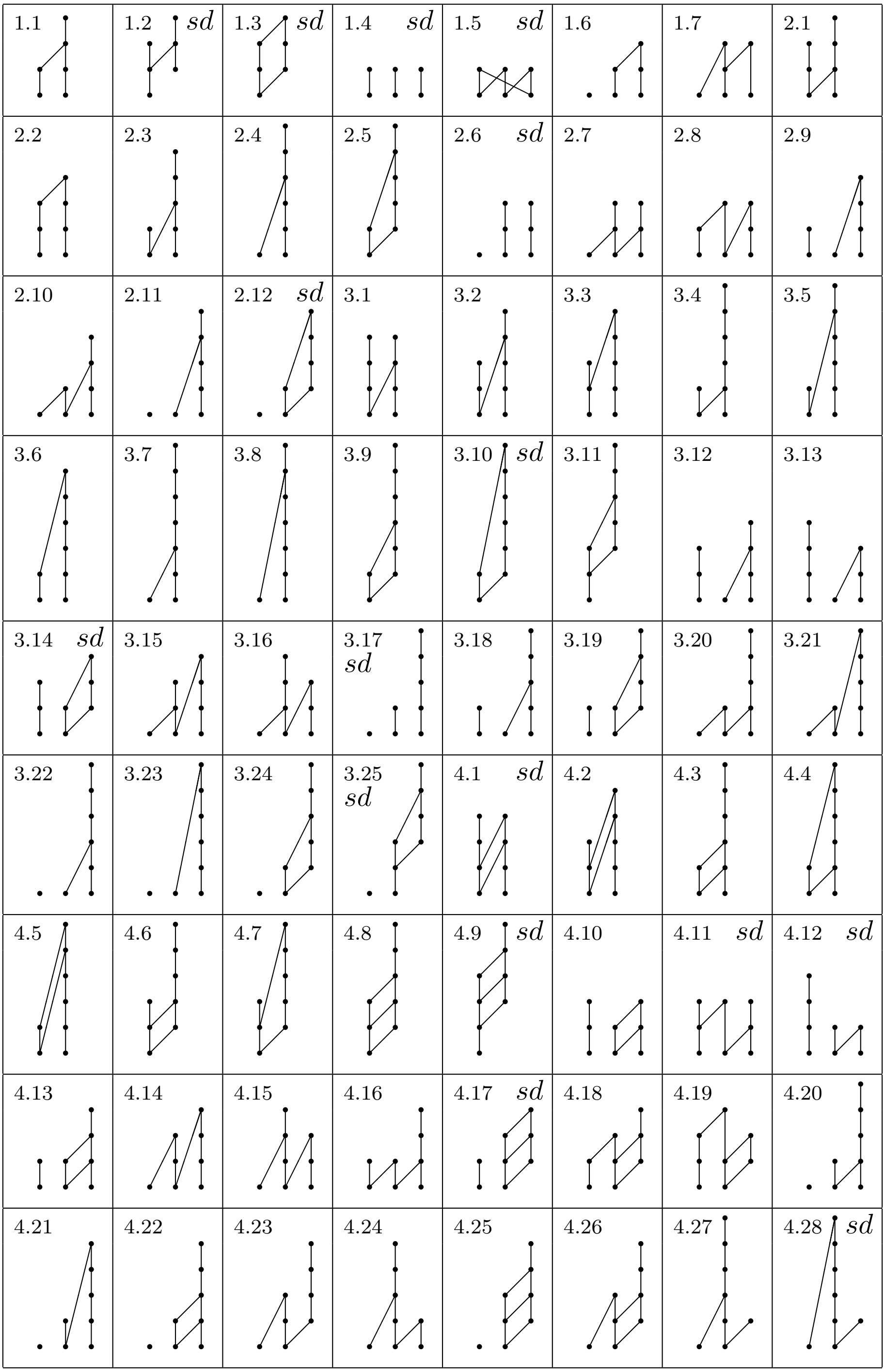
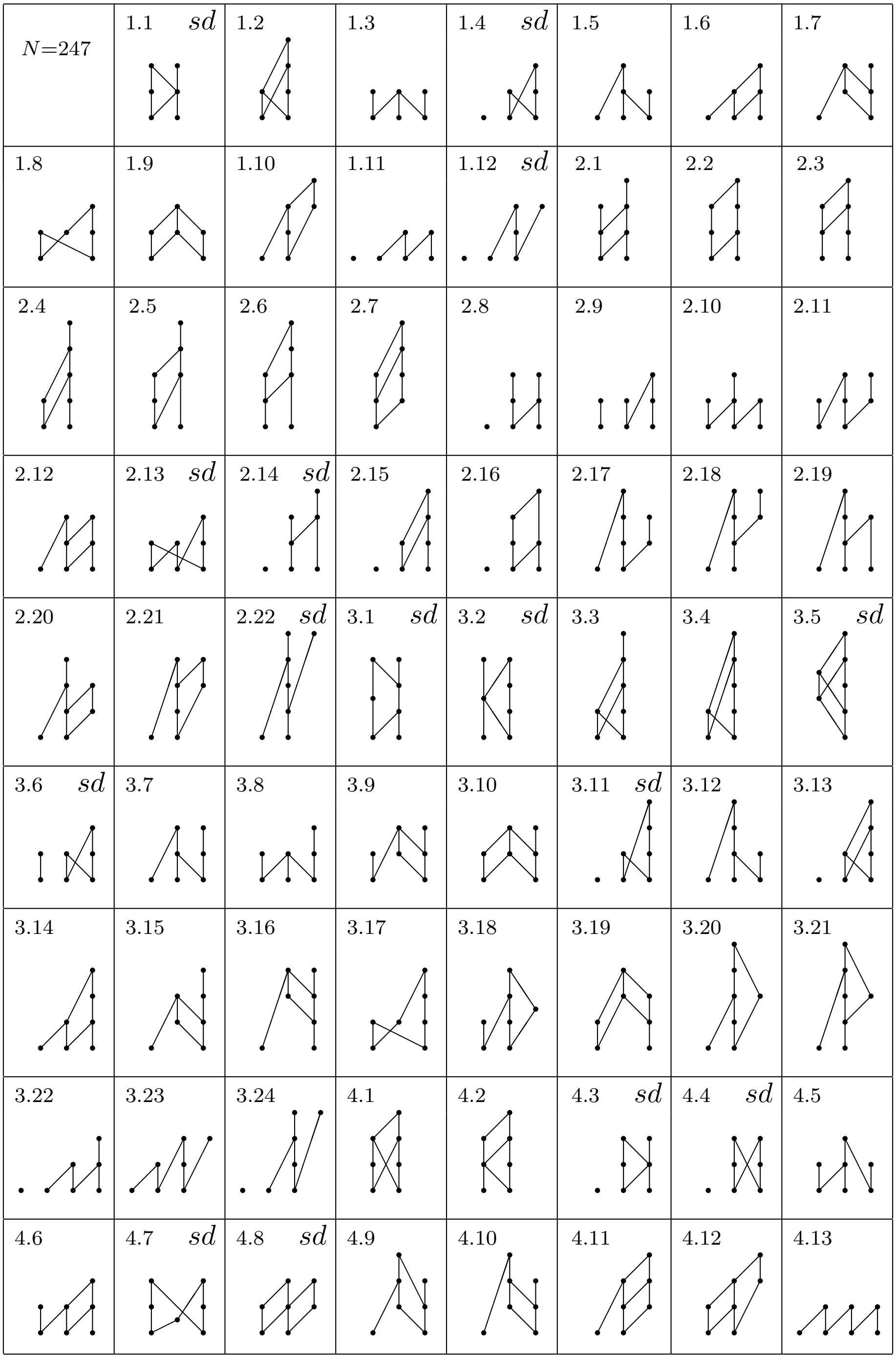
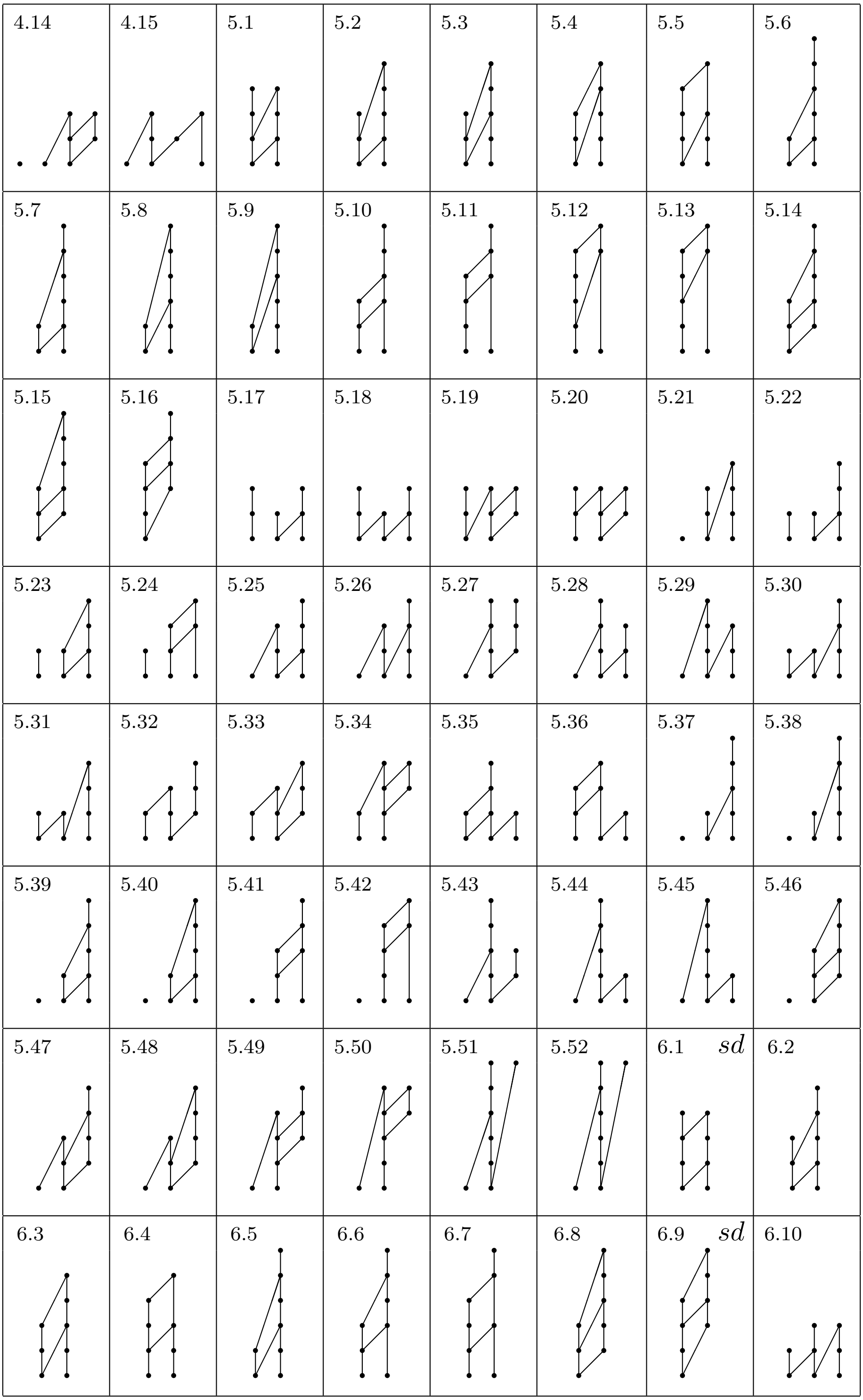


Table 2. The essential almost positive posets $AP i.j$**.**



N=247	1.1 *sd*	1.2	1.3	1.4 *sd*	1.5	1.6	1.7
1.8	1.9	1.10	1.11	1.12 *sd*	2.1	2.2	2.3
2.4	2.5	2.6	2.7	2.8	2.9	2.10	2.11
2.12	2.13 *sd*	2.14 *sd*	2.15	2.16	2.17	2.18	2.19
2.20	2.21	2.22 *sd*	3.1 *sd*	3.2 *sd*	3.3	3.4	3.5 *sd*
3.6 *sd*	3.7	3.8	3.9	3.10	3.11 *sd*	3.12	3.13
3.14	3.15	3.16	3.17	3.18	3.19	3.20	3.21
3.22	3.23	3.24	4.1	4.2	4.3 *sd*	4.4 *sd*	4.5
4.6	4.7 *sd*	4.8 *sd*	4.9	4.10	4.11	4.12	4.13


4.14
4.15
5.1
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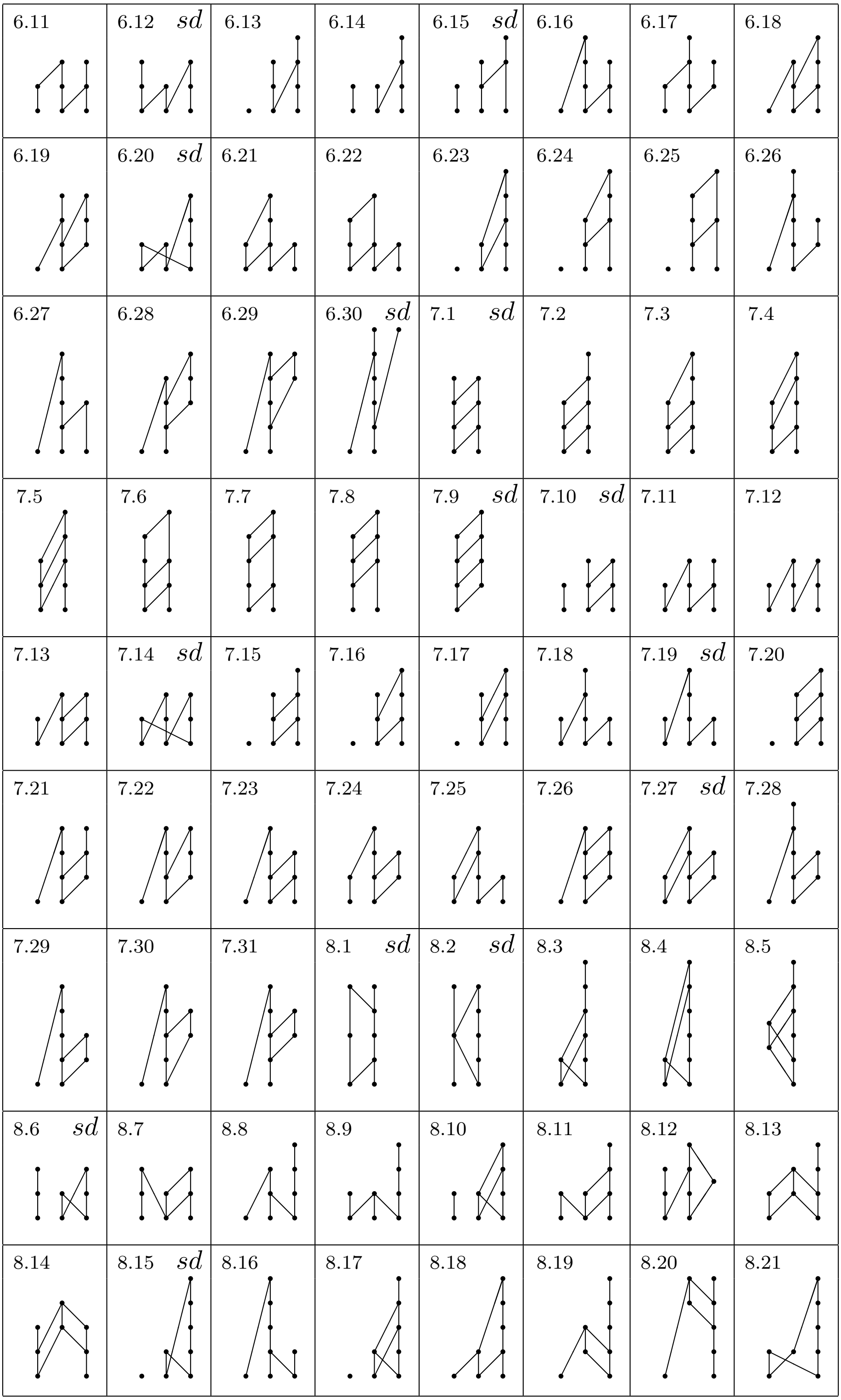
5.44
5.45
5.46
5.47
5.48
5.49
5.50
5.51
5.52
6.1 sd
6.2
6.3
6.4
6.5
6.6
6.7
6.8
6.9 sd
6.10


6.11
6.12 sd
6.13
6.14
6.15 sd
6.16
6.17
6.18
6.19
6.20 sd
6.21
6.22
6.23
6.24
6.25
6.26
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6.28
6.29
6.30 sd
7.1 sd
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8.1 sd
8.2 sd
8.3
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8.6 sd
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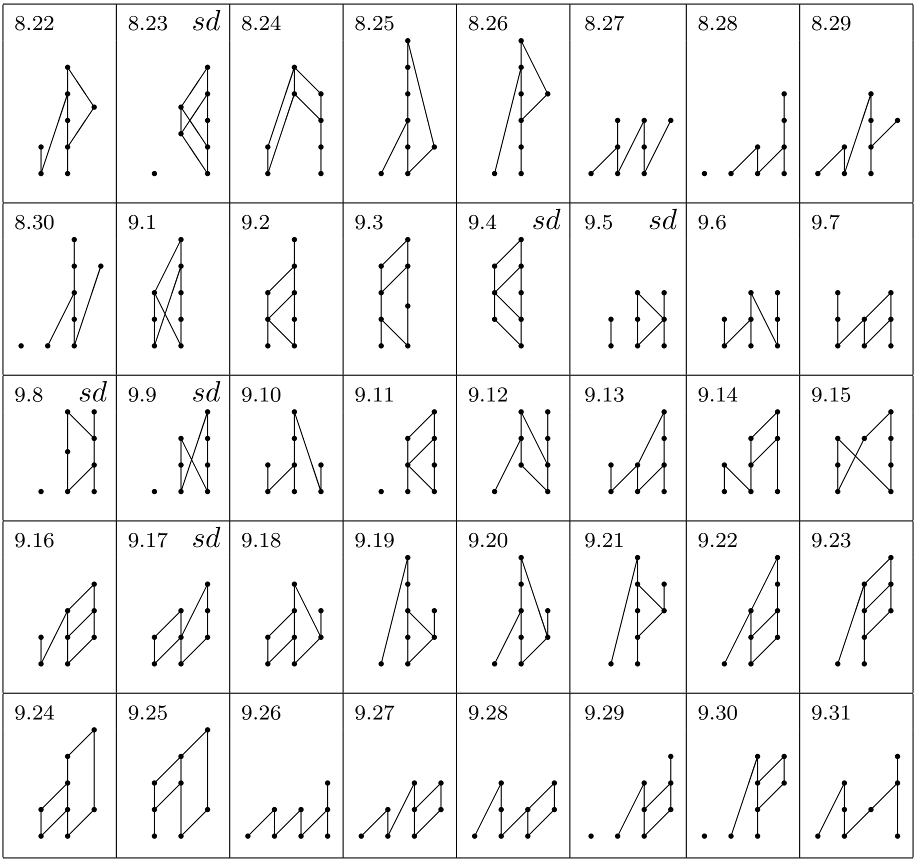

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