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The decreasing and monotone injective partial monoid on a finite chain

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Abstract. In this paper, we consider the monoid $\mathcal{DORI}_n$ consisting of all monotone and order-decreasing partial injective transformations, $I(n,p) = \{\alpha \in \mathcal{DORI}_n : |\operatorname{Im}\alpha| \leq p\}$ the two-sided ideal of $\mathcal{DORI}_n$, and $RQ_p(n)$ the Rees quotient of $I(n,p)$ on a chain with n elements. We calculate the cardinality of $\mathcal{DORI}_n$, characterize the Green's relations and their starred analogue for any structure $S \in \{\mathcal{DORI}_n, I(n,p), RQ_p(n)\}$. We demonstrate that for any structure S among $\{\mathcal{DORI}_n, I(n,p), RQ_p(n)\}$, the structure is abundant for all values of n; specifically, $\mathcal{DORI}_n$ is shown to be an ample monoid, and compute the rank of the Rees quotient $RQ_p(n)$ and the two-sided ideal $I(n,p)$; as a special case, we obtain the rank of the monoid $\mathcal{DORI}_n$ to be $3n-2$. Finally, we characterize all the maximal subsemigroups of the structure S among $\{\mathcal{DORI}_n, I(n,p), RQ_p(n)\}$.

Introduction

For a natural number n, let $[n]$ denote an n element chain $\{1, 2, \ldots, n\}$. Let $\mathcal{I}_n$ be the set of all injective partial transformations on $[n]$. This

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collection is known as *the symmetric inverse semigroup.* A map $\rho \in \mathcal{I}_n$ is called *order decreasing* if (for all $x \in \text{Dom } \rho$) $x\rho \leq x$; an *isotone* map (resp., an *anti-tone* map) if (for all $a, b \in \text{Dom } \rho$) $a \leq b$ implies $a\rho \leq b\rho$ (resp., $a\rho \geq b\rho$). The notations $\mathcal{DI}_n$ and $\mathcal{OI}_n$ usually denote *the semigroup of all order-decreasing injective partial transformations* on $[n]$ and *the semigroup of all isotone injective partial transformations* on $[n]$, respectively. Moreover, let $\mathcal{DORP}_n$ be the *monoid of all monotone and decreasing partial transformations* on $[n]$. The monoid $\mathcal{DORP}_n$ first appeared in [33], where its rank and that of its two-sided ideals where computed. Furthermore, let $\mathcal{IC}_n$ denote the *semigroup of all isotone and order-decreasing injective partial transformations* on $[n]$. The order of $\mathcal{IC}_n$ was obtained in [6, Theorem 14.2.8], as c_{n+1}, where

$$c_n = \frac{1}{n}\binom{2n}{n-1} \tag{1}$$

is known as the *n-th Catalan number.* The moinoid $\mathcal{IC}_n$ is referred to as the *injective partial Catalan monoid* defined as:

$$\mathcal{IC}_n = \mathcal{OI}_n \cap \mathcal{DI}_n. \tag{2}$$

The monoids $\mathcal{IC}_n$, $\mathcal{OI}_n$ and $\mathcal{DI}_n$ have been widely studied in various contexts, see for example [3, 6, 8, 17, 20, 23, 25, 27, 29, 33].

Now let

$$\mathcal{DORI}_n = \mathcal{DORP}_n \cap \mathcal{I}_n \tag{3}$$

be the *monoid of all monotone and decreasing injective partial transformations* on $[n]$. The monoid $\mathcal{DORI}_n$ seems not to have been discussed in the existing literature. In this article, we are going to investigate order, rank and some algebraic properties of the monoid $\mathcal{DORI}_n$, as well as the rank of its two-sided ideals $I(n, p)$ and their Rees quotients.

Remark 1. Notice that for $0 \leq n \leq 2$, $\mathcal{DORI}_n = \mathcal{IC}_n$.

The product (or composition) of two transformations ρ and σ in $\mathcal{I}_n$ is defined as $a(\rho \circ \sigma) = ((a)\rho)\sigma$ for all $a \in \text{Dom } \rho$. To prevent confusion, we will use the notation $\rho\sigma$ to denote $\rho \circ \sigma$. Additionally, we will adopt the following notations: $F(\rho) = \{x \in \text{Dom } \rho : x\rho = x\}$ for the set of fixed points of ρ, $f(\rho)$ for the number of fixed points of ρ (i.e., $|F(\rho)|$), $\text{id}_{[n]}$ for the identity transformation on $[n]$, $\text{Im}\,\rho$ for the image set of the transformation ρ, $\text{Dom } \rho$ for the domain set of the map ρ, $b(\rho) = |\text{ Dom } \rho|$ and $h(\rho) = |\text{ Im } \rho|$ for the height of ρ. A subset A of $[n]$ is called *convex* if, for any $a, b \in A$ such that $a \leq b$, and for any $c \in [n]$, if $a < c < b$, then $c \in A$. An element $a \in S$ is called an *idempotent* if $a^2 = a$, the collection of all idempotents in S shall as usual be denoted by $E(S)$.

Now, for $1 \leq p \leq n$, let

$$I(n, p) = \{\rho \in \mathcal{DORI}_n : |\operatorname{Im} \rho| \leq p\}, \tag{4}$$

be the two-sided ideal of $\mathcal{DORI}_n$ consisting of all monotone and decreasing injective partial functions with a height of at most p. Moreover, for $1 \leq p \leq n$, let

$$RQ_n(p) = I(n, p)/I(n, p-1); \tag{5}$$

denote the Rees quotient of $I(n, p)$. The elements of $RQ_n(p)$ can be viewed as the elements of $\mathcal{DORI}_n$ that have an exact height of p. The composition of two elements in $RQ_n(p)$ is $\mathbf{0}$ if their product in $RQ_n(p)$ has a height that is strictly below p; else, it remains in $RQ_n(p)$.

In this article, we are going to characterize the Green's equivalences and their starred counterpart, and compute the ranks of the monoid $\mathcal{DORI}_n$ and its two-side ideals. For more details about basic concepts in semigroup theory, we recommend that the reader consult the books [11, 15] by Howie and Higgins, respectively.

This section includes definitions of fundamental terms, and we compute the order of the monoid $\mathcal{DORI}_n$. In Section 2, we explore all of Green's relations and their starred counterparts within the monoid $\mathcal{DORI}_n$ and its two-sided ideal $I(n, p)$. Additionally, we show that both the monoid $\mathcal{DORI}_n$ and its two-sided ideals are abundant semigroups; in particular, we demonstrate that $\mathcal{DORI}_n$ is an ample monoid. Section 3 focuses on determining the rank of the Rees quotient semigroup $RQ_n(p)$, the two-sided ideal $I(n, p)$, and the monoid $\mathcal{DORI}_n$. Finally, in Section 4, we characterize all the maximal subsemigroups of the Rees quotients $RQ_n(p)$, the two-sided ideal $I(n, p)$, and in particular, the monoid $\mathcal{DORI}_n$.

We shall represent every $\rho \in \mathcal{IC}_n$ in two line notation as:

$$\rho = \begin{pmatrix} x_1 & \dots & x_p \\ a_1 & \dots & a_p \end{pmatrix} \quad (1 \leq p \leq n), \tag{6}$$

where $a_i \leq x_i$ for all $1 \leq i \leq p$. Moreover, we may, without loss of generality, suppose that $1 \leq x_1 < \ldots < x_p \leq n$ and $1 \leq a_1 < a_2 < \ldots < a_p \leq n$, since ρ is an isotone map.

The monoid $\mathcal{DORI}_n$ can equivalently be obtained as:

$$\begin{aligned}\mathcal{DORI}_n &= \{\rho \in \mathcal{I}_n : \rho \text{ is } \textit{monotone} \ \& \ \textit{decreasing}\} \\ &= \{\rho \in \mathcal{I}_n : \rho \text{ is } \textit{isotone} \ \& \ \textit{decreasing}\} \\ &\cup \{\rho \in \mathcal{I}_n : \rho \text{ is } \textit{antitone} \ \& \ \textit{decreasing}\}.\end{aligned}$$

Therefore, if we let $DRI_n = \{\rho \in \mathcal{I}_n : \rho \text{ is } \textit{antitone} \text{ and } \textit{decreasing}\}$, then

$$\mathcal{DORI}_n = \mathcal{IC}_n \cup DRI_n. \tag{7}$$

Now, let us briefly analyze the elements in DRI_n. First, observe that every element of height 1 in $\mathcal{DORI}_n$ is both isotone and antitone; that is, it belongs to both $\mathcal{IC}_n$ and DRI_n. However, antitones of height greater than or equal to 2 are never isotone; therefore, they must be included in DRI_n. Thus, we have the following remark.

Remark 2. Every element in DRI_n of height $1 < p < n$ of the form

$$\rho = \begin{pmatrix} x_1 & \cdots & x_p \\ a_p & \cdots & a_1 \end{pmatrix} \tag{8}$$

possesses the following properties:

(i) $1 \leq a_1 < \cdots < a_p \leq x_1 < \cdots < x_p \leq n$;

(ii) Product of two isotone maps or two antitone maps, is isotone; product of an isotone map and an antitone map, is antitone. Similarly, product of an antitone map and isotone map yields an antitone map.

Lemma 1. *Every element in $\mathcal{DORI}_n$ with height greater than $\lceil \frac{n}{2} \rceil$ is necessarily isotone.*

Proof. Let $\rho \in \mathcal{DORI}_n$ as expressed in (6) be of height $p > \lceil \frac{n}{2} \rceil$. Suppose by way of contradiction that ρ is antitone. Thus, we must have $1 \leq a_1 < \cdots < a_p \leq x_1 < \cdots < x_p \leq n$. Now $p > \lceil \frac{n}{2} \rceil$ implies $|\operatorname{Im} \rho| > \lceil \frac{n}{2} \rceil$, i.e., the number of elements in the image set $\{a_1, \ldots, a_p\}$ is greater than $\lceil \frac{n}{2} \rceil$. Notice that either $\operatorname{Im} \rho \cap \operatorname{Dom} \rho = \{x_1\}$ or $\operatorname{Im} \rho \cap \operatorname{Dom} \rho = \emptyset$. So, there are two cases to consider.

Case i. If $\operatorname{Im} \rho \cap \operatorname{Dom} \rho = \{x_1\}$. Then the domain set of ρ can be obtain from $[n] \setminus \{a_1, \ldots, a_p\} \cup \{x_1\}$. So that the total number of elements in $[n] \setminus \{a_1, \ldots, a_p\} \cup \{x_1\}$ is $n - |\operatorname{Im} \rho| + 1$. However, since $|\operatorname{Im} \rho| > \lceil \frac{n}{2} \rceil$, we see that:

$$n - |\operatorname{Im} \rho| + 1 < n - \left\lceil \frac{n}{2} \right\rceil + 1.$$

Thus, if n is odd, we have

$$n - |\operatorname{Im}\rho| + 1 < n - \left\lceil \frac{n}{2} \right\rceil + 1 = n - \frac{n+1}{2} + 1 = \frac{n+1}{2} = \left\lceil \frac{n}{2} \right\rceil < p.$$

This means that the number of elements in the set $[n]\backslash\{a_1, \ldots, a_p\} \cup \{x_1\}$ (where the domain elements can be selected) is less than the number of the elements fixed in the image set, which is a contradiction.
Similarly, if n is even, we have

$$n - |\operatorname{Im}\rho| + 1 < n - \left\lceil \frac{n}{2} \right\rceil + 1 = n - \frac{n}{2} + 1 = \frac{n+2}{2} = \left\lceil \frac{n}{2} \right\rceil + 1 < p + 1.$$

This implies that $n - |\operatorname{Im}\rho| < p$, which is also a contradiction.
Case ii. Using a similar argument as in Case i, we arrived at a contradiction; hence, ρ is isotone. □

Consequently, every element in DRI_n has height p that satisfies $1 \leq p \leq \lceil \frac{n}{2} \rceil$. We now have the following definition. An element $\rho \in \mathcal{DORI}_n$ as expressed in (6) is called *reversible* if $\begin{pmatrix} x_1 & \cdots & x_p \\ a_p & \cdots & a_1 \end{pmatrix} \in \mathcal{DORI}_n$. The following lemma is important.

Lemma 2. *An element $\rho \in \mathcal{IC}_n$, as expressed in (6), is reversible; specifically, $\begin{pmatrix} x_1 & \cdots & x_p \\ a_p & \cdots & a_1 \end{pmatrix} \in \mathcal{DORI}_n$ if and only if $a_p \leq x_1$ and $1 \leq p \leq \lceil \frac{n}{2} \rceil$.*

Proof. Notice that $x_1 = \min(\operatorname{Dom}\rho) \geq a_p$, by the order decreasing property. However, since ρ is antitone, then $n - p + 1 \geq x_1$ and $x_1 \geq p$, and so, $n-p+1 \geq x_1 \geq p$, which implies $n+1 \geq 2p$. Hence $p \leq \lfloor \frac{n+1}{2} \rfloor = \lceil \frac{n}{2} \rceil$. The converse is trivial. □

We note the following well known combinatorial identity, which is useful in our subsequent discussions.

Lemma 3 ([21], (3b), p. 8)**.** *For all natural numbers n, a, and b, we have*

$$\sum_{i=b}^{n} \binom{i}{b} \binom{n+a-i}{a} = \binom{n+a+1}{a+b+1}.$$

It is now clear that to obtain the size of $\mathcal{DORI}_n$, it is sufficient to compute the size of the set DRI_n. For a transformation ρ on a finite chain, let $b(\rho) = r$ and $h(\rho) = p$. Define the combinatorial function:

$$F(n, p) = |\{\rho \in DRI_n : h(\rho) = p\}|.$$

We then have the following result.

Theorem 1. *The number of elements in DRI_n of a fixed height p is*

$$F(n,p) = \binom{n+1}{2p}.$$

Proof. Let $\rho \in DRI_n$ be as expressed in (8), where $h(\rho) = p$. Moreover, let $r = x_1 = \min(\mathrm{Dom}\ \rho)$. Note that $p \leq r \leq n-p+1$, from the proof of Lemma 2, and for all $x_i \in \mathrm{Dom}\ \rho$ and $a_j \in \mathrm{Im}\ \rho$, we have $a_j \leq x_i$ since ρ is antitone. To count the number of $\rho \in DRP_n$, we first choose the domain elements. Now, since $r \in \mathrm{Dom}\ \rho$, we can choose the remaining $p-1$ elements from $[n] \setminus \{1, \ldots, r\}$, i.e., in $\binom{n-r}{p-1}$ ways. Then, we can choose the p images from the set $\{1, \ldots, r\}$ in $\binom{r}{p}$ ways. Finally, taking the sum of the product: $\binom{n-r}{p-1}\binom{r}{p}$, from $r = p$ to $r = n-p+1$ produces

$$F(n,p) = \sum_{r=p}^{n-p+1} \binom{n-r}{p-1}\binom{r}{p} = \binom{n+1}{2p} \quad \text{(by Lemma 3)},$$

as required. □

Now, let

$$a_n = \sum_{p=2}^{\lceil \frac{n}{2} \rceil} F(n,p) = \sum_{p=2}^{\lceil \frac{n}{2} \rceil} \binom{n+1}{2p} = 2^n - \frac{n(n+1)}{2} - 1.$$

We can now state the following result.

Theorem 2. *Let $\mathcal{DORI}_n$ be as defined in (7). Then*

$$|\mathcal{DORI}_n| = \frac{1}{n+1}\binom{2(n+1)}{n} + 2^n - \frac{n(n+1)}{2} - 1.$$

Proof. Notice that, as in (7), $\mathcal{DORI}_n = \mathcal{IC}_n \cup DRI_n$. However elements of height 1 are both isotone and antitone maps, and they are counted in $\mathcal{IC}_n$. Thus,

$$\begin{aligned} |\mathcal{DORI}_n| &= c_{n+1} + \sum_{p=2}^{\lceil \frac{n}{2} \rceil} F(n,p) \\ &= c_{n+1} + a_n \\ &= \frac{1}{n+1}\binom{2(n+1)}{n} + 2^n - \frac{n(n+1)}{2} - 1, \end{aligned}$$

as required. □

1. Green's and starred Green's relations

Within the framework of semigroup theory, Green's relations consist of five different types, represented as $\mathcal{L}, \mathcal{R}, \mathcal{D}, \mathcal{J}$, and $\mathcal{H}$. These relations are defined based on the principal ideals (left, right, or two-sided) generated by the elements of S. For a comprehensive discussion of these relations, readers are encouraged to refer to Howie [11]. As noted in [4], when there is potential for confusion, we denote a relation $\mathcal{K}$ on S as $\mathcal{K}(S)$. It is a well-known result in the study of a finite semigroup, $\mathcal{D} = \mathcal{J}$ (see [11, Proposition 2.1.4]). Consequently, we are going to concentrate on describing the equivalences $\mathcal{L}, \mathcal{R}, \mathcal{D}$, and $\mathcal{H}$ within the monoid of all decreasing and monotone injective partial transformations $\mathcal{DORI}_n$.

It is well known that the semigroup $\mathcal{DI}_n$ is $\mathcal{J}$-trivial (see [22, Lemma 2.2]). Notice that $\mathcal{DORI}_n$ is a subsemigroup of $\mathcal{DI}_n$, it follows that for any relation say $\mathcal{K} \in \{\mathcal{L}, \mathcal{R}, \mathcal{H}, \mathcal{J}\}$, we have

$$\mathcal{K}(\mathcal{DORI}_n) = \mathcal{K}(\mathcal{DI}_n) \cap (\mathcal{DORI}_n \times \mathcal{DORI}_n).$$

Consequently, we have the following result.

Theorem 3. *$\mathcal{DORI}_n$ is $\mathcal{J}$-trivial.*

In a semigroup S, an element $a \in S$ is termed *regular* if there exists another element $b \in S$ such that $a = aba$. Furthermore, a semigroup S is classified as a *regular semigroup* if all its elements are regular. We are now going to prove a general result about regular elements in an $\mathcal{R}$-trivial semigroup.

Lemma 4. *In an $\mathcal{R}$-trivial semigroup S, an element $a \in S$ is regular if and only if a is an idempotent.*

Proof. Let S be an $\mathcal{R}$-trivial semigroup and $a \in S$ be a regular element. This means $a = aba$ for some $b \in S$. Notice that $(ab)^2 = abab = (aba)b = ab$, so $ab \in E(S)$. Observe also that the equations $a = (ab)a$ and $ab = a(bab)$ ensures that $(a, ab) \in \mathcal{R}$. However, the fact that S is $\mathcal{R}$-trivial implies that $a = ab$, and so a is an idempotent. The converse is trivial. □

Remark 3. Now since every $\mathcal{J}$-trivial semigroup is $\mathcal{R}$-trivial, then Lemma 4 also holds on $\mathcal{DORI}_n$.

Consequently, we can immediately derive the following corollary from Theorem 3 and Lemma 4, and Remark 3.

Corollary 1. *An element $\rho \in \mathcal{DORI}_n$ is regular if and only if ρ is an idempotent. Hence, the monoid $\mathcal{DORI}_n$ is non-regular for $n \geq 2$.*

Consequently, based on Theorem 3, we derive the following descriptions of Green's equivalences in the semigroup S in $\{RQ_n(p), I(n, p)\}$.

Theorem 4. *Let $S \in \{RQ_n(p), I(n,p)\}$. Then S is $\mathcal{J}$-trivial. Hence, for $n \geq p \geq 2$, the semigroup S is non-regular.*

When a semigroup is not regular, it is common practice to examine the starred Green's relations to identify its algebraic structure. The five starred Green's equivalences include $\mathcal{L}^*, \mathcal{R}^*, \mathcal{D}^*, \mathcal{J}^*$, and $\mathcal{H}^*$. The relation $\mathcal{L}^*$ is defined as: for $a, b \in S$, $(a, b) \in \mathcal{L}^*$ if and only if a and b are related by $\mathcal{L}$ in some over-semigroup of S; the relation $\mathcal{R}^*$ is defined dually. The $\mathcal{D}^*$ relation is defined as the join of $\mathcal{L}^*$ and $\mathcal{R}^*$, while $\mathcal{H}^*$ is the intersection of $\mathcal{L}^*$ and $\mathcal{R}^*$. A semigroup S is described as *left abundant* if every $\mathcal{L}^*$-class contains at least one idempotent; it is called *right abundant* if every $\mathcal{R}^*$-class includes at least one idempotent; and it is called *abundant* if both every $\mathcal{L}^*$-class and every $\mathcal{R}^*$-class of S contain an idempotent [4,5]. Several classes of transformation semigroups have been identified as left abundant, right abundant, or abundant; refer to [2,18,23, 24,27,28,31,32,34] for examples. In this section, we aim to characterize the starred versions of Green's equivalences for $S \in \{\mathcal{DORI}_n, I(n, p)\}$.

We will need the following definition and lemmas from [2, 11, 24] for our subsequent discussions: A subsemigroup A of a semigroup S is called *full* if $E(A) = E(S)$; it is referred to as an *inverse ideal* of S if for every $a \in A$, there exists $a' \in S$ such that $aa'a = a$ and both $a'a$ and aa' are in A.

Lemma 5 ([24, Lemma 3.1.8.]). *Every inverse ideal A of S is abundant.*

Lemma 6 ([24, Lemma 3.1.9.]). *If A is an inverse ideal of S, then (1) $\mathcal{L}^*(A) = \mathcal{L}(S) \cap (A \times A)$; (2) $\mathcal{R}^*(A) = \mathcal{R}(S) \cap (A \times A)$; (3) $\mathcal{H}^*(A) = \mathcal{H}(S) \times (A \times A)$.*

Lemma 7 ([11, Exercise 5.11(2)]). *In the inverse semigroup $\mathcal{I}_n$, we have*

(a) $\rho \mathcal{L} \sigma$ *if and only if* $\operatorname{Im} \rho = \operatorname{Im} \sigma$;

(b) $\rho \mathcal{R} \sigma$ *if and only if* $\operatorname{Dom} \rho = \operatorname{Dom} \sigma$;

(c) $\rho \mathcal{D} \sigma$ *if and only if* $|\operatorname{Im} \rho| = |\operatorname{Im} \sigma|$;

(d) $\mathcal{D} = \mathcal{J}$.

We are now going to illustrate the following outcome.

Theorem 5. *Let $\mathcal{DORI}_n$ be as defined in (2). Then $\mathcal{DORI}_n$ is an inverse ideal of $\mathcal{I}_n$.*

Proof. Let $\rho \in \mathcal{DORI}_n$ be expressed as in (6). Then $\rho^{-1} \in \mathcal{I}_n$. Observe that for $1 \leq i \leq p$,

$$x_i\rho\rho^{-1}\rho = a_i\rho^{-1}\rho = x_i\rho.$$

Additionally, we have $\rho\rho^{-1} = \mathrm{Id}_{\mathrm{Dom}\ \rho} \in \mathcal{DORI}_n$ and $\rho^{-1}\rho = \mathrm{Id}_{\mathrm{Im}\ \rho} \in \mathcal{DORI}_n$, as required. □

Remark 4. It is straightforward to observe that $\mathcal{DORI}_n$ forms a full subsemigroup of $\mathcal{I}_n^-$ (where $\mathcal{I}_n^-$ represents the monoid of all decreasing partial injections on $[n]$, as referenced in [30]) in the sense that $E(\mathcal{DORI}_n) = E(\mathcal{I}_n^-)$. Consequently, $E(\mathcal{DORI}_n)$ constitutes a semilattice (by semilattice, we refer to a commutative semigroup in which each of its element is an idempotent).

Now, as a result of Theorem 5 and Lemma 5, we have the result below.

Theorem 6. *Let $\mathcal{DORI}_n$ be as defined in (2). Then $\mathcal{DORI}_n$ is abundant.*

Now, we present the result below.

Theorem 7. *Let $\mathcal{DORI}_n$ be as defined in (2). Then for $\rho, \sigma \in \mathcal{DORI}_n$, we have:*

(i) $\rho\mathcal{L}^*\sigma$ if and only if $\mathrm{Im}\,\rho = \mathrm{Im}\,\sigma$;

(ii) $\rho\mathcal{R}^*\sigma$ if and only if $\mathrm{Dom}\ \rho = \mathrm{Dom}\ \sigma$;

(iii) $\rho\mathcal{H}^*\sigma$ if and only if $\mathrm{Im}\,\rho = \mathrm{Im}\,\sigma$ and $\mathrm{Dom}\ \rho = \mathrm{Dom}\ \sigma$.

Proof. (i) Since $\mathcal{DORI}_n$ is an inverse ideal of $\mathcal{I}_n$ by Theorem 5, it follows from Lemma 6 that

$$\mathcal{L}^*(\mathcal{DORI}_n) = \mathcal{L}(\mathcal{I}_n) \cap (\mathcal{DORI}_n \times \mathcal{DORI}_n).$$

Thus, $(\rho, \sigma) \in \mathcal{L}^*(\mathcal{DORI}_n)$ if and only if $(\rho, \sigma) \in \mathcal{L}(\mathcal{I}_n) \cap (\mathcal{DORI}_n \times \mathcal{DORI}_n)$ if and only if $\mathrm{Im}\,\rho = \mathrm{Im}\,\sigma$ by Lemma 7(a).
(ii) The argument is analogous to (i).
(iii) follows from (i) and (ii). □

Let H^*_ρ denote the $\mathcal{H}^*-$class of ρ in $\mathcal{DORI}_n$. Then we have the following result.

Corollary 2. *Let $\mathcal{DORI}_n$ be as defined in (2). Then for any $\rho \in \mathcal{DORI}_n$,*

$$|H^*_\rho| = \begin{cases} 1, & \text{if } h(\rho) \in \{0, 1, \lceil \frac{n}{2} \rceil + 1, \ldots, n\}; \\ 2, & \text{if } 2 \leq h(\rho) \leq \lceil \frac{n}{2} \rceil. \end{cases}$$

Proof. Let $\rho \in \mathcal{DORI}_n$ be expressed as in (6). If $h(\rho) = 0, 1$, the result is trivial. Now, if $h(\rho) \in \{\lceil \frac{n}{2} \rceil + 1, \ldots, n\}$, $h(\rho) > \lceil \frac{n}{2} \rceil$, i.e., ρ is necessarily an isotone map by Lemma 1. So, ρ is not reversible by Lemma 2; that is to say the map,

$$\sigma = \begin{pmatrix} x_1 & \ldots & x_p \\ a_p & \ldots & a_1 \end{pmatrix} \notin \mathcal{DORI}_n.$$

This means that Dom ρ can only admit the image set $\{a_1, \ldots, a_p\}$ in one way. Thus, it follows from Theorem 7(iii) that $\rho \mathcal{H}^* \sigma$ if and only if $\rho = \sigma$, and consequently, $|H^*_\rho| = 1$.

On the other hand, if $2 \leq h(\rho) \leq \lceil \frac{n}{2} \rceil$, then by Lemma 2, ρ is reversible, and thus the map

$$\sigma = \begin{pmatrix} x_1 & \ldots & x_p \\ a_p & \ldots & a_1 \end{pmatrix} \in \mathcal{DORI}_n.$$

This means that Dom ρ can only admit the image set $\{a_1, \ldots, a_p\}$ in two ways. It follows from Theorem 7(iii) that $\rho \mathcal{H}^* \sigma$. Hence, $|H^*_\rho| = 2$, as required. □

An abundant semigroup S is called *adequate* if the set of idempotents $E(S)$ forms a semilattice. Inverse monoids (or semigroups) are typically examples of adequate monoids (or semigroups) because, in this case, we have $\mathcal{R}^* = \mathcal{R}$ and $\mathcal{L}^* = \mathcal{L}$ [5]. Consequently, as noted in Remark 4, the semigroup $\mathcal{DORI}_n$ is also adequate.

For an element a in an adequate semigroup S, we denote the (unique) idempotent in the $\mathcal{L}^*$-class (or $\mathcal{R}^*$-class) containing a as a^* (or a^+). An adequate semigroup S is called *ample* if for all elements $a \in S$ and all idempotents $e \in S$, the following conditions hold:

$$ae = (ae)^+a \text{ and } ea = a(ea)^*.$$

Since $\mathcal{DORI}_n$ is a subsemigroup of $\mathcal{I}^-_n$, and since $E(\mathcal{DORI}_n) = E(\mathcal{I}^-_n)$, and given that $\mathcal{I}^-_n$ is an ample semigroup, we can conclude from Remark 4 that the following result holds.

Theorem 8. *Let $\mathcal{DORI}_n$ be as defined in (2). Then $\mathcal{DORI}_n$ is an ample semigroup for all n.*

We demonstrate in the following lemma that the relations $\mathcal{L}^*$ and $\mathcal{R}^*$ in the monoid $\mathcal{DORI}_n$ do not commute for $n \geq 2$.

Lemma 8. *For $n \geq 2$, in the monoid $\mathcal{DORI}_n$, it holds that $\mathcal{R}^* \circ \mathcal{L}^* \neq \mathcal{L}^* \circ \mathcal{R}^*$.*

Proof. Consider

$$\rho = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ and } \sigma = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$

in $\mathcal{DORI}_n$. It follows from Theorem 7 that $(\rho, \sigma) \in \mathcal{L}^* \circ \mathcal{R}^*$, but $(\rho, \sigma) \notin \mathcal{R}^* \circ \mathcal{L}^*$, as required. □

Before we can characterize the relations $\mathcal{D}^*$ and $\mathcal{J}^*$, we must first present the following analogues of [30, Lemmas 2.7, 2.8, and 2.9] (respectively).

Lemma 9. *Let $\rho \in \mathcal{DORI}_n$ be as expressed in (6). Then there exists $\sigma \in \mathcal{DORI}_n$ with $\mathrm{Im}\ \sigma = \{1, \ldots, p\}$ such that $(\rho, \sigma) \in \mathcal{R}^*$.*

Proof. Let ρ be an element of $\mathcal{DORI}_n$. Then ρ is either isotone or antitone. In the former, let ρ be as expressed as in (6); and in the latter, let ρ be as expressed in (8). Now define

$$\sigma = \begin{pmatrix} x_1 & \ldots & x_p \\ 1 & \ldots & p \end{pmatrix}.$$

Clearly, σ is in $\mathcal{DORI}_n$, and in either of the cases $\mathrm{Dom}\ \rho = \mathrm{Dom}\ \sigma$, and so by Theorem 7(ii), we see that $(\rho, \sigma) \in \mathcal{R}^*$. □

Lemma 10. *Let $\rho \in \mathcal{DORI}_n$ be expressed as in (6). Then there exists $\sigma \in \mathcal{DORI}_n$ with $\mathrm{Dom}\ \sigma = \{n-p+1, \ldots, n\}$ such that $(\rho, \sigma) \in \mathcal{L}^*$.*

Proof. Let ρ be an element of $\mathcal{DORI}_n$. Then ρ is either isotone or antitone. In the former, let ρ be as expressed as in (6); and in the latter, let ρ be as expressed in (8). Now define

$$\sigma = \begin{pmatrix} n-p+1 & \ldots & n \\ a_1 & \ldots & a_p \end{pmatrix}.$$

Clearly, σ is in $\mathcal{DORI}_n$, and in either of the cases we see that $\mathrm{Im}\ \rho = \mathrm{Im}\ \sigma$, and so by Theorem 7(ii), $(\rho, \sigma) \in \mathcal{L}^*$, as required. □

Lemma 11. *Let ρ, $\sigma \in \mathcal{DORI}_n$. If $\rho \in J^*_\sigma$, then* $|\,\mathrm{Im}\ \rho| \leq |\,\mathrm{Im}\ \sigma|$.

Proof. Let $\rho \in J^*(\sigma)$. Then, according to [5, Lemma 1.7], there exist $\beta_0, \beta_1, \ldots, \beta_n \in \mathcal{DORI}_n$, $\gamma_1, \ldots, \gamma_n$, $\tau_1, \ldots, \tau_n$ in $\mathcal{DORI}_n$ such that $\rho = \beta_0$, $\sigma = \beta_n$, and $(\beta_i, \gamma_i\beta_{i-1}\tau_i) \in \mathcal{D}^*$ for $i = 1, \ldots, n$. Thus,

$$|\ \mathrm{Im}\ \beta_i\ |=|\ \mathrm{Im}\ \gamma_i\beta_{i-1}\tau_i\ |\leq|\ \mathrm{Im}\ \beta_{i-1}\ |,$$

so that

$$|\ \mathrm{Im}\ \rho\ |\leq|\ \mathrm{Im}\ \sigma\ | .$$

The result now follows. □

Within the semigroup $\mathcal{DORI}_n$, we establish a relation $\mathcal{K}$ on $\mathcal{DORI}_n$ defined by the condition that $(\rho, \sigma) \in \mathcal{K}$ if and only if $|\,\mathrm{Im}\ \rho| = |\,\mathrm{Im}\ \sigma|$. It follows that $\mathcal{D} \subseteq \mathcal{K}$. By utilizing Theorem 7 and Lemmas 9–11, and noting that all the maps in $\mathcal{DORI}_n$ are decreasing maps, the next results are the direct corollaries to [25, Lemmas 2.10 & 2.14].

Corollary 3. *For the monoid $\mathcal{DORI}_n$, it follows that $\mathcal{D}^* = \mathcal{R}^* \circ \mathcal{L}^* \circ \mathcal{R}^* = \mathcal{L}^* \circ \mathcal{R}^* \circ \mathcal{L}^*$.*

Corollary 4. *Let $\rho, \sigma \in \mathcal{DORI}_n$. Then $(\rho, \sigma) \in \mathcal{D}^*$ if and only if* $|\,\mathrm{Im}\ \rho| = |\,\mathrm{Im}\ \sigma|$.

Corollary 5. *On the monoid $\mathcal{DORI}_n$, we have $\mathcal{J}^* = \mathcal{D}^*$.*

Now we deduce that:

Lemma 12. *On the semigroups $RQ_n(p)$ and $I(n, p)$, it holds that $\mathcal{D}^* = \mathcal{R}^* \circ \mathcal{L}^* \circ \mathcal{R}^* = \mathcal{L}^* \circ \mathcal{R}^* \circ \mathcal{L}^*$.*

Lemma 13. *If $S \in \{RQ_n(p),\ I(n, p)\}$, then S is abundant.*

Remark 5. (i) It is evident from Theorem 7(i) and (ii) that for each $1 \leq p \leq n$, the number of $\mathcal{L}^*$-classes is equal to the number of $\mathcal{R}^*$-classes in $J^*_p = \{\rho \in \mathcal{DORI}_n : |\,\mathrm{Im}\ \rho| = p\}$. This quantity corresponds to the total number of p size subsets contained in $[n]$, given by $\binom{n}{p}$.

(ii) For S within the set $\{RQ_n(p),\ I(n,\ p)\}$, the characterizations of the starred Green's relations found in Theorem 7 are valid in S as well.

(iii) If S is a semigroup in $\{RQ_n(p),\ I(n,\ p)\}$, then S is adequate because $E(S)$ is a semilattice, making it an ample semigroup.

A semigroup S with $\mathbf{0}$ is called $0-^*$*bisimple* if it has a unique nonzero $\mathcal{D}^*-$class [23]. Therefore, we have established the following result.

Theorem 9. *Let $RQ_n(p)$ be as defined in (5). Then $RQ_n(p)$ is a non-regular $0-^*$bisimple ample semigroup.*

Thus, the semigroup $I(n, p)$, like $\mathcal{DORI}_n$ is the union of the $\mathcal{J}^*$ classes $J_0^*, J_1^*, \ldots, J_p^*$, where

$$J_p^* = \{\rho \in I(n, p) : |\operatorname{Im} \rho| = p\}.$$

Moreover, the ideal $I(n, p)$ has $\binom{n}{p}$ $\mathcal{R}^*$-classes, and $\binom{n}{p}$ $\mathcal{L}^*$-classes in each J_p^*. Consequently, the Rees quotient $RQ_n(p)$ has $\binom{n}{p} + 1$ $\mathcal{R}^*$-classes and $\binom{n}{p}+1$ $\mathcal{L}^*$-classes. (The term 1 results from the singleton class containing the zero element in every instance.) We now present the following lemma, which follows from Remark 5.

Lemma 14. *For all $0 \leq p \leq n$, if $J_p^* = \{\rho \in \mathcal{DORI}_n : |\operatorname{Im} \rho| = p\}$, then $|E(J_p^*)| = \binom{n}{p}$.*

It is a known fact from [6, Proposition 14.3.1] that $|E(\mathcal{IC}_n)| = 2^n$, thus since $E(\mathcal{DORI}_n) = E(\mathcal{IC}_n)$, it follows that $|E(\mathcal{DORI}_n)| = 2^n$.

2. Rank properties

Let A be a nonempty subset of a semigroup S. The *smallest subsemigroup* of S that includes A is referred to as the *subsemigroup generated by* A, typically represented by the notation $\langle A\rangle$. If A is a finite subset of S such that $\langle A\rangle = S$, then S is termed a *finitely-generated semigroup*. The *rank* of a finitely generated semigroup S is defined as the smallest size of a subset A for which $\langle A\rangle$ is equal to S. Specifically,

$$\operatorname{rank}(S) = \min\{|A| : \langle A\rangle = S\}.$$

For a more in-depth examination of ranks in semigroup theory, we recommend that readers refer to [12, 13]. Numerous authors have explored the ranks of various classes of transformation semigroups on the finite chain $[n]$. In particular, we would like to highlight the contributions of Gomes and Howie [7–9,14], Umar [23,25,26], and Zubairu *et al.* [31–34] and the references therein. To the best of our knowledge, the monoid $\mathcal{DORI}_n$ seem not to have appeared in the literature and so the rank of the monoid $\mathcal{DORI}_n$, its Rees quotient $RQ_n(p)$, and its two-sided ideal

$I(n, p)$ have not yet been discussed. In this section, we aim to address these questions.

We will begin our findings by first noting the following definition from [19, Introduction] which also hold for the monoid $\mathcal{DORI}_n$.

Definition 1. An element $\rho \in \mathcal{DORI}_n$ is called *quasi-idempotent* if ρ^2 is an idempotent. Equivalently, ρ in $\mathcal{DORI}_n$ is *quasi-idempotent* if $\rho^4 = \rho^2$.

Remark 6. (i) On the semigroup $\mathcal{DORI}_n$, every quasi-idempotent of height p and of shift 1 (shift of an element say α is the cardinal of the set $S(\alpha) = \{x \in [n] : x\alpha \neq x\}$, see [30]) is of the form:

$$\varepsilon = \begin{pmatrix} y_1 & \cdots & y_{i-1} & y_i & y_{i+1} & \cdots & y_p \\ y_1 & \cdots & y_{i-1} & y_i\varepsilon & y_{i+1} & \cdots & y_p \end{pmatrix}, \tag{9}$$

where $1 \leq y_1 < \cdots < y_{i-1} < y_i\varepsilon < y_i < \cdots < y_p \leq n$. Notice also that quasi-idempotents of shift 1 are not idempotents.

(ii) Every idempotent element is quasi-idempotent but the converse is not necessarily true.

We now introduce the following definition.

Definition 2. An antitone map $\rho \in \mathcal{DORI}_n$, of height $2 \leq p \leq \lceil \frac{n}{2} \rceil$, is said to be a *vital element* if Dom ρ is convex and min Dom ρ is a fixed point. That is to say ρ is of the form:

$$\rho = \begin{pmatrix} y_p & y_p + 1 & \cdots & y_p + p - 1 \\ y_p & y_{p-1} & \cdots & y_1 \end{pmatrix}, \tag{10}$$

where $1 \leq y_1 < \cdots < y_p \leq n$.

The following lemma can be stated immediately.

Lemma 15. *For $2 \leq p \leq \lceil \frac{n}{2} \rceil$, every $\mathcal{L}^*$-class of an element, say ρ, of height p in the monoid $\mathcal{DORI}_n$ contains a unique vital element in L^*_ρ.*

Proof. Let $\rho \in \mathcal{DORI}_n$ with $h(\rho) = p$, where $2 \leq p \leq \lceil \frac{n}{2} \rceil$. Then ρ is either an isotone map or an antitone map. In the former, let ρ be as expressed as in (6); and in the latter, let ρ be as expressed in (8). Consider L^*_ρ, now it is clear that in either case, $a_p = \max(\text{Im } \alpha)$. Moreover, $a_1 < \cdots < a_p < a_p + 1 < \cdots < a_p + p - 1$. Thus, the map

$$\delta = \begin{pmatrix} a_p & a_p + 1 & \cdots & a_p + p - 1 \\ a_p & a_{p-1} & \cdots & a_1 \end{pmatrix}$$

is an antitone map in $\mathcal{DORI}_n$, where $\text{Im }\rho = \text{Im }\delta$, and so $\delta \in L^*_\rho$. Clearly Dom δ is convex, and $a_p = \min(\text{Dom }\delta)$ is a fixed point of δ. Therefore, δ is the unique vital element in L^*_α. □

The next lemma concerns factorizations of antitone maps with a convex domain in the monoid $\mathcal{DORI}_n$.

Lemma 16. *Every antitone element $\rho \in \mathcal{DORI}_n$ with a convex domain is a product of quasi-idempotents and the unique vital element that belongs to L^*_ρ.*

Proof. Consider an antitone map of height p with a convex domain, say ρ, in $\mathcal{DORI}_n$ of the form:

$$\rho = \begin{pmatrix} t & t+1 & \cdots & t+p-1 \\ a_p & a_{p-1} & \cdots & a_1 \end{pmatrix},$$

where $2 \le p \le \lceil \frac{n}{2} \rceil$ and $1 \le a_1 < \cdots < a_p \le t < \cdots < t+p-1 \le n$ for some $t \in [n]$. Now, for $1 \le i \le p$, define:

$$\varepsilon_i = \left(\begin{array}{cccc|c|ccc} a_p & a_p+1 & \cdots & a_p+i-2 & t+i-1 & t+i & \cdots & t+p-1 \\ a_p & a_p+1 & \cdots & a_p+i-2 & a_p+i-1 & t+i & \cdots & t+p-1 \end{array} \right)$$

$$\& \ \delta = \begin{pmatrix} a_p & a_p+1 & \cdots & a_p+p-1 \\ a_p & a_{p-1} & \cdots & a_1 \end{pmatrix}.$$

Notice that $a_p \le t$ implies $a_p + i - 1 \le t + i - 1$ for all $1 \le i \le p$, and so ε_i is an isotone idempotent (if $a_p = t$) or quasi-idempotent (if $a_p < t$) element of shift 1 and height p in $\mathcal{DORI}_n$. Moreover, it is clear that $a_1 < \cdots < a_p < a_p + 1 < \cdots < a_p + p - 1$. Thus, δ is an antitone map with a convex domain that fixes $a_p = \min \text{Dom }\delta$. Therefore, δ is a vital element. Furthermore, $\text{Im }\rho = \text{Im }\delta$ implies $\delta \in L^*_\rho$, which is unique by Lemma 15.

At this juncture, observe that:

$$\begin{aligned} \varepsilon_1\varepsilon_2\cdots\varepsilon_p\delta &= \begin{pmatrix} t & t+1 & \cdots & t+p-1 \\ a_p & t+1 & \cdots & t+p-1 \end{pmatrix} \begin{pmatrix} a_p & t+1 & t+2 & \cdots & t+p-1 \\ a_p & a_p+1 & t+2 & \cdots & t+p-1 \end{pmatrix} \cdots \\ &\quad \begin{pmatrix} a_p & a_p+1 & \cdots & a_p+p-2 & t+p-1 \\ a_p & a_p+1 & \cdots & a_p+p-2 & a_p+p-1 \end{pmatrix} \begin{pmatrix} a_p & a_p+1 & \cdots & a_p+p-1 \\ a_p & a_{p-1} & \cdots & a_1 \end{pmatrix} \\ &= \begin{pmatrix} t & t+1 & \cdots & t+p-1 \\ a_p & a_{p-1} & \cdots & a_1 \end{pmatrix} = \rho, \end{aligned}$$

as postulated. □

Theorem 10. *The monoid $\mathcal{DORI}_n$ is generated by quasi-idempotents of shift 1 and vital elements.*

Proof. Let $\rho \in \mathcal{DORI}_n$. Then, ρ is either an isotone map or an antitone map.
(i) If ρ is an isotone map, then $\rho \in \mathcal{IC}_n$. Thus, by [1, Lemma 3.3], ρ is quasi-idempotent-generated.
(ii) Now suppose ρ is an antitone map of the form

$$\rho = \begin{pmatrix} x_1 & \cdots & x_p \\ a_p & \cdots & a_1 \end{pmatrix}$$

of height $2 \leq p \leq \lceil \frac{n}{2} \rceil$, where $a_1 < \cdots < a_p < x_1 < \cdots < x_p$. Now suppose $s_{i+1} = x_{i+1} - x_i$ for all $1 \leq i \leq p-1$. Next, observe that:

$$\begin{aligned} x_1 + s_2 + s_3 + \cdots + s_{i+1} &= x_1 + (x_2 - x_1) + (x_3 - x_2) + \cdots \\ &\quad + (x_i - x_{i-1}) + (x_{i+1} - x_i) \\ &= x_{i+1}. \end{aligned}$$

This shows that for each $1 \leq i \leq p-1$, $x_{i+1} = x_1 + s_2 + s_3 + \cdots + s_{i+1} \geq x_1 + i$. So, the map

$$\xi_i = \left(\begin{array}{ccc|c|ccc} x_1 & \cdots & x_1 + i - 1 & x_{i+1} & x_{i+2} & \cdots & x_p \\ x_1 & \cdots & x_1 + i - 1 & x_1 + i & x_{i+2} & \cdots & x_p \end{array} \right),$$

is an idempotent if $x_{i+1} = x_1 + i$; and it is quasi-idempotent of shift 1 and height p in $\mathcal{DORI}_n$ if $x_{i+1} > x_1 + i$ for all $1 \leq i \leq p$. Notice also that since $a_p \leq x_1$, it follows that $a_p \leq x_1 < x_1 + 1 < \cdots < x_1 + p - 1$. Therefore, the map σ defined as

$$\sigma = \begin{pmatrix} x_1 & x_1 + 1 & \cdots & x_1 + p - 1 \\ a_p & a_{p-1} & \cdots & a_1 \end{pmatrix}$$

is an antitone map in $\mathcal{DORI}_n$ with convex domain, and so by Lemma 16, σ is generated by quasi-idempotents and the unique vital element in L^*_σ. Now observe that:

$$\begin{aligned} \xi_1 \cdots \xi_p \sigma &= \begin{pmatrix} x_1 & x_2 & x_3 & \cdots & x_p \\ x_1 & x_1 + 1 & x_3 & \cdots & x_p \end{pmatrix} \begin{pmatrix} x_1 & x_1 + 1 & x_3 & x_4 & \cdots & x_p \\ x_1 & x_1 + 1 & x_1 + 2 & x_4 & \cdots & x_p \end{pmatrix} \\ &\quad \cdots \begin{pmatrix} x_1 & x_1 + 1 & \cdots & x_1 + p - 2 & x_p \\ x_1 & x_1 + 1 & \cdots & x_1 + p - 2 & x_1 + p - 1 \end{pmatrix} \begin{pmatrix} x_1 & x_1 + 1 & \cdots & x_1 + p - 1 \\ a_p & a_{p-1} & \cdots & a_1 \end{pmatrix} \\ &= \begin{pmatrix} x_1 & x_2 & \cdots & x_p \\ a_p & a_{p-1} & \cdots & a_1 \end{pmatrix} = \rho, \end{aligned}$$

as required.
Clearly, Im ρ = Im σ, and so $L^*_\sigma = L^*_\rho$. Hence, ρ is generated by quasi-idempotent and the unique vital element in L^*_ρ. □

To investigate the minimum generating set of $\mathcal{DORI}_n$, we first note the following definition from [1].

Definition 3 ([1, Definition 3.5])**.** A quasi-idempotent element ε of shift 1 as expressed in (9) is called an *essential element* if $y_i\varepsilon = y_i - 1$, and so essential elements of height p are of the form:

$$\varepsilon = \begin{pmatrix} y_1 & \cdots & y_{i-1} & y_i & y_{i+1} & \cdots & y_p \\ y_1 & \cdots & y_{i-1} & y_i - 1 & y_{i+1} & \cdots & y_p \end{pmatrix}. \tag{11}$$

Remark 7. For $1 \leq n \leq 2$, the quasi-idempotent elements in $\mathcal{DORI}_n$ are all essential elements. However, for $n \geq 3$, not every quasi-idempotent element of shift 1 is an essential element. For example, the element

$$\alpha = \begin{pmatrix} 1 & 4 & 5 \\ 1 & 2 & 5 \end{pmatrix} \in \mathcal{DORI}_5$$

is a quasi-idempotent of shift 1, but since $4\alpha = 2 \neq 3$, then it is not an essential element. Moreover, every quasi-idempotent of height $n-1$ is necessarily essential.

At this juncture, we introduce the following definition.

Definition 4. A vital element δ is called *convex* if $\operatorname{Im}\delta$ is convex, i.e., if δ is of the form

$$\delta = \begin{pmatrix} y_p & y_p + 1 & \cdots & y_p + p - 1 \\ y_p & y_p - 1 & \cdots & y_p - p + 1 \end{pmatrix},$$

where $1 \leq y_p - p + 1 < \cdots < y_p + p - 1 \leq n$.

Remark 8. Every convex vital element of height $2 \leq p \leq \lceil \frac{n}{2} \rceil$ is of the form:

$$\delta_i = \begin{pmatrix} i & i+1 & \cdots & i+p-1 \\ i & i-1 & \cdots & i-p+1 \end{pmatrix}, \tag{12}$$

where $p \leq i \leq n - p + 1$. Moreover, it is worth noticing that since every vital element is an antitone map, and there are no antitone map of height $p > \lceil \frac{n}{2} \rceil$ by the contrapositive of Lemma 1, then there are no convex vital elements of height $p > \lceil \frac{n}{2} \rceil$.

We now have the following lemma.

Lemma 17. *Every non-convex vital element in $\mathcal{DORI}_n$ is a product of a convex vital element and essential elements.*

Proof. Let δ be a non-convex vital element of height $2 \le p \le \lceil \frac{n}{2} \rceil$, as expressed in (10). This means that there exists $0 \le i \le p-2$ such that $\{y_{p-i}, y_{p-i+1}, \ldots, y_p\}$ is convex and $y_{p-(i+1)} - y_{p-i} > 1$. That is to say,

$$\{y_{p-i}, y_{p-i+1}, \ldots, y_{p-1}, y_p\} = \{y_p - i, y_p - i + 1, \ldots, y_p - 1, y_p\},$$

so that δ is of the form

$$\delta = \left(\begin{array}{ccccc|ccc} y_p & y_p+1 & \cdots & y_p+i-1 & y_p+i & y_p+i+1 & \cdots & y_p+p-1 \\ y_p & y_p-1 & \cdots & y_p-i+1 & y_p-i & y_{p-(i+1)} & \cdots & y_1 \end{array} \right).$$

This means that the subset (of Im δ) $\{y_1, \ldots, y_{p-(i+1)}\}$ needs not be convex. Notice that

$$\{y_{p-i}-1,\, y_{p-i}-2,\, \ldots,\, y_{p-i}-(p-i-2),\, y_{p-i}-(p-i-1)\}$$

is a convex set consisting of translates of y_{p-i}, which can be replaced with $y_p - i$ (since $y_{p-i} = y_p - i$) as follows:

$$\{y_p - i - 1,\, y_p - i - 2,\, \ldots,\, y_p - p + 2,\; y_p - p + 1\}.$$

Thus, the set $\{y_p - p + 1,\, y_p - p + 2,\, \ldots, y_p - i - 1,\, y_p - i,\, y_p - i + 1,\, \ldots,\; y_p - 1,\; y_p\}$ is convex. Therefore, the map δ^* defined as:

$$\delta^* = \left(\begin{array}{ccccc|c|cc} y_p & y_p+1 & \cdots & y_p+i-1 & y_p+i & y_p+i+1 & \cdots & y_p+p-1 \\ y_p & y_p-1 & \cdots & y_p-i+1 & y_p-i & y_p-i-1 & \cdots & y_p-p+1 \end{array} \right)$$

is a convex vital element. Additionally, the map γ defined as

$$\gamma = \begin{pmatrix} y_p-p+1 & y_p-p+2 & \cdots & y_p-i-1 & y_p-i & y_p-i+1 & \cdots & y_p-1 & y_p \\ y_1 & y_2 & \cdots & y_{p-i-1} & y_{p-i} & y_{p-i+1} & \cdots & y_{p-1} & y_p \end{pmatrix}$$

is a decreasing isotone map which is generated by essential elements by [1, Theorem 3.7]. Now, clearly, $\delta^*\gamma = \delta$. The result now follows. □

We now give and prove a key result of this section.

Theorem 11. *The monoid $\mathcal{DORI}_n$ is generated by essential and convex vital elements.*

Proof. Notice from Theorem 10, $\mathcal{DORI}_n$ is generated by quasi-idempotents of shift 1 and vital elements. Now we need to show that quasi-idempotent elements of shift 1 can be generated by essential elements, and the vital elements can be generated by the convex vital elements. Now let $\varepsilon \in \mathcal{DORI}_n$ as expressed in (9) be a quasi-idempotent. Since

$y_i > y_i\varepsilon$, we can let $s = y_i - y_i\varepsilon$, so clearly $y_i\varepsilon + j - 1 < y_i\varepsilon + j$ for all $1 \leq j \leq s$ and also $y_i = y_i\varepsilon + s$. Now for $1 \leq j \leq s$ define ε_j as:

$$\varepsilon_j = \begin{pmatrix} y_1 & \cdots & y_{i-1} & y_i\varepsilon + j & y_{i+1} & \cdots & y_p \\ y_1 & \cdots & y_{i-1} & y_i\varepsilon + j - 1 & y_{i+1} & \cdots & y_p \end{pmatrix}.$$

Notice that $y_{i-1} < y_i\varepsilon < y_i\varepsilon + 1 < \cdots < y_i\varepsilon + s = y_i < y_{i+1}$, and so for each j, we can easily see that ε_j is an essential element. Moreover, it is not difficult to see that

$$\varepsilon_s\varepsilon_{s-1}\cdots\varepsilon_1 = \varepsilon.$$

Furthermore, by Lemma 17 the vital elements can be generated by convex vital elements. The result follows. □

Lemma 18. *Every element in $S \in \{RQ_n(p), I(n,p)\}$ of height p can be expressed as a product of essential and convex vital elements in S, each of height p.*

Proof. Let $\alpha \in S \in \{RQ_n(p), I(n,p)\}$. Then clearly α is of height p, and so since in the proofs of Lemma 17 and Theorem 10, $h(\rho) = h(\xi_i) = h(\delta) = h(\delta^*) = h(\sigma) = p$ for all $i \in \{1,\ldots,p\}$, then the result easily follows. □

Now, for $2 \leq p \leq \lceil \frac{n}{2} \rceil$, let $O(p)$ be the collection of all convex vital elements in $RQ_n(p)$, and let $E(RQ_n(p) \setminus \{0\})$ be the collection of all nonzero idempotents in $RQ_n(p)$ and $M(p)$ be the collection of all essential elements in $RQ_n(p)$. It is important to note from definition that no essential element is an idempotent and vice-versa. Similarly, no essential is a convex vital element. Thus, $O(p) \cap M(p) = \emptyset$, $O(p) \cap E(RQ_n(p) \setminus \{0\}) = \emptyset$ and $M(p) \cap E(RQ_n(p) \setminus \{0\}) = \emptyset$. Then, we present the following lemma.

Lemma 19. *(i) For $2 \leq p \leq \lceil \frac{n}{2} \rceil$, $|O(p)| = n - 2p + 2$.*

(ii) For each $1 \leq p \leq n$, we have $|E(RQ_n(p) \setminus \{0\})| = \binom{n}{p}$.

(iii) $|M(p)| = (n-1)\binom{n-2}{p-1}$.

Proof. **(i)** Notice that by Remark 8, the domain of each convex vital element of height $2 \leq p \leq \lceil \frac{n}{2} \rceil$ is of the form $\{i, i+1, \ldots, i+p-1\}$, where $p \leq i \leq n - p + 1$. Clearly the domain set has its minimum element, i, within the range $p \leq i \leq n - p + 1$. This number is equivalent to counting all the possible minimum elements (which are within the range $p \le i \le n-p+1$) of the possible domains set.

(ii) Notice that since every idempotent element is an injective map, and to form an idempotent of height p is to select p elements out of the domain $[n]$, then fix each of these elements to form an idempotent element. This number is equivalent to $\binom{n}{p}$ $\mathcal{R}^*$-classes in J_p^*.

(iii) Notice that for $1 \le i \le n-1$, the element $i+1$ can be paired with i (which is equivalent to mapping $i+1$ to i) in $n-1$ ways. Now, from the remaining $n-2$ elements (i.e., elements of $[n] \setminus \{i, i+1\}$), we can select $p-1$ elements as our fixed points to form an essential element in $\binom{n-2}{p-1}$ ways. Hence we have altogether $(n-1)\binom{n-2}{p-1}$ essential elements, as required. □

Now, let

$$G(p) = \begin{cases} E(RQ_n(p) \setminus \{0\}) \cup M(p), & \text{if } p \in \{1, \lceil \frac{n}{2} \rceil + 1, \ldots, n\}; \\ M(p) \cup E(RQ_n(p) \setminus \{0\}) \cup O(p), & \text{if } 2 \le p \le \lceil \frac{n}{2} \rceil. \end{cases}$$

Consequently, we have the following result.

Corollary 6. *On the Rees quotient semigroup $RQ_n(p)$, we have:*

$$|G(p)| = \begin{cases} \binom{n}{p} + (n-1)\binom{n-2}{p-1}, & \text{if } p \in \{1, \lceil \frac{n}{2} \rceil + 1, \ldots, n\}; \\ \binom{n}{p} + (n-1)\binom{n-2}{p-1} + n - 2p + 2, & \text{if } 2 \le p \le \lceil \frac{n}{2} \rceil. \end{cases}$$

Proof. The result follows directly from Lemma 19. □

The forthcoming result establishes that $G(p)$ is the minimum generating set of $RQ_n(p) \setminus \{0\}$.

Lemma 20. *Let ρ, σ be elements in $RQ_n(p) \setminus \{0\}$ $(1 \le p \le n)$. Then $\rho\sigma \in G(p)$ if and only if $\rho, \sigma \in G(p)$ and $\rho\sigma = \rho$ or $\rho\sigma = \sigma$.*

Proof. Suppose $\rho\sigma \in G(p)$. Thus either $\rho\sigma \in E(RQ_n(p) \setminus \{0\})$ or $\rho\sigma \in M(p)$ or $\rho\sigma \in O(p)$. We consider the three cases separately.

Case i. If $\rho\sigma \in E(RQ_n(p) \setminus \{0\})$. Then

$$p = f(\rho\sigma) \le f(\rho) \le |\operatorname{Im} \rho| = p,$$

$$p = f(\rho\sigma) \le f(\sigma) \le |\operatorname{Im} \sigma| = p.$$

This ensures that

$$F(\rho) = F(\rho\sigma) = F(\sigma),$$

and so ρ, $\sigma \in E(RQ_n(p) \setminus \{0\})$ and $\rho\sigma = \rho$.

Case ii. Now suppose $\rho\sigma \in M(p)$. Thus $\rho\sigma$ is an essential. Thus $\rho\sigma$ is an essential element which can be expressed as in (11), that is:

$$\rho\sigma = \begin{pmatrix} y_1 & \cdots & y_{i-1} & y_i & y_{i-1} & \cdots & y_p \\ y_1 & \cdots & y_{i-1} & y_i - 1 & y_{i+1} & \cdots & y_p \end{pmatrix}.$$

This means that Dom ρ = Dom $\rho\sigma$, Im σ = Im $\rho\sigma$, and Im ρ = Dom σ. Thus,

$$\rho = \begin{pmatrix} y_1 & \cdots & y_{i-1} & y_i & y_{i+1} & \cdots & y_p \\ y_1 & \cdots & y_{i-1} & y_i\rho & y_{i+1} & \cdots & y_p \end{pmatrix}$$

and

$$\sigma = \begin{pmatrix} y_1 & \cdots & y_{i-1} & y_i\rho & y_{i+1} & \cdots & y_p \\ y_1 & \cdots & y_{i-1} & y_i - 1 & y_{i+1} & \cdots & y_p \end{pmatrix}.$$

Notice that since ρ and σ are decreasing maps, we must have $y_i\rho \leq y_i$ and $y_i - 1 \leq y_i\rho$, and so $y_i - 1 \leq y_i\rho \leq y_i$. However, the fact that $\rho\sigma$ is an essential element, implies either $y_i\rho = y_i - 1$ or $y_i\rho = y_i$. We consider the two subcases separately.

Subcase a. If $y_i\rho = y_i - 1$, then ρ and σ are obviously essential and idempotent elements, respectively, and so $\rho\sigma = \rho$ and $\sigma \in E(RIC_n(p) \setminus \{0\})$.

Subcase b. If $y_i\rho = y_i$, then ρ and σ are clearly idempotent and essential elements, respectively. Thus, it follows easily that $\rho\sigma = \sigma$ and $\rho \in E(RIC_n(p) \setminus \{0\})$. In either of the subcases, we see that $\rho, \sigma \in G(p)$ and either $\rho\sigma = \sigma$ or $\rho\sigma = \rho$.

Case iii. Now suppose $\rho\sigma \in O(p)$. Thus, $\rho\sigma$ is a convex vital element, which has the form

$$\rho\sigma = \begin{pmatrix} i & i+1 & \cdots & i+p-1 \\ i & i-1 & \cdots & i-p+1 \end{pmatrix},$$

where $p \leq i \leq n - p + 1$. This means that Dom ρ = Dom $\rho\sigma$, Im σ = Im $\rho\sigma$, and Im ρ = Dom σ. Thus,

$$\rho = \begin{pmatrix} i & i+1 & \cdots & i+p-1 \\ i\rho & (i+1)\rho & \cdots & (i+p-1)\rho \end{pmatrix}$$

and

$$\sigma = \begin{pmatrix} i\rho & (i+1)\rho & \cdots & (i+p-1)\rho \\ i & i-1 & \cdots & i-p+1 \end{pmatrix}.$$

The claim here is that ρ must be an idempotent. Notice that ρ and σ are decreasing maps. Thus, $i\rho \leq i$ and $i = (i\rho)\sigma \leq i\rho$. This ensures that $i\rho = i$. Moreover, for any $i-p+1 \leq j \leq i-1$ we see that $j \leq (j+2)\rho \leq j+2$. This means that either $(j+2)\rho = j$ or $(j+2)\rho = j+1$ or $(j+2)\rho = j+2$.

Notice that if $(j+2)\rho = j$, then in particular, if $j = i-2$, we see that $(i-2+2)\rho = i-2$, that is, $i\rho = i-2$, which contradicts the fact that $i\rho = i$.

Now, if $(j+2)\rho = j+1$ for all j, then in particular, if $j = i-1$, we have $(i-1+2)\rho = i-1+1$, that is, $(i+1)\rho = i$, which also contradicts the fact that $i\rho = i$. Hence, we conclude that $(j+2)\rho = j+2$ for all j. This ensures that

$$\rho = \begin{pmatrix} i & i+1 & \cdots & i+p-1 \\ i & i+1 & \cdots & i+p-1 \end{pmatrix} \text{ and } \sigma = \begin{pmatrix} i & i+1 & \cdots & i+p-1 \\ i & i-1 & \cdots & i-p+1 \end{pmatrix}.$$

Therefore, $\sigma \in O(p) \subset G(p)$ and $\rho \in E(RQ_n(p) \setminus \{0\}) \subset G(p)$, and also $\rho\sigma = \sigma$.

The converse follows easily. □

Thus, we state one of the main results in this section.

Theorem 12. *Let $RQ_n(p)$ be as defined in (5). Then*

$$\text{rank } RQ_n(p) = \begin{cases} \binom{n}{p} + (n-1)\binom{n-2}{p-1}, & \text{if } p \in \{1, \lceil \frac{n}{2} \rceil + 1, \ldots, n\}; \\ \binom{n}{p} + (n-1)\binom{n-2}{p-1} + n - 2p + 2, & \text{if } 2 \leq p \leq \lceil \frac{n}{2} \rceil. \end{cases}$$

Proof. By Lemma 20, we know that $G(p)$ is the minimal generating set of $RQ_n(p)$, and Corollary 6 provides its order. □

For $1 \leq p \leq n-1$ let

$$J_p^* = \{\rho \in \mathcal{DORI}_n : |\operatorname{Im} \rho| = p\},$$

and, for $2 \leq p \leq \lceil \frac{n}{2} \rceil$ let $O(p)$ and $M(p)$ be the collection of all convex vital and essential elements in J_p^*, and let

$$G(p) = \begin{cases} E(J_p^*) \cup M(p), & \text{if } p \in \{1, \lceil \frac{n}{2} \rceil + 1, \ldots, n\}; \\ M(p) \cup E(J_p^*) \cup O(p), & \text{if } 2 \leq p \leq \lceil \frac{n}{2} \rceil. \end{cases}$$

The following lemma is needed in the quest to determine the ranks of the monoid $\mathcal{DORI}_n$ and its two-sided ideal $I(n, p)$.

Lemma 21. *In each J_p^*, we have:*

$$|G(p)| = \begin{cases} \binom{n}{p} + (n-1)\binom{n-2}{p-1}, & \text{if } p \in \{1, \lceil \frac{n}{2} \rceil + 1, \ldots, n\}; \\ \binom{n}{p} + (n-1)\binom{n-2}{p-1} + n - 2p + 2, & \text{if } 2 \leq p \leq \lceil \frac{n}{2} \rceil. \end{cases}$$

Proof. It is important to note that $J_p^* = RQ_n(p) \setminus \{\mathbf{0}\}$, and so $G(p)$ defined on J_p^* and $RQ_n(p)$ are equal. Thus, the result follows from Corollary 6. □

Let us recall from (12) that, for $2 \leq p \leq \lceil \frac{n}{2} \rceil$ and any $p \leq i \leq n-p+1$ the convex vital element of height p has the form:

$$\delta_i = \begin{pmatrix} i & i+1 & \cdots & i+p-1 \\ i & i-1 & \cdots & i-p+1 \end{pmatrix}.$$

We now give the following definition.

Definition 5. A convex vital element δ_i $(p \leq i \leq n-p+1)$ is called *extreme* if the union of its domain and image set contains 1 or n, i.e., if $n \in \text{Dom } \delta_i \cup \text{Im } \delta_i$ or $1 \in \text{Dom } \delta_i \cup \text{Im } \delta_i$.

For the purpose of illustrations, consider the following convex vital elements of height 3 in $\mathcal{DORI}_7$: $\delta_1 = \begin{pmatrix} 3 & 4 & 5 \\ 3 & 2 & 1 \end{pmatrix}$, $\delta_2 = \begin{pmatrix} 4 & 5 & 6 \\ 4 & 3 & 2 \end{pmatrix}$, and $\delta_3 = \begin{pmatrix} 5 & 6 & 7 \\ 5 & 4 & 3 \end{pmatrix}$. Clearly, δ_1 and δ_2 are extreme vital elements. However, δ_2 is not.

Considering the convex vital element δ_i $(p \leq i \leq n-p+1)$ defined as in (12), its two extreme cases are when $i = p$ and $i = n-p+1$, i.e.,

$$\delta_p = \begin{pmatrix} p & p+1 & \cdots & 2p-1 \\ p & p-1 & \cdots & 1 \end{pmatrix}$$

and

$$\delta_{n-p+1} = \begin{pmatrix} n-p+1 & n-p+2 & \cdots & n-1 & n \\ n-p+1 & n-p & \cdots & n-2p+1 & n-2p+2 \end{pmatrix},$$

respectively. These elements have the following property:

If n is odd and $p = \lceil \frac{n}{2} \rceil = \frac{n+1}{2}$. Then,

$$n - p + 1 = n - \frac{n+1}{2} + 1 = \frac{n+1}{2} = \lceil \frac{n}{2} \rceil = p.$$

Hence, if n is odd, $n > 1$ and $p = \lceil \frac{n}{2} \rceil$, then it is easy to see that there is only one extreme convex vital element which is $\delta_{\lceil \frac{n}{2} \rceil}$.

However, if n is even, $n > 2$ and $p = \lceil \frac{n}{2} \rceil$, then there are two extreme convex vital elements: $\delta_{\frac{n}{2}}$ and $\delta_{\frac{n+2}{2}}$.

We now have the following lemma.

Lemma 22. *Let $2 \leq p \leq \lceil \frac{n}{2} \rceil - 1$. Then, for $p + 1 \leq i \leq n - p$, the convex vital element δ_i in J^*_p as expressed in (12), can be expressed as a product of a convex vital element and idempotent elements in J^*_{p+1}.*

Proof. Let $\delta_{p,\,i}$ be a convex vital element in J^*_p as expressed in (12), where $p + 1 \leq i \leq n - p$. Notice that adding and subtracting p to the range $p + 1 \leq i \leq n - p$ implies $2p + 1 \leq i + p \leq n$ and $1 \leq i - p \leq n - 2p$, respectively. Thus, we see that $i + p \leq n$ and $i - p \geq 1$, which ensures that the map defined as

$$\delta'_{p,\,i} = \begin{pmatrix} i & i+1 & \cdots & i+p-1 & i+p \\ i & i-1 & \cdots & i-p+1 & i-p \end{pmatrix}$$

is a convex vital element in J^*_{p+1}. Notice that the map ϵ defined as

$$\epsilon = \begin{pmatrix} i-p+1 & i-p+2 & \cdots & i & i+1 \\ i-p+1 & i-p+2 & \cdots & i & i+1 \end{pmatrix}$$

is in $E(J^*_{p+1})$. Now it is easy to see that

$$\begin{aligned} \delta'_{p,\,i}\epsilon &= \begin{pmatrix} i & i+1 & \cdots & i+p-1 & i+p \\ i & i-1 & \cdots & i-p+1 & i-p \end{pmatrix} \begin{pmatrix} i-p+1 & i-p+2 & \cdots & i & i+1 \\ i-p+1 & i-p+2 & \cdots & i & i+1 \end{pmatrix} \\ &= \begin{pmatrix} i & i+1 & \cdots & i+p-1 \\ i & i-1 & \cdots & i-p+1 \end{pmatrix} = \delta_{p,\,i} \end{aligned}$$

as required. ∎

Remark 9. It is crucial to highlight that there is no convex vital element and idempotent of height greater than p that can generate the extreme convex vital elements of height p, δ_p and δ_{n-p+1}.

We now present the following lemma.

Lemma 23.

$$J^*_p \subset \begin{cases} \langle J^*_{p+1} \rangle, & \text{if } p \in \{0, 1, \lceil \frac{n}{2} \rceil + 1, \ldots, n-2\}; \\ \langle J^*_{p+1} \cup \{\delta_p,\ \delta_{n-p+1}\} \rangle, & \text{if } 2 \leq p \leq \lceil \frac{n}{2} \rceil. \end{cases}$$

Proof. **(i)** If $\lceil \frac{n}{2} \rceil + 1 \leq p \leq n-2$ or $p = 0, 1$ then using Theorem 10, it is enough to establish that every element in $G(p)$ can be written as a product of elements in $G(p+1)$. That is to say, every essential or idempotent element of height p can be written as a product of essential or idempotent elements of height $p+1$. Thus, we consider the elements of $E(J_p^*)$ and $M(p)$ separately.

a. (1) The elements in $E(J_p^*)$:

Let $\epsilon \in E(J_p^*)$ be expressed as:

$$\epsilon = \begin{pmatrix} y_1 & \cdots & y_p \\ y_1 & \cdots & y_p \end{pmatrix},$$

where $1 \leq y_1 < \cdots < y_p \leq n$. Since $p \leq n-2$, it follows that $(\text{Dom } \epsilon)'$ contains at least two elements, say c and d. Without loss of generality, suppose $c < d$. Let $A = \text{Dom } \epsilon \cup \{c\}$ and $B = \text{Dom } \epsilon \cup \{d\}$, and define ϵ_1 and ϵ_2 as follows:

For $x \in A$ and $y \in B$

$$x\epsilon_1 = \begin{cases} x, & \text{if } x \neq c; \\ c, & \text{if } x = c \end{cases} \quad \text{and} \quad y\epsilon_2 = \begin{cases} y, & \text{if } y \neq d; \\ d, & \text{if } y = d. \end{cases}$$

Clearly, ϵ_1 and ϵ_2 are idempotents in $E(J_{p+1}^*)$, and one can easily show that $\epsilon = \epsilon_1\epsilon_2$.

(2) The elements in $M(p)$:

Let ε be an essential element of height p as expressed in (11), which has the form:

$$\varepsilon = \begin{pmatrix} y_1 & \cdots & y_{i-1} & y_i & y_{i+1} & \cdots & y_p \\ y_1 & \cdots & y_{i-1} & y_i - 1 & y_{i+1} & \cdots & y_p \end{pmatrix}.$$

Now, since $p \leq n-2$, it follows that $(\text{Dom } \varepsilon \cup \text{Im } \varepsilon)'$ contains at least one element, say d. Notice that $y_i - 1 \notin \text{Dom } \varepsilon$. Now let $A = \text{Dom } \varepsilon \cup \{d\}$ and $B = \text{Dom } \varepsilon \cup \{y_i - 1\}$, and define ε' and ϵ as follows:

For $x \in A$ and $y \in B$

$$x\varepsilon' = \begin{cases} x\varepsilon, & \text{if } x \neq d; \\ d & \text{if } x = d \end{cases} \quad \text{and} \quad y\epsilon = y.$$

Notice that $\epsilon \in E(J_{p+1}^*) \subset G(p+1)$, and it is not difficult to see that ε' is an essential element in $M(p+1) \subset G(p+1)$. One can now easily show that $\varepsilon = \varepsilon'\epsilon$.

b. Now suppose $2 \leq p \leq \lceil \frac{n}{2} \rceil$. The essential and idempotent elements in $G(p)$ within this range have been addressed by (i) above. So it is sufficient to show that every convex vital element in $G(p)$ can be expressed as a product of elements in $G(p+1) \cup \{\delta_p,\ \delta_{n-p+1}\}$.

Let δ_i be a convex vital element of height p in $G(p)$, as expressed in (12). It follows from Lemma 22 that for all $p+1 \leq i \leq n-p$, δ_i is a product of a convex vital element and an idempotent element, each of height $p+1$. Notice that the remaining convex vital elements δ_p and δ_{n-p+1} in $G(p)$, which by Remark 9 are not expressible as products of elements in $G(p+1)$, as such, every convex vital element in $G(p)$ can be expressed as a product of elements in $G(p+1) \cup \{\delta_p,\ \delta_{n-p+1}\}$. We have now finalized the proof of the lemma. □

Remark 10. It becomes clear from the above lemma that for $2 \leq p \leq \lceil \frac{n}{2} \rceil$, any minimum generating set of $I(n,p)$ must contain all the extreme convex vital elements of height below p, i.e., the elements δ_i and δ_{n-i+1} for all $2 \leq i \leq p-1$.

Now, on $I(n,p)$ let

$$W(p) = \begin{cases} G(p), & \text{if } p = 0, 1; \\ G(p) \cup O(\lceil \frac{n}{2} \rceil) \cup \{\delta_i,\ \delta_{n-i+1} : 2 \leq i \leq \lceil \frac{n}{2} \rceil - 1\}, & \text{if } p \in \{\lceil \frac{n}{2} \rceil + 1, \ldots, n-1\}; \\ G(p) \cup \{\delta_i,\ \delta_{n-i+1} : 2 \leq i \leq p-1\}, & \text{if } 2 \leq p \leq \lceil \frac{n}{2} \rceil. \end{cases}$$

We now present the following lemmas.

Lemma 24. *For $0 \leq p \leq n-1$, $W(p)$ is the minimum generating set of $I(n,p)$.*

Proof. The result follows from the fact that $G(p)$ is the minimum generating set $\langle J_p^* \rangle$ by Lemma 20, and also from the fact that each minimum generating set of the ideal $I(n,p)$ must contain all the extreme convex vital elements below height p, as stated in Remark 10. □

Lemma 25. *For $1 \leq p \leq n-1$, we have $|W(p)| = (n-2) + \binom{n}{p} + (n-1)\binom{n-2}{p-1}$.*

$$|W(p)| = \begin{cases} 1, & \textit{if } p = 0; \\ (n-2) + \binom{n}{p} + (n-1)\binom{n-2}{p-1}, & \textit{if } 1 \leq p \leq n-1. \end{cases}$$

Proof. If $p = 0$, the result is trivial. Now, if $p \in \{1, \lceil \frac{n}{2} \rceil + 1, \ldots, n-1\}$, we see that

$$|W(p)| = |E(J_p^*)| + |O\Big(\lceil \frac{n}{2} \rceil\Big)| + |\{\delta_i,\, \delta_{n-i+1} : 2 \leq i \leq \lceil \frac{n}{2} \rceil - 1\}|.$$

Thus, by Lemma 21, it follows that

$$\begin{aligned} |W(p)| &= 2\Big(\lceil \frac{n}{2} \rceil - 2\Big) + \Big(n - 2\Big(\lceil \frac{n}{2} \rceil\Big) + 2\Big) + \binom{n}{p} + (n-1)\binom{n-2}{p-1} \\ &= (n-2) + \binom{n}{p} + (n-1)\binom{n-2}{p-1}. \end{aligned}$$

Similarly, if $2 \leq p \leq \lceil \frac{n}{2} \rceil$, then

$$|W(p)| = |G(p)| + |\{\delta_i,\ \delta_{n-i+1} : 2 \leq i \leq p-1\}|.$$

Thus, by Lemma 21, we see that

$$\begin{aligned} |W(p)| &= 2(p-2) + (n - 2p + 2) + \binom{n}{p} + (n-1)\binom{n-2}{p-1} \\ &= (n-2) + \binom{n}{p} + (n-1)\binom{n-2}{p-1}. \end{aligned}$$

The result now follows. □

Therefore, we have the following result.

Theorem 13. *Let $I(n,p)$ be as defined in (4). Then,*

$$\textit{rank } I(n,p) = \begin{cases} 1, & \textit{if } p = 0; \\ (n-2) + \binom{n}{p} + (n-1)\binom{n-2}{p-1}, & \textit{if } 1 \leq p \leq n-1. \end{cases}$$

Proof. If $p = 0$, the result is trivial. Now, observe that by Lemma 23, $\langle J_p^* \cup \{\delta_i,\ \delta_{n-i+1} : 2 \leq i \leq p-1\}\rangle = I(n,p)$ for all $1 \leq p \leq n-1$. Notice that if $2 \leq p \leq \lceil \frac{n}{2} \rceil$, then $\langle G(p) \cup \{\delta_i,\ \delta_{n-i+1} : 2 \leq i \leq p-1\}\rangle = \langle J_p^* \cup \{\delta_i,\ \delta_{n-i+1} : 2 \leq i \leq p-1\}\rangle$; and if $\lceil \frac{n}{2} \rceil + 1 \leq p \leq n-1$, then $\langle E(J_p^*) \cup O(\lceil \frac{n}{2} \rceil) \cup \{\delta_i,\ \delta_{n-i+1} : 2 \leq i \leq \lceil \frac{n}{2} \rceil - 1\}\rangle = \langle J_p^* \cup O(\lceil \frac{n}{2} \rceil) \cup \{\delta_i,\ \delta_{n-i+1} : 2 \leq i \leq \lceil \frac{n}{2} \rceil - 1\}\rangle$. In either case, the result follows from Lemmas 24 and 25. □

We now present the following result.

Theorem 14. *Let $\mathcal{DORI}_n$ be as defined in (3). Then, the rank $\mathcal{DORI}_n$ $= 3n - 2$.*

Proof. Notice that for any $\rho \in W(n-1)$, $\rho \mathrm{id}_{[n]} = \mathrm{id}_{[n]}\rho = \rho$. This means that the product of identity element and any other element of $W(n-1)$ does not generate any new element apart from elements in $W(n-1)$. Notice also that $\langle W(n-1)\rangle = I(n, n-1)$ and $I(n, n-1) \cup \{\mathrm{id}_{[n]}\} = \mathcal{DORI}_n$. Since $I(n, n-1)$ and $\{\mathrm{id}_{[n]}\}$ are disjoint, it follows that

$$\text{rank } \mathcal{DORI}_n = \text{ rank } I(n, n-1) + \text{ rank } |\{\mathrm{id}_{[n]}\}|.$$

Thus, using Theorem 13 we see that

$$\begin{aligned}\text{rank } \mathcal{DORI}_n &= (n-2) + \binom{n}{n-1} + (n-1)\binom{n-2}{n-2} + 1 \\ &= 3n-2,\end{aligned}$$

as required. □

3. The maximal subsemigroups of the Rees quotient $RQ_n(p)$, the two-sided ideal $I(n, p)$ and the monoid $DORI_n$

A subsemigroup $M \subseteq S$ is called *maximal* provided that $M \neq S$ and for any subsemigroup $X \subseteq S$, the inclusion $M \subseteq X$ implies $M = X$ or $X = S$. That is, a proper non-empty subsemigroup M of a semigroup S, is maximal if $M \subseteq T \subseteq S$ for some subsemigroup T of S, we have $M = T$ or $T = S$ [6, 10]. In other words, a proper subsemigroup of a semigroup S is considered maximal if it is not contained in any other proper subsemigroup of S. We would like to highlight [16, Table 1], which presents a list of different transformation semigroups along with the number of their maximal subsemigroups. An element $a \in S$ is said to be *indecomposable* if there are no elements $b, c \in S \setminus \{a\}$ such that $a = bc$. Now let Base(S) denote the collection of all indecomposable elements of S, usually known as *base* of the semigroup S, i.e.,

$$\mathrm{Base}(S) = S \setminus S^2 = \{x \in S | \, \forall y, z \in S : x \neq yz\}.$$

We now prove a more general result.

Proposition 1. *Let S be a semigroup and Base(S) be the base of S. Then for any $a \in$ Base(S), $S \setminus \{a\}$ is a maximal subsemigroup of S.*

Proof. Let $x, y \in S \setminus \{a\}$, where $a \in \mathrm{Base}(S)$. Thus, $xy \neq a$ and so $xy \in S \setminus \{a\}$, which implies that $S \setminus \{a\}$ is a subsemigroup of S. Now to show that $S \setminus \{a\}$ is maximal, let M be any subsemigroup of S containing $S \setminus \{a\}$, i.e.,

$$S \setminus \{a\} \subseteq M \subseteq S.$$

(i) If $a \in M$ then it follows easily that $M = S$, which will implies that $S \setminus \{a\}$ is maximal.
(ii) Now suppose $a \notin M$, then $M \setminus \{a\} = M$ and so,

$$S \setminus \{a\} \subseteq M \setminus \{a\} \subseteq S.$$

Notice that $M \subseteq S$ implies $M \setminus \{a\} \subseteq S \setminus \{a\}$. Thus,

$$S \setminus \{a\} \subseteq M \setminus \{a\} \subseteq S \setminus \{a\},$$

which implies that $S \setminus \{a\} = M \setminus \{a\} = M$, i.e., $S \setminus \{a\} = M$. Hence $S \setminus \{a\}$ is maximal. □

We start our findings by first referring back to Lemmas 20 and 24, which state that the sets $G(p)$ and $W(p)$ are the minimum generating sets of $RQ_n(p)$ and the two-sided ideal $I(n,p)$. Moreover, in particular, the set $W(n-1) \cup \{\mathrm{id}_{[n]}\}$ is the minimum generating set of $\mathcal{DORI}_n$. We now present the following lemma.

Lemma 26. *An element $\rho \in RQ_n(p)$ is indecomposable if and only if $\rho \in G(p)$.*

Proof. Let $\rho \in RQ_n(p)$ be an indecomposable element. Clearly, ρ must be included in any generating set of $RQ_n(p)$ because it cannot be generated by any other elements, and so $\rho \in G(p)$. Conversely, suppose $\rho \in G(p)$. Notice that $G(p)$ is the minimum generating set of $RQ_n(p)$ by Lemma 20, so that for any $\sigma, \delta \in G(p) \setminus \{\rho\}$, we either have $\sigma\delta = \sigma$ or $\sigma\delta = \delta$, and so $\sigma\delta \neq \rho$. Hence ρ is not decomposable, as required. □

The following result now follows.

Corollary 7. *An element $\rho \in I(n,p)$ is indecomposable if and only if $\rho \in W(p)$. In particular, an element $\rho \in \mathcal{DORI}_n$ is indecomposable if and only if $\rho \in W(n-1) \cup \{id_{[n]}\}$.*

Recall also that in the semigroup $\mathcal{DORI}_n$, the set $W(n-1)$ consists of the essential elements, the idempotent elements of J^*_{n-1}, the convex vital elements of height $\lceil \frac{n}{2} \rceil$, and all the extreme convex vital elements of height below $\lceil \frac{n}{2} \rceil$. We now state and prove the following result.

Theorem 15. *Let $RQ_n(p)$ be as defined in (5). Then,*

(a) For $p \in \{1, \lceil \frac{n}{2} \rceil + 1, \ldots, n\}$, a subsemigroup M of $RQ_n(p)$ is maximal if and only if M belongs to one of the following two types:

(i) $M_\epsilon = RQ_n(p) \setminus \{\epsilon\}$, where $\epsilon \in E(RQ_n(p))$;

(ii) $M_\varepsilon = RQ_n(p) \setminus \{\varepsilon\}$, where $\varepsilon \in M(p)$.

(b) For $p \in \{2, \ldots, \lceil \frac{n}{2} \rceil\}$, a subsemigroup M of $RQ_n(p)$ is maximal if and only if M belongs to one of the following three types:

(i) $M_\epsilon = RQ_n(p) \setminus \{\epsilon\}$, where $\epsilon \in E(RQ_n(p))$;

(ii) $M_\varepsilon = RQ_n(p) \setminus \{\varepsilon\}$, where $\varepsilon \in M(p)$;

(iii) $M_\delta = RQ_n(p) \setminus \{\delta\}$, where $\delta \in O(p)$.

Proof. **(a)** If $p \in \{1, \lceil \frac{n}{2} \rceil + 1, \ldots, n\}$, then by Proposition 1 the result follows from the fact that $G(p) = E(RQ_n(p)) \cup M(p)$ is the minimum generating set by Lemma 20, consisting of all the indecomposable elements of $RQ_n(p)$ by Lemma 26.
(b) Now, if $p \in \{2, \ldots, \lceil \frac{n}{2} \rceil\}$, then the result follows from the fact that $G(p) = E(RQ_n(p)) \cup M(p) \cup O(p)$ is the minimum generating set by Lemma 20 consisting of all the indecomposable elements of $RQ_n(p)$ by Lemma 26. □

Remark 11. Note that if the generating set is only minimal, the statement in Lemma 26 and Corollary 7 may not be true. To see this, consider the cyclic group $C_{12} = \langle x : x^{12} = e \rangle$. Then $\langle x^3, x^4 \rangle = C_{12}$. However, $\langle \{x^3, x^4\} \setminus \{x^3\} \rangle = \langle x^4 \rangle = C_3 \subset C_6 \subset C_{12}$, is not maximal.

We now have the following corollary.

Corollary 8. *If $p \in \{1, \lceil \frac{n}{2} \rceil + 1, \ldots, n\}$, then the semigroup $RQ_n(p)$ contains exactly $\binom{n}{p} + (n-1)\binom{n-2}{p-1}$ maximal subsemigroups; and if $p \in \{2, \ldots, \lceil \frac{n}{2} \rceil\}$ then the semigroup $RQ_n(p)$ contains exactly $\binom{n}{p} + (n-1)\binom{n-2}{p-1} + n - 2p + 2$ maximal subsemigroups.*

Proof. The result follows from counting all the maximal subsemigroups as stated in Theorem 15. □

The next result characterizes the maximal subsemigroups of the two-side ideal $I(n, p)$.

Theorem 16. *Let $I(n, p)$ be as defined in (4). Then,*

(a) *For* $p \in \{1, \lceil \frac{n}{2} \rceil + 1, \ldots, n\}$, *a subsemigroup* M *of* $I(n,p)$ *is maximal if and only if* M *belongs to one of the following four types:*

(i) $M_\epsilon = I(n,p) \setminus \{\epsilon\}$, *where* $\epsilon \in E(J_p^*)$;

(ii) $M_\varepsilon = I(n,p) \setminus \{\varepsilon\}$, *where* $\varepsilon \in M(p)$;

(iii) $M_\delta = I(n,p) \setminus \{\delta\}$, *where* $\delta \in O(\lceil \frac{n}{2} \rceil)$;

(iv) $M_{\delta'} = I(n,p) \setminus \{\delta'\}$, *where* $\delta' \in \{\delta_i, \delta_{n-i+1} : 2 \leq i \leq \lceil \frac{n}{2} \rceil - 1\}$.

(b) *For* $p \in \{2, \ldots, \lceil \frac{n}{2} \rceil\}$, *a subsemigroup* M *of* $I(n,p)$ *is maximal if and only if* M *belongs to one of the following four types:*

(i) $M_\epsilon = I(n,p) \setminus \{\epsilon\}$ *where* $\epsilon \in E(J_p^*)$;

(ii) $M_\varepsilon = I(n,p) \setminus \{\varepsilon\}$, *where* $\varepsilon \in M(p)$;

(iii) $M_\delta = I(n,p) \setminus \{\delta\}$, *where* $\delta \in O(p)$;

(iv) $M_{\delta'} = I(n,p) \setminus \{\delta'\}$, *where* $\delta' \in \{\delta_i, \delta_{n-i+1} : 2 \leq i \leq p-1\}$.

Proof. **(a)** If $p \in \{1, \lceil \frac{n}{2} \rceil + 1, \ldots, n-1\}$, the result follows from the fact that $W(p) = G(p) \cup O(\lceil \frac{n}{2} \rceil) \cup \{\delta_i,\ \delta_{n-i+1} : 2 \leq i \leq \lceil \frac{n}{2} \rceil - 1\}$ is the minimum generating set by Lemma 24, consisting of all the indecomposable elements of $I(n,p)$ by Corollary 7.
(b) Now, if $p \in \{2, \ldots, \lceil \frac{n}{2} \rceil\}$, then the result also follows from the fact that $W(p) = G(p) \cup \{\delta_i,\ \delta_{n-i+1} : 2 \leq i \leq p-1\}$ is the minimum generating set by Lemma 24, consisting of all the indecomposable elements of $I(n,p)$ by Corollary 7. □

We now have the following corollary.

Corollary 9. *For* $1 \leq p \leq n-1$, *the semigroup* $I(n,p)$ *contains exactly* $(n-2) + \binom{n}{p} + (n-1)\binom{n-2}{p-1}$ *maximal subsemigroups.*

Proof. The result follows from counting all the maximal subsemigroups as stated in Theorem 16. □

In conclusion, we wrap up the paper with the following results without proof.

Theorem 17. *Let* $\mathcal{DORI}_n$ *be as defined in* (3). *Then, a subsemigroup* M *of* $\mathcal{DORI}_n$ *is maximal if and only if* M *belongs to one of the following five types:*

(i) $M_{id_{[n]}} = \mathcal{DORI}_n \setminus \{id_{[n]}\}$;

(ii) $M_\epsilon = \mathcal{DORI}_n \setminus \{\epsilon\}$, *where* $\epsilon \in E(J^*_{n-1})$;

(iii) $M_\varepsilon = \mathcal{DORI}_n \setminus \{\varepsilon\}$, *where* $\varepsilon \in M(p)$;

(iv) $M_\delta = \mathcal{DORI}_n \setminus \{\delta\}$, *where* $\delta \in O(\lceil \frac{n}{2} \rceil)$;

(v) $M_{\delta'} = \mathcal{DORI}_n \setminus \{\delta'\}$, *where* $\delta' \in \{\delta_i, \delta_{n-i+1} : 2 \leq i \leq \lceil \frac{n}{2} \rceil - 1\}$.

Corollary 10. *The monoid* $\mathcal{DORI}_n$ *contains exactly* $3n-2$ *maximal subsemigroups.*

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