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par-Functions of square matrices

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Abstract. New functions of square matrices are constructed, and some of their properties and connections with Hessenberg matrices are investigated.

Introduction

Matrix analysis helps to generalize and unify, systematize, and classify various mathematical objects. At the same time, it serves as a powerful analytical tool for mathematical research. Therefore, its further development is a relevant task. The main functions of matrices are determinants and permanents. Pfaffians, which Cayley used to study the properties of skew-symmetric matrices [1], have had a somewhat lesser impact on mathematics.

This paper discusses multilinear functions of square matrices that are constructed based on the following three conditions:

- Each term of the multilinear polynomial function of the matrix is a product of elements from a transversal of the matrix, meaning a product of elements taken one from each row and column.
- Each element of the matrix affects the value of its function.
- Half of the terms in the function's polynomial appear with a positive sign, while the other half appear with a negative sign.

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Since the determinant and permanent of a square matrix are constructed based on the set S_n, which consists of all permutations of the first n positive integers, it is natural to expect that a new multilinear function of a matrix, built on a subset of S_n, will have some connection with determinants and permanents.

The goal of this paper is to construct new functions of square matrices related to ordered partitions of a natural number into natural summands and linear recursions. We call these functions par−functions, or partitioners. They have a structure similar to the determinant functions of Hessenberg matrices [2], [3].

1. Construction of *par*−Functions of Square Matrices

Let a matrix be given by

$$A_n = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}_n , \tag{1}$$

where its elements belong to some numerical field K.

We construct a bijection between the elements $(n_1, n_2, \ldots, n_r) \vdash n$ of the set of ordered partitions $C(n,+)$ and the elements $(i_1, i_2, \ldots, i_n)_r$ (see [4]) of a certain subset $CS_n \subset S_n$ of all permutations of the first n natural numbers according to the following rule:

$$(n_1, n_2, \ldots, n_r) \vdash n \longleftrightarrow$$

$$(n_1, n_1 - 1, \ldots, 1; n_1 + n_2, n_1 + n_2 - 1, \ldots, n_1 + 1; \ldots; n_1 + \ldots + n_r, \tag{2}$$

$$n_1 + \ldots + n_r - 1, \ldots, n_1 + \ldots + n_{r-1} + 1).$$

In this bijection, the first component n_1 of the partition of n corresponds to the first n_1 elements of the permutation. These elements are separated from the rest of the permutation elements by a semicolon. The second component corresponds to the next n_2 elements of the permutation, which are also separated by a semicolon, and so on.

For example, the ordered partition $(1, 3, 1, 2, 1)$ of the natural number 8 corresponds to the permutation $(1; 4, 3, 2; 5; 7, 6; 8)$.

It is known that

$$|C(n,+)| = |CS_n| = 2^{n-1}. \tag{3}$$

Here is a table showing the correspondence between all partitions in the set $C(4,+)$ and their corresponding permutations in the set CS_4, constructed according to the correspondence (2):

Partition α		*Permutation* (i_1, i_2, i_3, i_4)
(1,1,1,1)	$\leftrightarrow$	$(1; 2; 3; 4)$
$(2, 1, 1)$	$\leftrightarrow$	$(2, 1; 3; 4)$
$(1, 2, 1)$	$\leftrightarrow$	$(1; 3, 2; 4)$
$(3, 1)$	$\leftrightarrow$	$(3, 2, 1; 4)$
$(1, 1, 2)$	$\leftrightarrow$	$(1; 2; 4, 3)$
$(2, 2)$	$\leftrightarrow$	$(2, 1; 4, 3)$
$(1, 3)$	$\leftrightarrow$	$(1; 4, 3, 2)$
(4)	$\leftrightarrow$	$(4, 3, 2, 1)$

From permutations $(i_1, i_2, \ldots, i_n) \in CS_n$, we transition to the corresponding substitutions and then to transversals on square matrices of order n.

For example, the ordered partition $(1, 2, 1)$ of the natural number 4 corresponds to the transversal

$$\begin{pmatrix} \bullet & \circ & \circ & \circ \\ \circ & \circ & \bullet & \circ \\ \circ & \bullet & \circ & \circ \\ \circ & \circ & \circ & \bullet \end{pmatrix}.$$

Here, the elements of the transversal are marked with black circles.

To construct a transversal of a matrix corresponding to an ordered partition

$$(n_1, n_2, \ldots, n_r) \vdash n,$$

we select the first superdiagonal parallel to the secondary diagonal that contains n_1 elements and move to a new matrix without the first n_1 rows and columns. In this new matrix, we select n_2 elements from the superdiagonal, and so on.

Definition 1.1. *The par−function of the square matrix* (1) *is defined as the multilinear polynomial*

$$par(A_n) = \sum_{r=1}^{n} \sum_{(i_1,i_2,\ldots,i_n)_r \in CS_n} (-1)^{n-r} a_{1i_1} a_{2i_2} \cdots a_{ni_n}. \tag{4}$$

Note that due to (3), the multilinear polynomial of the *par*−function of an n-th order matrix contains 2^{n-1} terms. The first five values of the *par*−functions for matrices $A_i,\ i = 1, 2, 3, 4, 5$ are given below:

$$\mathrm{par}(A_1) = a_{11};$$

$$\mathrm{par}(A_2) = a_{11}a_{22} - a_{12}a_{21};$$

$$\mathrm{par}(A_3) = a_{11}a_{22}a_{33} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32} + a_{13}a_{22}a_{31};$$

$$\mathrm{par}(A_4) = a_{11}a_{22}a_{33}a_{44} - a_{12}a_{21}a_{33}a_{44} - a_{11}a_{23}a_{32}a_{44}+$$

$$a_{13}a_{22}a_{31}a_{44} - a_{11}a_{22}a_{34}a_{43} + a_{12}a_{21}a_{34}a_{43} + a_{11}a_{24}a_{33}a_{42} - a_{14}a_{23}a_{32}a_{41};$$

$$\mathrm{par}(A_5) = a_{11}a_{22}a_{33}a_{44}a_{55} - a_{11}a_{22}a_{33}a_{45}a_{54} - a_{11}a_{22}a_{34}a_{43}a_{55}+$$

$$a_{11}a_{22}a_{35}a_{44}a_{53} - a_{11}a_{23}a_{32}a_{44}a_{55} + a_{11}a_{23}a_{32}a_{45}a_{54}+$$

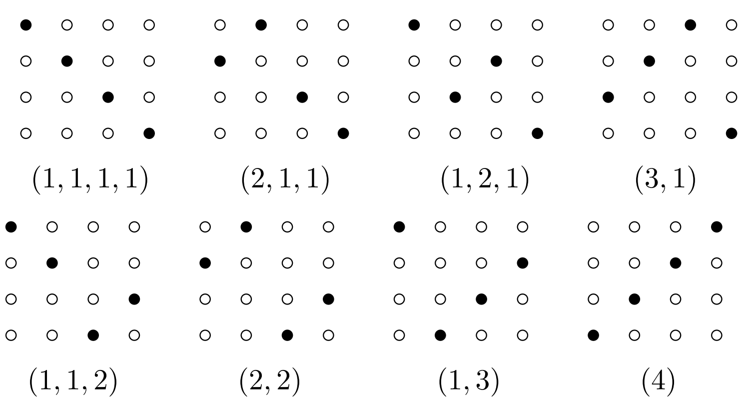
$$a_{11}a_{24}a_{33}a_{42}a_{55} - a_{11}a_{25}a_{34}a_{43}a_{52} - a_{12}a_{21}a_{33}a_{44}a_{55}+$$

$$a_{12}a_{21}a_{33}a_{45}a_{54} + a_{12}a_{21}a_{34}a_{43}a_{55} - a_{12}a_{21}a_{35}a_{44}a_{53}+$$

$$a_{13}a_{22}a_{31}a_{44}a_{55} - a_{13}a_{22}a_{31}a_{45}a_{54} - a_{14}a_{23}a_{32}a_{41}a_{55}+$$

$$a_{15}a_{24}a_{33}a_{42}a_{51}.$$

Example 1.1. We illustrate the one-to-one correspondence between all ordered partitions of the number 4 and the terms of the *par*-function of a fourth-order matrix using the following schemes:



Img. 1.1.

Theorem 1.1. *The following identity holds:*

$$\operatorname{par}(A_n) = \sum_{i=1}^{n} \prod_{j=0}^{i-1} (-1)^{i-1} a_{n-j,n-i+j+1} \operatorname{par}(A_{n-i}), \tag{5}$$

where $p(A_0) = 1$ *and* $p(A_{<0}) = 0$.

Proof. An ordered partition of a natural number n into natural summands can be represented as

$$n = (n-i) + i, \quad i = 1, 2, \ldots, n,$$

that is, each ordered partition can be written as an ordered partition of $(n-i)$ with an additional summand i. Thus, there are exactly $c(n-i)$ ordered partitions of n with the last summand i. Here, we assume that $c(0) = 1$.

We associate the last summand i of the ordered partition of n with the product of all i elements lying on a diagonal parallel to the secondary diagonal. Thus, the i elements of the transversals are uniquely determined. The remaining $(n-i)$ elements of the transversals are selected according to the ordered partitions of $(n-i)$ in the first $(n-i)$ rows and the first $(n-i)$ columns of the matrix.

By the definition of the partitioner (1.1), the sign of each term in its multilinear polynomial depends on the matrix order n and the number of components in the ordered partition r, and it is determined by the factor $(-1)^{n-r}$. Previously, we considered the partition $(n-i) + i \vdash n$, where the component i contributes the factor $(-1)^{i-1}$ to the sign of the r-partition. The factor $(-1)^{n-i-(r-1)}$ corresponding to the component $(n-i)$ is included in the partitioner $\operatorname{par} A_{n-i}$. Thus, the equality (5) holds. □

We now write the values of the decompositions of the par-functions of the matrices (A_n) for $n = 1, 2, 3, 4, 5$:

$$\operatorname{par}(A_1) = a_{11};$$

$$\operatorname{par}(A_2) = a_{22}\operatorname{par}(A_1) - a_{21}a_{12};$$

$$\operatorname{par}(A_3) = a_{33}\operatorname{par}(A_2) - a_{32}a_{23}\operatorname{par}(A_1) + a_{31}a_{22}a_{13};$$

$$\operatorname{par}(A_4) = a_{44}\operatorname{par}(A_3) - a_{43}a_{34}\operatorname{par}(A_2) + a_{42}a_{33}a_{24}\operatorname{par}(A_1) - a_{41}a_{32}a_{23}a_{14};$$

$$\operatorname{par}(A_5) = a_{55}\operatorname{par}(A_4) - a_{54}a_{45}\operatorname{par}(A_3) + a_{53}a_{44}a_{35}\operatorname{par}(A_2) -$$

$$-a_{52}a_{43}a_{34}a_{25}\mathrm{par}(A_1) + a_{51}a_{42}a_{33}a_{24}a_{15}.$$

We establish the connection between *par*-functions of square matrices and Hessenberg matrices.

Theorem 1.2. *The following identity holds for all elements a_{ij}, $i, j = 1, 2, \ldots, n$, belonging to some numerical field:*

$$det(H_n) =$$

$$\begin{vmatrix} h_{11} & 1 & 0 & \cdots & 0 & 0 \\ h_{21} & h_{22} & 1 & \cdots & 0 & 0 \\ h_{31} & h_{32} & h_{33} & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ h_{n-1,1} & h_{n-1,2} & h_{n-1,3} & \cdots & h_{n-1,n-1} & 1 \\ h_{n1} & h_{n2} & h_{n3} & \cdots & h_{n,n-1} & h_{nn} \end{vmatrix}$$

$$= \mathrm{par}\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}, \tag{6}$$

where

$$h_{ij} = \prod_{k=0}^{i-j} a_{i-k,j+k}. \tag{7}$$

Proof. For *par*-functions of matrices, according to Theorem 1.1, the recursion (5) holds. For Hessenberg matrices, a similar recursion holds:

$$\det(H_n) = h_{nn}\det(H_{n-1}) - h_{n,n-1}\det(H_{n-2}) + \ldots + (-1)^{n-1}h_{n1}\det(H_0),$$

where $\det(H_0) = 1$. Therefore, when equality (7) holds, the determinant values of the Hessenberg matrix coincide with the values of the par-functions of the corresponding matrices. □

Alongside par-functions of square matrices, we can also consider par$^+$-functions of matrices, where all 2^{n-1} terms are positive, i.e.,

$$\mathrm{par}^+(A_n) = \sum_{r=1}^{n} \sum_{(i_1,i_2,\ldots,i_n)_r \in CS_n} a_{1i_1}a_{2i_2}\cdots a_{ni_n}.$$

For *par*$^+$-functions, the following recurrence relation holds:

$$\mathrm{par}^+(A_n) = \sum_{i=1}^{n}\prod_{j=0}^{i-1} a_{n-j,n-i+j+1}\mathrm{par}^+(A_{n-i}),$$

where $\text{par}^+(A_0) = 1,\ \text{par}^+(A_{<0}) = 0$.

The validity of this recurrence relation is proved similarly to the proof of the recurrence relation in Theorem 1.1, the equality (5).

For the classical determinant and permanent functions of a square matrix, an important problem is Polya's problem (see [5, 6]) about a transformation of the matrix that allows computing the permanent by calculating the determinant of a transformed matrix. For partitioners, such a matrix transformation is possible.

Theorem 1.3. *Let*

$$A_n^{'} = \begin{pmatrix} a_{11} & -a_{12} & a_{13} & \cdots & a_{1,n-1} & a_{1n} \\ a_{21} & a_{22} & -a_{23} & \cdots & a_{1,n-1} & a_{1n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n-1,1} & a_{n-1,2} & a_{n-1,3} & \cdots & a_{n-1,n-1} & -a_{n-1,n} \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{n,n-1} & a_{nn} \end{pmatrix}$$

and

$$A_n = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1,n-1} & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{1,n-1} & a_{1n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n-1,1} & a_{n-1,2} & a_{n-1,3} & \cdots & a_{n-1,n-1} & a_{n-1,n} \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{n,n-1} & a_{nn} \end{pmatrix}.$$

Then, the following identity holds:

$$\text{par}(A_n^{'}) = \text{par}^+(A_n).$$

Proof. By Theorem 1.1, the equality (5) holds. We apply this theorem to the expansion of the partitioner of the matrix $A_n^{'}$. In this matrix, all elements $a_{i,i+1}$, $i = 1, 2, \ldots, n-1$ have a negative sign. These elements have the property that the difference between their column and row indices is equal to one. The coefficient of a general term in the expansion of matrix $A_n^{'}$ in terms of the elements of the last row has the form:

$$(-1)^{n-j} \prod_{r=0}^{n-j} a_{n-r,j+r}.$$

We analyze under which conditions a negative element of matrix $A_n^{'}$ appears in the product

$$\prod_{r=0}^{n-j} a_{n-r,j+r}.$$

The difference between the column and row indices of elements in this product is $(j-n)+2r$, and the parity of this difference depends on the parity of $(j-n)$. Thus, a negative element of matrix $A_n^{'}$ will be a factor of this product if and only if $(j-n)$ takes an odd value, which proves the theorem. $\square$

Assertion 1.1. *Let the elements $a_{i,i+1}$, $i=1,2,\ldots,n-1$ in the matrix (1) be zeros, then the following equality holds:*

$$\operatorname{par}(A_n) = \operatorname{par}^+(A_n). \tag{8}$$

Proof. In the expansion of the par-function of the given matrix by the elements of the last row, it is evident that only terms with a positive sign remain on both sides of the equality (see the proof of Theorem 1.3). $\square$

Example 1.2. *Let the given matrix be*

$$A_n = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\ 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\ 0 & 1 & 1 & \cdots & 1 & 1 & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 & 1 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 1 \end{pmatrix}_n$$

We compute $par^+(A_n)$.

Expanding the par^+-function of this matrix by the elements of the last row:

$$par^+(A_n) = par^+(A_{n-1}) + par^+(A_{n-2}).$$

However,

$$par^+(A_1) = 1 = F_2, \quad par^+(A_2) = 2 = F_3.$$

Thus,

$$par^+(A_n) = F_{n+1},$$

where F_n is the n-th Fibonacci number.

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