© Algebra and Discrete Mathematics
Volume **39** (2025). Number 1, pp. 110–129
DOI:10.12958/adm2385

RESEARCH ARTICLE

On the structure of the algebra of derivation of some non-nilpotent Leibniz algebras

Leonid A. Kurdachenko, Mykola M. Semko, and Igor Ya. Subbotin

Abstract. We investigate the algebras of derivations of certain non-nilpotent Leibniz algebras of low dimensions. Our focus is on the structural properties and classifications of these derivations, providing insights into their algebraic behaviors.

Introduction

Let V be a vector space over a field F. Denote by $End_F(V)$ the set of all linear transformations of V. Then $End_F(V)$ is an associative algebra by the operations $+$ and $\circ$. As usual, $End_F(V)$ is a Lie algebra by the operations $+$ and $[,]$ where

$$[f, g] = f \circ g - g \circ f$$

for all $f, g \in End_F(V)$.

Let L be an algebra over a field F with the operations $+$ and $[,]$. Recall that a linear transformation f of an algebra L is called a *derivation* if

$$f([a, b]) = [f(a), b] + [a, f(b)] \text{ for all } a, b \in L.$$

The first author is very grateful to the Conselleria d'Innovació, Universitats, Ciència i Societat Digital of the Generalitat (Valencian Community, Spain) and the Universitat de València for their financial support to host researchers affected by the war in Ukraine in research centres of the Valencian Community (covered by the "Programa Operativo 2014-2020" of the European Union fund FEDER REACT-EU).

2020 Mathematics Subject Classification: *17A32, 17A60, 17A99.*

Key words and phrases: *Leibniz algebra, Lie algebra, derivation, endomorphism.*

Derivations play a very important role in studying the structure of many types of non-associative algebras. In particular, such is especially true for Lie and Leibniz algebras.

Let L be an algebra over a field F with the binary operations $+$ and $[,]$. Then L is called a *left Leibniz algebra* if it satisfies the left Leibniz identity,

$$[[a, b], c] = [a, [b, c]] - [b, [a, c]],$$

for all $a, b, c \in L$. We will also use another form of this identity:

$$[a, [b, c]] = [[a, b], c] + [b, [a, c]].$$

Derivations play a crucial role in studying the structure of many types of non-associative algebras. This is particularly true for Lie and Leibniz algebras, where derivations help unveil their structural and functional properties.

Leibniz algebras first appeared in the paper of A. Blokh [2], but the term "Leibniz algebra" appears in the book of J.-L. Loday [15] and his article [16]. Some of the results of this theory were presented in the books [1,6].

Let $Der(L)$ be the subset of all derivations of a Leibniz algebra L. It can be proven that $Der(L)$ is a subalgebra of the Lie algebra $End_F(L)$. $Der(L)$ is called the *algebra of derivations* of the Leibniz algebra L.

The influence on the structure of a Leibniz algebra of its algebra of derivations can be observed in the following result: If A is an ideal of a Leibniz algebra, then the factor-algebra of L by the annihilator of A is isomorphic to some subalgebra of $Der(A)$ [4, Proposition 3.2].

The structure of the algebra of derivations of finite-dimensional one-generator Leibniz algebras was described in the papers [10,17], and that of infinite-dimensional one-generator Leibniz algebras was delineated in the paper [14].

The question about the algebras of derivations of Leibniz algebras of small dimensions naturally arises. In contrast to Lie algebras, the situation with Leibniz algebras of dimension 3 is very diverse. The Leibniz algebras of dimension 3 have been described, and their most detailed description can be found in [5]. The algebras of derivations of the Leibniz algebras of dimension 3 were studied in the papers [6–9, 11, 12].

The study continues in the present work. The next step involves investigating the algebra of derivations of other Leibniz algebras. We will analyze their structure step by step, focusing on the remaining classes of Leibniz algebras.

1. Some preliminaries and remarks

We will need some general properties of an algebra of derivations of a Leibniz algebra. Here, we show some basic elementary properties of derivations that have been proven in the paper [12]. First, let us recall some definitions.

Every Leibniz algebra L has a specific ideal. Denote by $Leib(L)$ the subspace generated by the elements $[a, a]$, $a \in L$. It is possible to prove that $Leib(L)$ is an ideal of L. The ideal $Leib(L)$ is called the *Leibniz kernel* of algebra L. By the definition, factor-algebra $L/Leib(L)$ is a Lie algebra. Conversely, if K is an ideal of L such that L/K is a Lie algebra, then K includes the Leibniz kernel.

Let L be a Leibniz algebra. Define the *lower central series* of L,

$$L = \gamma_1(L) \geq \gamma_2(L) \geq \dots \gamma_\alpha(L) \geq \gamma_{\alpha+1}(L) \geq \dots \gamma_\delta(L),$$

by the following rule: $\gamma_1(L) = L$, $\gamma_2(L) = [L, L]$, recursively, $\gamma_{\alpha+1}(L) = [L, \gamma_\alpha(L)]$ for every ordinal α, and $\gamma_\lambda(L) = \bigcap_{\mu<\lambda} \gamma_\mu(L)$ for every limit ordinal λ. The last term $\gamma_\delta(L) = \gamma_\infty(L)$ is called the *lower hypocenter* of L. We have: $\gamma_\delta(L) = [L, \gamma_\delta(L)]$.

As usual, we say that a Leibniz algebra L is called *nilpotent* if a positive integer k exists, such that $\gamma_k(L) = \langle 0 \rangle$. More precisely, L is said to be *nilpotent of nilpotency class c* if $\gamma_{c+1}(L) = \langle 0 \rangle$ but $\gamma_c(L) \neq \langle 0 \rangle$.

The *left* (respectively *right*) *center* $\zeta^{\text{left}}(L)$ (respectively $\zeta^{\text{right}}(L)$) of a Leibniz algebra L is defined by the rule below:

$$\zeta^{\text{left}}(L) = \{x \in L |\ [x, y] = 0 \text{ for each element } y \in L\}$$

(respectively

$$\zeta^{\text{right}}(L) = \{x \in L |\ [y, x] = 0 \text{ for each element } y \in L\}).$$

It is not hard to prove that the left center of L is an ideal, but this is not true for the right center. Moreover, $Leib(L) \leq \zeta^{\text{left}}(L)$, so that $L/\zeta^{\text{left}}(L)$ is a Lie algebra. The right center is a subalgebra of L; the left and right centers are generally different; they may even have different dimensions (see [4]).

The center of L is defined by the rule below:

$$\zeta(L) = \{x \in L |\ [x, y] = 0 = [y, x] \text{ for each element } y \in L\}.$$

The center is an ideal of L.

Lemma 1. *Let L be a Leibniz algebra over a field F and f be a derivation of L. Then $f(\zeta^{\text{left}}(L)) \leq \zeta^{\text{left}}(L)$, $f(\zeta^{\text{right}}(L)) \leq \zeta^{\text{right}}(L)$ and $f(\zeta(L)) \leq \zeta(L)$.*

Corollary 1. *Let L be a Leibniz algebra over a field F and f be a derivation of L. Then $f(\zeta_\alpha(L)) \leq \zeta_\alpha(L)$ for every ordinal α.*

Lemma 2. *Let L be a Leibniz algebra over a field F and f be a derivation of L. Then $f([L, L]) \leq [L, L]$.*

Corollary 2. *Let L be a Leibniz algebra over a field F and f be a derivation of L. Then $f(\gamma_\alpha(L)) \leq \gamma_\alpha(L)$ for every ordinal α.*

Denote by Ξ the classic monomorphism of $End(L)$ in $M_3(F)$ (i.e., the mapping, assigning to each endomorphism its matrix concerning the basis $\{a_1, a_2, a_3\}$).

Let L be a non-nilpotent Leibniz algebra of dimension 3 having a Leibniz kernel of dimension 2. Moreover, suppose that is not generated by any of all its elements.

The fact that $L/[L, L]$ is abelian, in particular, it is a Lie algebra, implies that $Leib(L) \leq [L, L]$. On the other hand, L is soluble, so that $L \neq [L, L]$. Then the fact that $dim_F(Leib(L)) = 2$ implies that $Leib(L) = [L, L]$.

Since L is not a Lie algebra, L has an element a such that $[a, a] = c \neq 0$. In particular, it follows that $a \notin Leib(L)$, and hence $Fa \cap Fc = \langle 0 \rangle$. Since L is not one-generated, subalgebra $\langle a, c \rangle$ has dimension 2. Taking into consideration the information about the structure of a Leibniz algebra of dimension 2 (see, for example, survey [13]) we can suppose that either $[a, c] = 0$ or $[a, c] = c$. The fact that $Leib(L)$ has dimension 2 implies that $Leib(L) = Fc \oplus Fb$ for some element b. Then $[b, c] = [c, b] = 0$; in particular, $C = Fc$ is an ideal of L. Assume first that $[a, c] = 0$. In this case we obtain that $c \in \zeta(L)$.

The fact that $dim_F(Leib(L)) = 2$ implies that the factor-algebra L/C is not a Lie algebra. The inclusion $C \leq \zeta(L)$ shows that the factor-algebra L/C is not nilpotent. Taking into consideration the information about the structure of a Leibniz algebra of dimension 2 we obtain that $L/C = \langle u + C, v + C \rangle$ and $[u + C, u + C] = v + C$, $[u + C, v + C] = v + C$, $[v + C, u + C] = C = [v + C, v + C]$. Then $\langle v, C \rangle$ is an ideal of L, having dimension 2. The factor-algebra $L/\langle v, C \rangle$ has dimension 1, in particular, it is abelian. It follows that $[L, L] \leq \langle v, C \rangle$ and hence $[L, L] = \langle v, C \rangle$. In turn out it follows that $\langle v, C \rangle = Leib(L)$. In particular, $\langle v, C \rangle$ is abelian.

Then $[v,v]=[v,c]=[c,v]=0$. Also, we have $\langle v,C\rangle$ is $\zeta^{\text{left}}(L)$, so that $[v,u]=0$. The equality $[u+C,u+C]=v+C$ implies that $[u,u]=v+\rho_1 c$ for some scalar $\rho_1\in F$. An equality $[u+C,v+C]=v+C$ implies that $[u,v]=v+\rho_2 c$ for some scalar $\rho_2\in F$. We have

$$[u,[u,u]]=[u,v+\rho_1 c]=[u,v]+\rho_1[u,c]=[u,v]=v+\rho_2 c\in\langle u\rangle.$$

Since $v+\rho_1 c=[u,u]\in\langle u\rangle$, $v+\rho_1 c-(v+\rho_2 c)=(\rho_1-\rho_2)c\in\langle u\rangle$. If $\rho_1\neq\rho_2$, then we obtain that $c\in\langle u\rangle$. In turn, it follows that $v\in\langle u\rangle$, and hence $\langle u\rangle=Fu\oplus Fv\oplus Fc=L$. This contradiction shows that $\rho_1=\rho_2$.

The equality $L=Fu\oplus Fv\oplus Fc$ implies that $a=\alpha u+\beta v+\gamma c$ for some scalars $\alpha,\beta,\gamma\in F$. Then we obtain

$$\begin{aligned}c=[a,a]&=[\alpha u+\beta v+\gamma c,\alpha u+\beta v+\gamma c]=\alpha^2[u,u]+\alpha\beta[u,v]+\alpha\gamma[u,c]\\&=\alpha^2(v+\rho_1 c)+\alpha(v+\rho_1 c)=\alpha(\alpha+\beta)v+\alpha(\alpha+\beta)\rho_1 c.\end{aligned}$$

It follows that $\alpha(\alpha+\beta)=0$. But in this case, we obtain that $[a,a]=0$, and we obtain a contradiction.

This contradiction shows that $[a,c]=c$. We have $[a,b]=\kappa_1 b+\kappa_2 c$ for some scalars $\kappa_1,\kappa_2\in F$. Since $b\in Leib(L)$, $[b,a]=0$. If $\kappa_1=0$, then $[a,b]\in C$. Together with $[a,a]\in C$ it follows that A/C is abelian, and we obtain a contradiction. This contradiction shows that $\kappa_1\neq 0$. Without loss of generality, we can assume that $\kappa_1=1$ (otherwise, instead of b we will consider $\kappa_1^{-1}b$).

Suppose first that $\kappa_2\neq 0$, so that $[a,a]=[a,c]=c$, and $[a,b]=b+\kappa_2 c$. Let $x=\alpha a+\beta b+\gamma c$ be an arbitrary element of L such that $x\notin Leib(L)$, $\alpha,\beta,\gamma\in F$. It follows that $\alpha\neq 0$. If we suppose that $\beta=0$, then $x\in\langle a\rangle$ and, in this case, $\langle x\rangle$ is a proper subalgebra. Therefore, we will assume that $\beta\neq 0$. We have

$$\begin{aligned}[x,x]&=[\alpha a+\beta b+\gamma c,\alpha a+\beta b+\gamma c]\\&=\alpha^2[a,a]+\alpha\beta[a,b]+\alpha\gamma[a,c]\\&=\alpha^2 c+\alpha\beta(b+\kappa_2 c)+\alpha\gamma c=\alpha\beta b+\alpha(\alpha+\beta\kappa_2+\gamma)c.\end{aligned}$$

Since $\alpha\neq 0$, α^{-1} exists. We have $\alpha^{-1}[x,x]=\beta b+(\alpha+\beta\kappa_2+\gamma)c=d\in\langle x\rangle\cap[L,L]$. Then

$$\begin{aligned}d_1=x-d&=\alpha a+\beta b+\gamma c-\beta b-(\alpha+\beta\kappa_2+\gamma)c\\&=\alpha a-(\alpha+\beta\kappa_2)c\in\langle x\rangle.\end{aligned}$$

Put $\lambda = \alpha + \beta\kappa_2$. Furthermore,

$$\begin{aligned}[x, d_1] &= [\alpha a + \beta b + \gamma c, \alpha a - \lambda c] \\ &= \alpha^2[a, a] - \alpha\lambda[a, c] \\ &= \alpha^2 c - \alpha\lambda c = \alpha(\alpha - \lambda)c = \alpha(\alpha - \alpha - \beta\kappa_2)c \\ &= -\alpha\beta\kappa_2 c.\end{aligned}$$

We note that $[x, d_1] \in \langle x \rangle \cap [L, L]$. The matrix

$$\begin{pmatrix} \beta & \alpha + \beta\kappa_2 + \gamma \\ 0 & -\alpha\beta\kappa_2 \end{pmatrix}$$

is not degenerate. It follows that $\langle x \rangle \cap [L, L]$ has dimension 2, so that $[L, L] = \langle x \rangle \cap [L, L]$. Since $x \notin [L, L]$, $L = \langle x \rangle$, and again we obtain a contradiction. This contradiction shows that $\kappa_2 = 0$. In other words, $[a, b] = b$, $[a, c] = c$. It follows that every subspace of $Leib(L) = Fb \oplus Fc$ is L-invariant. Then $F[x, x]$ is an ideal for every element $x \in L$. It follows that in this case L cannot be one-generator. Thus, we come to the following Leibniz algebra:

$$\begin{gathered} Lei_{10}(3, F) = Fa_1 \oplus Fa_2 \oplus Fa_3 \text{ where} \\ [a_1, a_1] = a_3 = [a_1, a_3], [a_1, a_2] = a_2, \\ [a_2, a_3] = [a_3, a_2] = [a_2, a_2] = [a_3, a_3] = [a_2, a_1] = [a_3, a_1] = 0. \end{gathered}$$

In this case $Leib(L) = Fa_2 \oplus Fa_3 = [L, L] = \zeta^{\text{left}}(L)$, every subspace of $Leib(L)$ is an ideal of L, L is a sum of two non-nilpotent subalgebras $A_1 = Fa_1 \oplus Fa_3$ and $A_2 = Fa_1 \oplus Fa_2$ of dimension 2. Note also that $\zeta^{\text{right}}(L) = \zeta(L)$.

Let's verify that an algebra with such defining relations is indeed a Leibniz algebra.

Let x, y, z be arbitrary elements of L,

$$\begin{aligned} x &= \xi_1 a_1 + \xi_2 a_2 + \xi_3 a_3, \\ y &= \eta_1 a_1 + \eta_2 a_2 + \eta_3 a_3, \\ z &= \sigma_1 a_1 + \sigma_2 a_2 + \sigma_3 a_3, \end{aligned}$$

where $\xi_1, \xi_2, \xi_3, \eta_1, \eta_2, \eta_3, \sigma_1, \sigma_2, \sigma_3$ are arbitrary scalars. Then

$$\begin{aligned}[x, y] &= [\xi_1 a_1 + \xi_2 a_2 + \xi_3 a_3, \eta_1 a_1 + \eta_2 a_2 + \eta_3 a_3] \\ &= \xi_1\eta_1[a_1, a_1] + \xi_1\eta_2[a_1, a_2] + \xi_1\eta_3[a_1, a_3]\end{aligned}$$

$$\begin{aligned}
&= \xi_1\eta_1 a_3 + \xi_1\eta_2 a_2 + \xi_1\eta_3 a_3 \\
&= \xi_1\eta_2 a_2 + (\xi_1\eta_1 + \xi_1\eta_3)a_3, \\
[x,z] &= \xi_1\sigma_2 a_2 + (\xi_1\sigma_1 + \xi_1\sigma_3)a_3, \\
[y,z] &= \eta_1\sigma_2 a_2 + (\eta_1\sigma_1 + \eta_1\sigma_3)a_3, \\
[x,[y,z]] &= [\xi_1 a_1 + \xi_2 a_2 + \xi_3 a_3, \eta_1\sigma_2 a_2 + (\eta_1\sigma_1 + \eta_1\sigma_3)a_3] \\
&= \xi_1\eta_1\sigma_2[a_1,a_2] + \xi_1(\eta_1\sigma_1 + \eta_1\sigma_3)[a_1,a_3] \\
&= \xi_1\eta_1\sigma_2 a_2 + \xi_1(\eta_1\sigma_1 + \eta_1\sigma_3)a_3, \\
[[x,y],z] &= [\xi_1\eta_2 a_2 + (\xi_1\eta_1 + \xi_1\eta_3)a_3, \sigma_1 a_1 + \sigma_2 a_2 + \sigma_3 a_3] = 0, \\
[y,[x,z]] &= [\eta_1 a_1 + \eta_2 a_2 + \eta_3 a_3, \xi_1\sigma_2 a_2 + (\xi_1\sigma_1 + \xi_1\sigma_3)a_3] \\
&= \eta_1\xi_1\sigma_2[a_1,a_2] + \eta_1(\xi_1\sigma_1 + \xi_1\sigma_3)[a_1,a_3] \\
&= \eta_1\xi_1\sigma_2 a_2 + \eta_1(\xi_1\sigma_1 + \xi_1\sigma_3)a_3.
\end{aligned}$$

Thus we can see that

$$[x,[y,z]] = [[x,y],z] + [y,[x,z]],$$

so that $Lei_{10}(3,F)$ is really a Leibniz algebra.

It remains to consider the algebras of derivations of a non-nilpotent Leibniz algebra of dimension 3 having Leibniz kernel of dimension 1. In this paper we consider only the case when the center of the Leibniz algebra includes the Leibniz kernel. Consider the structure of such Leibniz algebras.

Suppose now that L is a non-nilpotent Leibniz algebra of dimension 3, having a Leibniz kernel of dimension 1. Let $K = Leib(L)$, then L/K is a Lie algebra of dimension 2. Let c be an element such that $K = Fc$. Then $S = Ann_L^{\text{left}}(K)$ is an ideal of L such that L/S is isomorphic to some subalgebra of derivation of K [4, Proposition 3.2]. It follows that either $S = L$ or $dim_F(L/S) = 1$. Since $Leib(L) \le \zeta^{\text{left}}(L)$, $S = Ann_L(K)$.

Suppose that $S = L$. Then $K \le \zeta(L)$. Since L is not nilpotent, the factor-algebra L/K is non-abelian. In this case L/K has an ideal $A/K = F(a+K)$ and has an element $b+K$ such that $[a+K, b+K] = a+K$ (see, for example, the book [3, Chapter 1, § 4]). We have $[c,x] = [x,c] = 0$ for every element $x \in L$.

The equality $S = L$ implies that $K \le \zeta(L)$. If A is not a Lie algebra, then $[a,a] = \xi c \ne 0$. Clearly, without loss of generality, we may suppose that $[a,a] = c$. Let x be an element such that $x \notin A$. Since A is an ideal of L, $[a,x] = \alpha a + \nu c$ for some scalars $\alpha, \nu \in F$. Since A/K is not nilpotent, $\alpha \ne 0$. The fact that L/K is a Lie algebra implies that

$[x, a] = -\alpha a + \nu_1 c$ for some $\nu_1 \in F$. We have

$$[x, [a, x]] = [[x, a], x] + [a, [x, x]] = [[x, a], x].$$

Then we obtain

$$\alpha[x, a] = [x, \alpha a] = [x, \alpha a + \nu c] = [x, [a, x]] = [[x, a], x]$$
$$= [-\alpha a + \nu_1 c, x] = [-\alpha a, x] = -\alpha[a, x].$$

Since $\alpha \neq 0$, we obtain that $[x, a] = -[a, x]$.

The equality $[a + K, b + K] = a + K$ implies that $[a, b] = a + \kappa_1 c$ for some scalar $\kappa_1 \in F$. If $\kappa_1 = 0$, then put $a_1 = b$. If not, then put $a_1 = b - \kappa_1 a$. In this case, we have

$$[a, a_1] = [a, b - \kappa_1 a] = [a, b] - [a, \kappa_1 a] = a + \kappa_1 c - \kappa_1 c = a.$$

As proven above $[a_1, a] = -a$. Put $a_2 = a$, $a_3 = c$. Thus, we come to the following Leibniz algebra:

$$Lei_{11}(3, F) = Fa_1 \oplus Fa_2 \oplus Fa_3 \text{ where}$$
$$[a_2, a_2] = a_3, [a_2, a_1] = a_2, [a_1, a_2] = -a_2,$$
$$[a_2, a_3] = [a_3, a_2] = [a_1, a_3] = [a_3, a_1] = [a_3, a_3] = 0,$$
$$[a_1, a_1] = \gamma a_3, \gamma \in F.$$

In this case, $Leib(L) = Fa_3 = \zeta(L) = \zeta^{\text{left}}(L) = \zeta^{\text{right}}(L)$, $[L, L] = A = Fa_2 \oplus Fa_3$ is a nilpotent radical of L, moreover, $[L, L]$ is a one-generator nilpotent Leibniz subalgebra of L.

Let's verify that an algebra with such defining relations is indeed a Leibniz algebra.

Let x, y, z be arbitrary elements of L,

$$x = \xi_1 a_1 + \xi_2 a_2 + \xi_3 a_3,$$
$$y = \eta_1 a_1 + \eta_2 a_2 + \eta_3 a_3,$$
$$z = \sigma_1 a_1 + \sigma_2 a_2 + \sigma_3 a_3,$$

where $\xi_1, \xi_2, \xi_3, \eta_1, \eta_2, \eta_3, \sigma_1, \sigma_2, \sigma_3$ are arbitrary scalars. Then

$$[x, y] = [\xi_1 a_1 + \xi_2 a_2 + \xi_3 a_3, \eta_1 a_1 + \eta_2 a_2 + \eta_3 a_3]$$
$$= \xi_1\eta_1[a_1, a_1] + \xi_1\eta_2[a_1, a_2] + \xi_2\eta_1[a_2, a_1] + \xi_2\eta_2[a_2, a_2]$$
$$= \xi_1\eta_1\gamma a_3 - \xi_1\eta_2 a_2 + \xi_2\eta_1 a_2 + \xi_2\eta_2 a_3$$

$$\begin{aligned}
&= (\xi_2\eta_1 - \xi_1\eta_2)a_2 + (\xi_1\eta_1\gamma + \xi_2\eta_2)a_3, \\
[x, z] &= (\xi_2\sigma_1 - \xi_1\sigma_2)a_2 + (\xi_1\sigma_1\gamma + \xi_2\sigma_2)a_3, \\
[y, z] &= (\eta_2\sigma_1 - \eta_1\sigma_2)a_2 + (\eta_1\sigma_1\gamma + \eta_2\sigma_2)a_3, \\
[x, [y, z]] &= [\xi_1 a_1 + \xi_2 a_2 + \xi_3 a_3, (\eta_2\sigma_1 - \eta_1\sigma_2)a_2 + (\eta_1\sigma_1\gamma + \eta_2\sigma_2)a_3] \\
&= \xi_1(\eta_2\sigma_1 - \eta_1\sigma_2)[a_1, a_2] + \xi_2(\eta_2\sigma_1 - \eta_1\sigma_2)[a_2, a_2] \\
&= -\xi_1(\eta_2\sigma_1 - \eta_1\sigma_2)a_2 + \xi_2(\eta_2\sigma_1 - \eta_1\sigma_2)a_3 \\
&= \xi_1(\eta_1\sigma_2 - \eta_2\sigma_1)a_2 + \xi_2(\eta_2\sigma_1 - \eta_1\sigma_2)a_3, \\
[[x, y], z] &= [(\xi_2\eta_1 - \xi_1\eta_2)a_2 + (\xi_1\eta_1\gamma + \xi_2\eta_2)a_3, \sigma_1 a_1 + \sigma_2 a_2 + \sigma_3 a_3] \\
&= \sigma_1(\xi_2\eta_1 - \xi_1\eta_2)[a_2, a_1] + \sigma_2(\xi_2\eta_1 - \xi_1\eta_2)[a_2, a_2] \\
&= \sigma_1(\xi_2\eta_1 - \xi_1\eta_2)a_2 + \sigma_2(\xi_2\eta_1 - \xi_1\eta_2)a_3, \\
[y, [x, z]] &= [\eta_1 a_1 + \eta_2 a_2 + \eta_3 a_3, (\xi_2\sigma_1 - \xi_1\sigma_2)a_2 + (\xi_1\sigma_1\gamma + \xi_2\sigma_2)a_3] \\
&= \eta_1(\xi_2\sigma_1 - \xi_1\sigma_2)[a_1, a_2] + \eta_2(\xi_2\sigma_1 - \xi_1\sigma_2)[a_2, a_2] \\
&= -\eta_1(\xi_2\sigma_1 - \xi_1\sigma_2)a_2 + \eta_2(\xi_2\sigma_1 - \xi_1\sigma_2)a_3 \\
&= \eta_1(\xi_1\sigma_2 - \xi_2\sigma_1)a_2 + \eta_2(\xi_2\sigma_1 - \xi_1\sigma_2)a_3.
\end{aligned}$$

Thus,

$$\begin{gathered}
[[x, y], z] + [y, [x, z]] = \\
\sigma_1(\xi_2\eta_1 - \xi_1\eta_2)a_2 + \sigma_2(\xi_2\eta_1 - \xi_1\eta_2)a_3 \\
+\eta_1(\xi_1\sigma_2 - \xi_2\sigma_1)a_2 + \eta_2(\xi_2\sigma_1 - \xi_1\sigma_2)a_3 \\
= (\sigma_1\xi_2\eta_1 - \sigma_1\xi_1\eta_2 + \eta_1\xi_1\sigma_2 - \eta_1\xi_2\sigma_1)a_2 \\
+(\sigma_2\xi_2\eta_1 - \sigma_2\xi_1\eta_2 + \eta_2\xi_2\sigma_1 - \eta_2\xi_1\sigma_2)a_3 \\
= (\eta_1\xi_1\sigma_2 - \sigma_1\xi_1\eta_2)a_2 + (\sigma_2\xi_2\eta_1 - \sigma_2\xi_1\eta_2 + \eta_2\xi_2\sigma_1 - \eta_2\xi_1\sigma_2)a_3.
\end{gathered}$$

Thus, we can see that

$$[[x, y], z] + [y, [x, z]] = [x, [y, z]] + 2\xi_1\eta_2\sigma_2 a_3.$$

It follows that $Lei_{11}(3, F)$ is a Leibniz algebra if $char(F) = 2$.

Suppose now that A is a Lie algebra, then $[a, a] = 0$. It follows that A is abelian. Consider the mapping $L_b : A \to A$, defined by the rule: $L_b(y) = [b, y]$, $y \in A$. Clearly L_b is a linear mapping, $Ker(L_b) = Fc$, $Im(L_b) = [b, A]$. An isomorphism

$$[b, A] = Im(L_b) \cong A/Ker(L_b) = A/Fc$$

shows that $[b, A]$ has dimension 1. It is not hard to see that $[b, A]$ is an ideal of L. Clearly $[b, A] \cap \zeta(L) = \langle 0 \rangle$. In the factor-algebra L/K we have

$$([b, A] + K)/K = A/K.$$

Choose in the coset $a + K$ an element a_2 such that $a_2 \in [b, A]$. The equalities $[a + K, b + K] = a + K$ and $[b, A] \cap K = \langle 0 \rangle$ imply that $[a_2, b] = a_2$. As above we obtain that $[b, a_2] = -a_2$. The factor-algebra $L/[b, A]$ is nilpotent. We note that $L/[b, A]$ is not abelian, otherwise, we obtain that $Leib(L) = K \leq [b, A]$ and we come to a contradiction. It follows that $L/[b, A]$ is a nilpotent Leibniz algebra of dimension 2. Since $b \notin A$,

$$b + [b, A] \notin Leib(L/[b, A]) = (Leib(L) + [b, A])/[b, A].$$

It follows that $[b + [b, A], b + [b, A]]$ is not zero. Therefore, $[b, b] \neq 0$. Put $b = a_1$, $[b, b] = a_3$. Thus, we come to the following Leibniz algebra:

$$\begin{gathered} Lei_{12}(3, F) = Fa_1 \oplus Fa_2 \oplus Fa_3 \text{ where} \\ [a_1, a_1] = a_3, [a_1, a_2] = -a_2, [a_2, a_1] = a_2, \\ [a_2, a_2] = [a_3, a_3] = [a_3, a_2] = [a_2, a_3] = [a_3, a_2] = [a_3, a_1] = 0. \end{gathered}$$

In thus case $Leib(L) = Fa_3 = \zeta(L) = \zeta^{\text{left}}(L) = \zeta^{\text{right}}(L)$, $[L, L] = A = Fa_2 \oplus Fa_3$ is an abelian ideal, moreover, Fa_2 is also an ideal of L, $Fa_1 \oplus Fa_3$ is a nilpotent subalgebra of L, $Fa_1 \oplus Fa_2$ is a non-nilpotent Lie subalgebra of L.

Let's verify that an algebra with such defining relations is indeed a Leibniz algebra.

Let x, y, z be arbitrary elements of L,

$$\begin{aligned} x &= \xi_1 a_1 + \xi_2 a_2 + \xi_3 a_3, \\ y &= \eta_1 a_1 + \eta_2 a_2 + \eta_3 a_3, \\ z &= \sigma_1 a_1 + \sigma_2 a_2 + \sigma_3 a_3, \end{aligned}$$

where $\xi_1, \xi_2, \xi_3, \eta_1, \eta_2, \eta_3, \sigma_1, \sigma_2, \sigma_3$ are arbitrary scalars. Then

$$\begin{aligned} [x, y] &= [\xi_1 a_1 + \xi_2 a_2 + \xi_3 a_3, \eta_1 a_1 + \eta_2 a_2 + \eta_3 a_3] \\ &= \xi_1 \eta_1 [a_1, a_1] + \xi_1 \eta_2 [a_1, a_2] + \xi_2 \eta_1 [a_2, a_1] \\ &= \xi_1 \eta_1 a_3 - \xi_1 \eta_2 a_2 + \xi_2 \eta_1 a_2 = (\xi_2 \eta_1 - \xi_1 \eta_2) a_2 + \xi_1 \eta_1 a_3, \\ [x, z] &= (\xi_2 \sigma_1 - \xi_1 \sigma_2) a_2 + \xi_1 \sigma_1 a_3, \\ [y, z] &= (\eta_2 \sigma_1 - \eta_1 \sigma_2) a_2 + \eta_1 \sigma_1 a_3, \\ [x, [y, z]] &= [\xi_1 a_1 + \xi_2 a_2 + \xi_3 a_3, (\eta_2 \sigma_1 - \eta_1 \sigma_2) a_2 + \eta_1 \sigma_1 a_3] \end{aligned}$$

$$
\begin{aligned}
&= \xi_1(\eta_2\sigma_1 - \eta_1\sigma_2)[a_1, a_2] = -\xi_1(\eta_2\sigma_1 - \eta_1\sigma_2)a_2 \\
&= (\xi_1\eta_1\sigma_2 - \xi_1\eta_2\sigma_1)a_2, \\
[[x, y], z] &= [(\xi_2\eta_1 - \xi_1\eta_2)a_2 + \xi_1\eta_1 a_3, \sigma_1 a_1 + \sigma_2 a_2 + \sigma_3 a_3] \\
&= \sigma_1(\xi_2\eta_1 - \xi_1\eta_2)[a_2, a_1] = \sigma_1(\xi_2\eta_1 - \xi_1\eta_2)a_2, \\
[y, [x, z]] &= [\eta_1 a_1 + \eta_2 a_2 + \eta_3 a_3, (\xi_2\sigma_1 - \xi_1\sigma_2)a_2 + \xi_1\sigma_1 a_3] \\
&= \eta_1(\xi_2\sigma_1 - \xi_1\sigma_2)[a_1, a_2] = -\eta_1(\xi_2\sigma_1 - \xi_1\sigma_2)a_2 \\
&= \eta_1(\xi_1\sigma_2 - \xi_2\sigma_1)a_2.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
[[x, y], z] + [y, [x, z]] &= \sigma_1(\xi_2\eta_1 - \xi_1\eta_2)a_2 + \eta_1(\xi_1\sigma_2 - \xi_2\sigma_1)a_2 \\
&= (\xi_2\eta_1\sigma_1 - \xi_1\eta_2\sigma_1 + \xi_1\eta_1\sigma_2 - \xi_2\eta_1\sigma_1)a_2 \\
&= (-\xi_1\eta_2\sigma_1 + \xi_1\eta_1\sigma_2)a_2.
\end{aligned}
$$

Thus, we can see that

$$[[x, y], z] + [y, [x, z]] = [x, [y, z]].$$

It follows that $Lei_{12}(3, F)$ really is a Leibniz algebra.

2. Main results

Theorem 1. *Let D be an algebra of derivations of the Leibniz algebra $Lei_{10}(3, F)$. Then D is isomorphic to a Lie subalgebra of $M_3(F)$ consisting of the matrices of the following form:*

$$
\begin{pmatrix} 0 & 0 & 0 \\ 0 & \beta & 0 \\ \gamma & \lambda & \gamma \end{pmatrix},
$$

$\beta, \gamma, \lambda \in F$. Furthermore, D is a semidirect sum of ideal E and one dimensional subalgebra, moreover E is a direct sum of two one dimensional ideals of D.

Proof. Let $L = Lei_{10}(3, F)$ and $f \in Der(L)$. By Lemma 1, $f(\zeta(L)) \leq \zeta(L) = Fa_3$ and $f(Fa_1 \oplus Fa_3) \leq Fa_1 \oplus Fa_3$. So that

$$
\begin{aligned}
f(a_1) &= \alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3, \\
f(a_2) &= \beta_2 a_2 + \beta_3 a_3, \\
f(a_3) &= \gamma a_3,
\end{aligned}
$$

$\alpha_1, \alpha_2, \alpha_3, \beta_2, \beta_3, \gamma \in F$. Then

$$\begin{aligned} f(a_3) &= f([a_1, a_1]) = [f(a_1), a_1] + [a_1, f(a_1)] \\ &= [\alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3, a_1] + [a_1, \alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3] \\ &= \alpha_1 [a_1, a_1] + \alpha_1 [a_1, a_1] + \alpha_2 [a_1, a_2] + \alpha_3 [a_1, a_3] \\ &= 2\alpha_1 a_3 + \alpha_2 a_2 + \alpha_3 a_3 = \alpha_2 a_2 + (2\alpha_1 + \alpha_3) a_3, \\ f(a_3) &= f([a_1, a_3]) = [f(a_1), a_3] + [a_1, f(a_3)] \\ &= [\alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3, a_3] + [a_1, \gamma a_3] \\ &= \alpha_1 [a_1, a_3] + \gamma [a_1, a_3] = \alpha_1 a_3 + \gamma a_3 = (\alpha_1 + \gamma) a_3, \\ f(a_2) &= f([a_1, a_2]) = [f(a_1), a_2] + [a_1, f(a_2)] \\ &= [\alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3, a_2] + [a_1, \beta_2 a_2 + \beta_3 a_3] \\ &= \alpha_1 [a_1, a_2] + \beta_2 [a_1, a_2] + \beta_3 [a_1, a_3] \\ &= \alpha_1 a_2 + \beta_2 a_2 + \beta_3 a_3 = (\alpha_1 + \beta_2) a_2 + \beta_3 a_3. \end{aligned}$$

Thus, we obtain

$$\alpha_2 a_2 + (2\alpha_1 + \alpha_3) a_3 = (\alpha_1 + \gamma) a_3 = \gamma a_3,$$
$$(\alpha_1 + \beta_2) a_2 + \beta_3 a_3 = \beta_2 a_2 + \beta_3 a_3.$$

It follows that

$$\alpha_2 = 0, 2\alpha_1 + \alpha_3 = \alpha_1 + \gamma, \alpha_1 + \beta_2 = \beta_2$$

and

$$\alpha_2 = 0, \alpha_3 = \gamma, \alpha_1 = 0.$$

Hence, $\Xi(f)$ is the following matrix:

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & \beta_2 & 0 \\ \gamma & \beta_3 & \gamma \end{pmatrix},$$

$\beta_2, \beta_3, \gamma \in F$.

Conversely, let x, y be arbitrary elements of L,

$$x = \xi_1 a_1 + \xi_2 a_2 + \xi_3 a_3,$$
$$y = \eta_1 a_1 + \eta_2 a_2 + \eta_3 a_3$$

where $\xi_1, \xi_2, \xi_3, \eta_1, \eta_2, \eta_3$ are arbitrary scalars. Then

$$[x, y] = [\xi_1 a_1 + \xi_2 a_2 + \xi_3 a_3, \eta_1 a_1 + \eta_2 a_2 + \eta_3 a_3]$$

$$\begin{aligned}
&= \xi_1\eta_1[a_1, a_1] + \xi_1\eta_2[a_1, a_2] + \xi_1\eta_3[a_1, a_3] \\
&= \xi_1\eta_1 a_3 + \xi_1\eta_2 a_2 + \xi_1\eta_3 a_3 \\
&= \xi_1\eta_2 a_2 + (\xi_1\eta_1 + \xi_1\eta_3)a_3, \\
f(x) &= f(\xi_1 a_1 + \xi_2 a_2 + \xi_3 a_3) = \xi_1 f(a_1) + \xi_2 f(a_2) + \xi_3 f(a_3) \\
&= \xi_1\gamma a_3 + \xi_2(\beta_2 a_2 + \beta_3 a_3) + \xi_3\gamma a_3 \\
&= \xi_2\beta_2 a_2 + (\xi_1\gamma + \xi_2\beta_3 + \xi_3\gamma)a_3, \\
f(y) &= \eta_2\beta_2 a_2 + (\eta_1\gamma + \eta_2\beta_3 + \eta_3\gamma)a_3, \\
f([x, y]) &= f(\xi_1\eta_2 a_2 + (\xi_1\eta_1 + \xi_1\eta_3)a_3) \\
&= \xi_1\eta_2 f(a_2) + (\xi_1\eta_1 + \xi_1\eta_3)f(a_3) \\
&= \xi_1\eta_2(\beta_2 a_2 + \beta_3 a_3) + (\xi_1\eta_1 + \xi_1\eta_3)\gamma a_3 \\
&= \xi_1\eta_2\beta_2 a_2 + (\xi_1\eta_2\beta_3 + \gamma\xi_1\eta_1 + \xi_1\eta_3\gamma)a_3.
\end{aligned}$$

Thus,

$$\begin{gathered}
[f(x), y] + [x, f(y)] \\
= [\xi_2\beta_2 a_2 + (\xi_1\gamma + \xi_2\beta_3 + \xi_3\gamma)a_3, \eta_1 a_1 + \eta_2 a_2 + \eta_3 a_3] \\
+[\xi_1 a_1 + \xi_2 a_2 + \xi_3 a_3, \eta_2\beta_2 a_2 + (\eta_1\gamma + \eta_2\beta_3 + \eta_3\gamma)a_3] \\
= \xi_1\eta_2\beta_2[a_1, a_2] + \xi_1(\eta_1\gamma + \eta_2\beta_3 + \eta_3\gamma)[a_1, a_3] \\
= \xi_1\eta_2\beta_2 a_2 + \xi_1(\eta_1\gamma + \eta_2\beta_3 + \eta_3\gamma)a_3,
\end{gathered}$$

so that $f([x, y]) = [f(x), y] + [x, f(y)]$.

Denote by A the subset of $M_3(F)$ consisting of the matrices of the following form:

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \gamma & 0 & \gamma \end{pmatrix},$$

$\gamma \in F$. Denote by C the subset of $M_3(F)$ consisting of the matrices of the following form:

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \lambda & 0 \end{pmatrix},$$

$\lambda \in F$. Denote by B the subset of $M_3(F)$ consisting of the matrices of the following form:

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$\beta \in F$.

It is not hard to see that A, C are one-dimensional ideals of $\Xi(L)$, B is an one-dimensional subalgebra of $\Xi(L)$ and $\Xi(L)$ is a semidirect sum of the ideal $A \oplus C$ and the subalgebra B. □

Theorem 2. *Let D be an algebra of derivations of the Leibniz algebra $Lei_{11}(3, F)$. Then $char(F) = 2$, and D is isomorphic to Lie subalgebra of $M_3(F)$ consisting of the matrices of the following form:*

$$\begin{pmatrix} 0 & 0 & 0 \\ \alpha & \beta & 0 \\ \nu & \alpha & 0 \end{pmatrix},$$

$\alpha, \beta, \nu \in F$. Furthermore, D is a semidirect sum of an abelian ideal A and a one-dimensional subalgebra, moreover, A is a direct sum of the one-dimensional center of D and a one-dimensional ideals of D.

Proof. Let $L = Lei_{11}(3, F)$ and $f \in Der(L)$. By Lemma 1, $f(\zeta(L)) \leq \zeta(L) = Fa_3$ and by Lemma 2, $f(Fa_1 \oplus Fa_3) \leq Fa_1 \oplus Fa_3$. So that

$$\begin{aligned} f(a_1) &= \alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3, \\ f(a_2) &= \beta_2 a_2 + \beta_3 a_3, \\ f(a_3) &= \tau a_3, \end{aligned}$$

$\alpha_1, \alpha_2, \alpha_3, \beta_2, \beta_3, \tau \in F$. Then

$$\begin{aligned} f(a_2) &= f([a_2, a_1]) = [f(a_2), a_1] + [a_2, f(a_1)] \\ &= [\beta_2 a_2 + \beta_3 a_3, a_1] + [a_2, \alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3] \\ &= \beta_2 [a_2, a_1] + \alpha_1 [a_2, a_1] + \alpha_2 [a_2, a_2] \\ &= \beta_2 a_2 + \alpha_1 a_2 + \alpha_2 a_3 = (\alpha_1 + \beta_2) a_2 + \alpha_2 a_3, \\ -f(a_2) &= f(-a_2) = f([a_1, a_2]) = [f(a_1), a_2] + [a_1, f(a_2)] \\ &= [\alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3, a_2] + [a_1, \beta_2 a_2 + \beta_3 a_3] \\ &= \alpha_1 [a_1, a_2] + \alpha_1 [a_2, a_2] + \beta_2 [a_1, a_2] \\ &= -\alpha_1 a_2 + \alpha_2 a_3 - \beta_2 a_2 = -(\alpha_1 + \beta_2) a_2 + \alpha_2 a_3, \\ f(a_3) &= f([a_2, a_2]) = [f(a_2), a_2] + [a_2, f(a_2)] \\ &= [\beta_2 a_2 + \beta_3 a_3, a_2] + [a_2, \beta_2 a_2 + \beta_3 a_3] \\ &= \beta_2 [a_2, a_2] + \beta_2 [a_2, a_2] = 2\beta_2 [a_2, a_2] = 0, \\ \tau f(a_3) &= f(\gamma a_3) = f([a_1, a_1]) = [f(a_1), a_1] + [a_1, f(a_1)] \\ &= [\alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3, a_1] + [a_1, \alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3] \\ &= \alpha_1 [a_1, a_1] + \alpha_2 [a_2, a_1] + \alpha_1 [a_1, a_1] + \alpha_2 [a_1, a_2] \\ &= \alpha_2 a_2 - \alpha_2 a_2 = 0. \end{aligned}$$

Thus, we obtain

$$\beta_2 a_2 + \beta_3 a_3 = (\alpha_1 + \beta_2)a_2 + \alpha_2 a_3 = (\alpha_1 + \beta_2)a_2 - \alpha_2 a_3,$$
$$0 = f(a_3) = \tau a_3.$$

It follows that $\alpha_1 = 0$, $\alpha_2 = \beta_3$, $0 = \tau$. Hence, $\Xi(f)$ is the following matrix:

$$\begin{pmatrix} 0 & 0 & 0 \\ \beta_3 & \beta_2 & 0 \\ \alpha_3 & \beta_3 & 0 \end{pmatrix},$$

$\alpha_3, \beta_2, \beta_3 \in F$.

Conversely, let x, y be arbitrary elements of L,

$$x = \xi_1 a_1 + \xi_2 a_2 + \xi_3 a_3,$$
$$y = \eta_1 a_1 + \eta_2 a_2 + \eta_3 a_3$$

where $\xi_1, \xi_2, \xi_3, \eta_1, \eta_2, \eta_3$ are arbitrary scalars. Then

$$\begin{aligned}
[x, y] &= [\xi_1 a_1 + \xi_2 a_2 + \xi_3 a_3, \eta_1 a_1 + \eta_2 a_2 + \eta_3 a_3] \\
&= \xi_1\eta_1[a_1, a_1] + \xi_1\eta_2[a_1, a_2] + \xi_2\eta_1[a_2, a_1] + \xi_2\eta_2[a_2, a_2] \\
&= \xi_1\eta_1\gamma a_3 + \xi_1\eta_2 a_2 + \xi_2\eta_1 a_2 + \xi_2\eta_2 a_3 \\
&= (\xi_1\eta_2 + \xi_2\eta_1)a_2 + (\xi_1\eta_1\gamma + \xi_2\eta_2)a_3
\end{aligned}$$

and

$$\begin{aligned}
f(x) &= f(\xi_1 a_1 + \xi_2 a_2 + \xi_3 a_3) = \xi_1 f(a_1) + \xi_2 f(a_2) + \xi_3 f(a_3) \\
&= \xi_1(\beta_2 a_2 + \beta_3 a_3) + \xi_2(\beta_2 a_2 + \beta_3 a_3) \\
&= (\xi_1\beta_3 + \xi_2\beta_2)a_2 + (\xi_1\alpha_3 + \xi_2\beta_3)a_3, \\
f(y) &= (\eta_1\beta_3 + \eta_2\beta_2)a_2 + (\eta_1\alpha_3 + \eta_2\beta_3)a_3, \\
f([x, y]) &= f((\xi_1\eta_2 + \xi_2\eta_1)a_2 + (\xi_1\eta_1\gamma + \xi_2\eta_2)a_3) \\
&= (\xi_1\eta_2 + \xi_2\eta_1)f(a_2) + (\xi_1\eta_1\gamma + \xi_2\eta_2)f(a_3) \\
&= (\xi_1\eta_2 + \xi_2\eta_1)(\beta_2 a_2 + \beta_3 a_3) \\
&= \beta_2(\xi_1\eta_2 + \xi_2\eta_1)a_2 + \beta_3(\xi_1\eta_2 + \xi_2\eta_1)a_3,
\end{aligned}$$

Thus,

$$\begin{aligned}
&[f(x), y] + [x, f(y)] \\
&= [(\xi_1\beta_3 + \xi_2\beta_2)a_2 + (\xi_1\alpha_3 + \xi_2\beta_3)a_3, \eta_1 a_1 + \eta_2 a_2 + \eta_3 a_3]
\end{aligned}$$

$$+[\xi_1 a_1 + \xi_2 a_2 + \xi_3 a_3, (\eta_1\beta_3 + \eta_2\beta_2)a_2 + (\eta_1\alpha_3 + \eta_2\beta_3)a_3]$$
$$= \eta_1(\xi_1\beta_3 + \xi_2\beta_2)[a_2, a_1] + \eta_2(\xi_1\beta_3 + \xi_2\beta_2)[a_2, a_2]$$
$$+\xi_1(\eta_1\beta_3 + \eta_2\beta_2)[a_1, a_2] + \xi_2(\eta_1\beta_3 + \eta_2\beta_2)[a_2, a_2]$$
$$= \eta_1(\xi_1\beta_3 + \xi_2\beta_2)a_2 + \eta_2(\xi_1\beta_3 + \xi_2\beta_2)a_3$$
$$+\xi_1(\eta_1\beta_3 + \eta_2\beta_2)a_2 + \xi_2(\eta_1\beta_3 + \eta_2\beta_2)a_3$$
$$= (\eta_1\xi_1\beta_3 + \eta_1\xi_2\beta_2 + \xi_1\eta_1\beta_3 + \xi_1\eta_2\beta_2)a_2$$
$$+(\eta_2\xi_1\beta_3 + \eta_2\xi_2\beta_2 + \xi_2\eta_1\beta_3 + \xi_2\eta_2\beta_2)a_3$$
$$= (\eta_1\xi_2\beta_2 + \xi_1\eta_2\beta_2)a_2 + (\eta_2\xi_1\beta_3 + \xi_2\eta_1\beta_3)a_3,$$

so that $f([x, y]) = [f(x), y] + [x, f(y)]$.

Denote by A the subset of $\Xi(L)$, consisting of the matrices, having the following form:

$$\begin{pmatrix} 0 & 0 & 0 \\ \beta & 0 & 0 \\ \alpha & \beta & 0 \end{pmatrix},$$

$\alpha, \beta \in F$. Denote by B the subset of $\Xi(L)$, consisting of the matrices, having the following form:

$$\begin{pmatrix} 0 & 0 & 0 \\ \beta_3 & 0 & 0 \\ 0 & \beta & 0 \end{pmatrix},$$

$\beta \in F$. Denote by C the subset of $\Xi(L)$, consisting of the matrices, having the following form:

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \alpha & 0 & 0 \end{pmatrix},$$

$\alpha \in F$. Denote by E the subset of $\Xi(L)$, consisting of the matrices, having the following form:

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & \nu & 0 \\ \alpha & 0 & 0 \end{pmatrix},$$

$\alpha, \nu \in F$.

It is not hard to see that the center of $\Xi(L)$ includes C. Since $char(F) = 2$, B is the ideal of $\Xi(L)$, $dim_F(C) = dim_F(B) = 1$, A is an abelian ideal of $\Xi(L)$, $A = C \oplus B$, E is a one-dimensional subalgebra of $\Xi(L)$, and $\Xi(L)$ is a semidirect sum of A and E. $\square$

Theorem 3. *Let D be an algebra of derivations of the Leibniz algebra $Lei_{12}(3, F)$. Then D is isomorphic to a Lie subalgebra of $M_3(F)$ consisting of the matrices having the following form:*

$$\begin{pmatrix} 0 & 0 & 0 \\ \alpha & \beta & 0 \\ \nu & 0 & 0 \end{pmatrix},$$

$\alpha, \beta, \nu \in F$. Furthermore, D is a semidirect sum of an abelian ideal A and a one-dimensional subalgebra. Moreover, A is a direct sum of the one-dimensional center of D and one-dimensional ideals of D.

Proof. Let $L = Lei_{12}(3, F)$ and $f \in Der(L)$. By Lemma 1, $f(\zeta(L)) \leq \zeta(L) = Fa_3$, and by Lemma 2, $f(Fa_1 \oplus Fa_3) \leq Fa_1 \oplus Fa_3$. So that

$$\begin{aligned} f(a_1) &= \alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3, \\ f(a_2) &= \beta_2 a_2 + \beta_3 a_3, \\ f(a_3) &= \gamma a_3, \end{aligned}$$

$\alpha_1, \alpha_2, \alpha_3, \beta_2, \beta_3, \gamma \in F$. Then

$$\begin{aligned} f(a_3) &= f([a_1, a_1]) = [f(a_1), a_1] + [a_1, f(a_1)] \\ &= [\alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3, a_1] + [a_1, \alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3] \\ &= \alpha_1 [a_1, a_1] + \alpha_2 [a_2, a_1] + \alpha_1 [a_1, a_1] + \alpha_2 [a_1, a_2] \\ &= \alpha_1 a_3 + \alpha_2 a_2 + \alpha_1 a_3 - \alpha_2 a_2 = 2\alpha_1 a_3, \\ f(a_2) &= f([a_2, a_1]) = [f(a_2), a_1] + [a_2, f(a_1)] \\ &= [\beta_2 a_2 + \beta_3 a_3, a_1] + [a_2, \alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3] \\ &= \beta_2 [a_2, a_1] + \alpha_1 [a_2, a_1] = \beta_2 a_2 + \alpha_1 a_2 = (\beta_2 + \alpha_1) a_2, \\ -f(a_2) &= f(-a_2) = f([a_1, a_2]) = [f(a_1), a_2] + [a_1, f(a_2)] \\ &= [\alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3, a_2] + [a_1, \beta_2 a_2 + \beta_3 a_3] \\ &= \alpha_1 [a_1, a_2] + \beta_2 [a_1, a_2] = -\alpha_1 a_2 - \beta_2 a_2 = -(\alpha_1 + \beta_2) a_2. \end{aligned}$$

Thus, we obtain

$$\beta_2 a_2 + \beta_3 a_3 = (\alpha_1 + \beta_2) a_2, \ 2\alpha_1 a_3 = \gamma a_3.$$

It follows that $\beta_3 = 0$, $\alpha_1 = 0$, $0 = 2\alpha_1 = \gamma$. Hence, $\Xi(f)$ is the following matrix:

$$\begin{pmatrix} 0 & 0 & 0 \\ \alpha_2 & \beta_2 & 0 \\ \alpha_3 & 0 & 0 \end{pmatrix},$$

$\alpha_2, \alpha_3, \beta_2 \in F$.

Conversely, let x, y be arbitrary elements of L,

$$x = \xi_1 a_1 + \xi_2 a_2 + \xi_3 a_3,$$
$$y = \eta_1 a_1 + \eta_2 a_2 + \eta_3 a_3$$

where $\xi_1, \xi_2, \xi_3, \eta_1, \eta_2, \eta_3$ are arbitrary scalars. Then

$$\begin{aligned}
[x, y] &= [\xi_1 a_1 + \xi_2 a_2 + \xi_3 a_3, \eta_1 a_1 + \eta_2 a_2 + \eta_3 a_3] \\
&= \xi_1 \eta_1 [a_1, a_1] + \xi_1 \eta_2 [a_1, a_2] + \xi_2 \eta_1 [a_2, a_1] \\
&= \xi_1 \eta_1 a_3 - \xi_1 \eta_2 a2 + \xi_2 \eta_1 a_2 \\
&= (\xi_2 \eta_1 - \xi_1 \eta_2) a_2 + \xi_1 \eta_1 a_3, \\
f(x) &= f(\xi_1 a_1 + \xi_2 a_2 + \xi_3 a_3) = \xi_1 f(a_1) + \xi_2 f(a_2) + \xi_3 f(a_3) \\
&= \xi_1 (\alpha_2 a_2 + \alpha_3 a_3) + \xi_2 (\beta_2 a_2) = \xi_1 \alpha_2 a_2 + \xi_1 \alpha_3 a_3 + \xi_2 \beta_2 a_2 \\
&= (\xi_1 \alpha_2 + \xi_2 \beta_2) a_2 + \xi_1 \alpha_3 a_3, \\
f(y) &= (\eta_1 \alpha_2 + \eta_2 \beta_2) a_2 + \eta_1 \alpha_3 a_3, \\
f([x, y]) &= f((\xi_2 \eta_1 - \xi_1 \eta_2) a_2 + \xi_1 \eta_1 a_3) \\
&= (\xi_2 \eta_1 - \xi_1 \eta_2) f(a_2) + \xi_1 \eta_1 f(a_3) = \beta_2 (\xi_2 \eta_1 - \xi_1 \eta_2) a_2.
\end{aligned}$$

Thus,

$$\begin{aligned}
[f(x), y] + [x, f(y)] &= [(\xi_1 \alpha_2 + \xi_2 \beta_2) a_2 + \xi_1 \alpha_3 a_3, \eta_1 a_1 + \eta_2 a_2 + \eta_3 a_3] \\
&\quad + [\xi_1 a_1 + \xi_2 a_2 + \xi_3 a_3, (\eta_1 \alpha_2 + \eta_2 \beta_2) a_2 + \eta_1 \alpha_3 a_3] \\
&= \eta_1 (\xi_1 \alpha_2 + \xi_2 \beta_2) [a_2, a_1] + \xi_1 (\eta_1 \alpha_2 + \eta_2 \beta_2) [a_1, a_2] \\
&= \eta_1 (\xi_1 \alpha_2 + \xi_2 \beta_2) a_2 - \xi_1 (\eta_1 \alpha_2 + \eta_2 \beta_2) a_2 \\
&= (\eta_1 \xi_1 \alpha_2 + \eta_1 \xi_2 \beta_2 - \xi_1 \eta_1 \alpha_2 - \xi_1 \eta_2 \beta_2) a_2 \\
&= (\eta_1 \xi_2 \beta_2 - \xi_1 \eta_2 \beta_2) a_2,
\end{aligned}$$

so that $f([x, y]) = [f(x), y] + [x, f(y)]$.

Denote by A the subset of $\Xi(L)$ consisting of the matrices having the following form:

$$\begin{pmatrix} 0 & 0 & 0 \\ \beta & 0 & 0 \\ \alpha & 0 & 0 \end{pmatrix},$$

$\alpha, \beta \in F$. Denote by B the subset of $\Xi(L)$ consisting of the matrices having the following form:

$$\begin{pmatrix} 0 & 0 & 0 \\ \beta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$\beta \in F$. Denote by C the subset of $\Xi(L)$, consisting of the matrices having the following form:

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \alpha & 0 & 0 \end{pmatrix},$$

$\alpha \in F$. Denote by E the subset of $\Xi(L)$, consisting of the matrices having the following form:

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & \nu & 0 \\ \alpha & 0 & 0 \end{pmatrix},$$

$\alpha, \nu \in F$. It is not hard to see that the center of $\Xi(L)$ includes C, B is the ideal of $\Xi(L)$, $dim_F(C) = dim_F(B) = 1$, A is an abelian ideal of $\Xi(L)$, $A = C \oplus B$, E is an one dimensional subalgebra of $\Xi(L)$, and $\Xi(L)$ is a semidirect sum of A and E. □

References

[1] Ayupov, Sh., Omirov, B., Rakhimov, I.: Leibniz Algebras: Structure and Classification. CRC Press, Taylor & Francis Group (2020). https://doi.org/10.1201/9780429344336

[2] Blokh, A.: On a generalization of the concept of Lie algebra. Dokl. Akad. Nauk SSSR **165**(3), 471–473 (1965)

[3] Jacobson, N.: Lie algebras. New York: Interscience (1962)

[4] Kurdachenko, L.A., Otal, J., Pypka, A.A.: Relationships between the factors of the canonical central series of Leibniz algebras. Eur. J. Math. **2**(2), 565–577 (2016). https://doi.org/10.1007/s40879-016-0093-5

[5] Kurdachenko, L.A., Pypka, O.O., Subbotin, I.Ya.: On the structure of low-dimensional Leibniz algebras: some revision. Algebra Discrete Math. **34**(1), 68–104 (2022). https://doi.org/10.12958/adm2036

[6] Kurdachenko, L.A., Pypka, O.O., Subbotin, I.Ya.: General Theory of Leibniz Algebras. Switzerland: Springer (2024). https://doi.org/10.1007/978-3-031-58148-9

[7] Kurdachenko, L.A., Semko, M.M., Subbotin, I.Ya.: On the algebra of derivations of some low-dimensional Leibniz algebras. Algebra Discrete Math. **36**(1), 43–60 (2023). https://doi.org/10.12958/adm2161

[8] Kurdachenko, L.A., Semko, M.M., Subbotin, I.Ya.: On the structure of algebras of derivations of some non-nilpotent Leibniz algebras. Algebra Discrete Math. **37**(2), 244–261 (2024). https://doi.org/10.12958/adm2227

[9] Kurdachenko, L.A., Semko, M.M, Subbotin, I.Ya.: On the algebra of derivations of some Leibniz algebras. Algebra Discrete Math. **38**(1), 63–86 (2024). https://doi.org/10.12958/adm2316

[10] Kurdachenko, L.A., Semko, M.M., Yashchuk, V.S.: On the structure of the algebra of derivations of cyclic Leibniz algebras. Algebra Discrete Math. **32**(2), 241–252 (2021). https://doi.org/10.12958/adm1898

[11] Kurdachenko, L.A., Semko, M.M., Yashchuk, V.S.: On the algebra of derivations of some nilpotent Leibniz algebras. Res. Math. **31**(1), 62–71 (2023). https://doi.org/10.15421/242306

[12] Kurdachenko, L.A., Semko, M.M., Yashchuk, V.S.: On the structure of the algebra of derivations of some low-dimensional Leibniz algebras. Ukr. Math. J. **76**(5), 728–742 (2024). https://doi.org/10.3842/umzh.v76i5.7573

[13] Kurdachenko, L.A., Subbotin, I.Ya., Semko, N.N: From groups to Leibniz algebras: Common approaches, parallel results. Adv. Group Theory Appl. **5**, 1–31 (2018). https://doi.org/10.4399/97888255161421

[14] Kurdachenko, L.A., Subbotin, I.Ya., Yashchuk, V.S.: On the endomorphisms and derivations of some Leibniz algebras. J. Algebra Appl. **23**(1), 2450002 (2024). https://doi.org/10.1142/S0219498824500026

[15] Loday, J.-L.: Cyclic homology. Grundlehren der Mathematischen Wissenschaften **301**. Springer Verlag (1992). https://doi.org/10.1007/978-3-662-11389-9

[16] Loday, J.-L.: Une version non commutative des algèbres de Lie; les algèbras de Leibniz. Enseign. Math. **39**, 269–293 (1993)

[17] Semko, M.M., Skaskiv, L.V., Yarovaya, O.A.: On the derivations of cyclic nilpotent Leibniz algebras. Carpathian Math. Publ. **14**(2), 345–353 (2022). https://doi.org/10.15330/cmp.14.2.345-353

Contact information

L. A. Kurdachenko — Oles Honchar Dnipro National University, Nauky Av. 72, Dnipro, 49045, Ukraine
E-Mail: lkurdachenko@gmail.com

M. M. Semko — State Tax University, Universitetska Str. 31, Irpin, 08205, Ukraine
E-Mail: dr.mykola.semko@gmail.com

I. Ya. Subbotin — National University, 5245 Pacific Concourse Drive, Los Angeles, CA 90045-6904, USA
E-Mail: isubboti@nu.edu

Received by the editors: 18.02.2025.