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Free products of semigroups defined by automata

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on the occasion of his 80th birthday

Abstract. It is shown that the free product of semigroups defined by (finite) automata over a finite alphabet is defined by (finite) automata over the same alphabet.

1. Introduction

Automata, especially finite, over finite alphabets give efficient way to define semigroups and inverse semigroups. Defined in this way they naturally act as endomorphisms or partial automorphisms of regular rooted trees respectively [5, 7–9]. One of the natural questions is to understand which semigroup theoretical operations preserve the property to be defined by automata. Semigroups defined by automata are rich of free subsemigroups [11]. They are convenient to generate monogenic free inverse semigroups [4, 6, 9].

More sophisticated approach to define semigroups by automata is to generate a semigroups using all states of a given automaton. Semigroups defined in this way are referred as automaton semigroups. In [1, 2] the authors show that the class of automaton semigroups is very close to

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be closed under the free product. However, the complete answer is still under investigation.

The main our result is Theorem 2. We show that for arbitrary semigroup defined by automata over a finite alphabet their free product is defined by automata over the same alphabet. Moreover, if the automata defining initial semigroups are finite then their free product is defined by finite automata as well. In other words, the finite state wreath power [10] of the symmetric transformation semigroup is closed under free products.

The paper is organized as follows. In Section 2 we recall basic definitions and notation involving transformations defined by automata over finite alphabets. For more detailed introduction to the topic one can refer to [3,5,9]. In Section 1 we prove an auxiliary statement that allows us to reduce the size of the alphabet. Section 4 contains the proof of the main result of the paper.

2. Preliminaries

Let X be a finite alphabet with $|\mathsf{X}| > 1$. The set $\mathsf{X}^* = \bigcup_{i=0}^{\infty} \mathsf{X}^i$, consisting of all finite words over X, including the empty word Λ, forms a free monoid with basis X under concatenation. The length of a word $w = x_1 \dots x_n \in \mathsf{X}^*$, where $x_1, \dots, x_n \in \mathsf{X}$, is n and is denoted by $|w|$; i.e., $w \in \mathsf{X}^{|w|}$.

The right Cayley graph of the monoid X^* with respect to the basis X defines a structure of a regular rooted tree $\mathcal{T}(\mathsf{X})$ on X^* as its vertex set. Two words $u, v \in \mathsf{X}^*$ are joined by an edge if and only if $u = vx$ or $v = ux$ for some $x \in \mathsf{X}$. The empty word Λ serves as the root of $\mathcal{T}(\mathsf{X})$.

For every $n \geq 0$, the set X^n forms the n-th level of this tree, and the union $\mathsf{X}^{(n)} = \bigcup_{i=0}^{n} \mathsf{X}^i$ represents the vertex set of a regular rooted subtree of depth n.

An automaton over alphabet X is a triple $\mathcal{A} = (Q, \lambda, \mu)$ such that Q is a non-empty set, the set of states, $\lambda : Q \times \mathsf{X} \to Q$ is the transition function, $\mu : Q \times \mathsf{X} \to Q$ is the output function. The automaton $\mathcal{A}$ is called finite if the set Q of its states is finite.

The transition function λ and the output function μ can be recursively extended to the set $Q \times \mathsf{X}^*$ using the equalities

$$\lambda(q, \Lambda) = q, \quad \lambda(q, xw) = \lambda(\lambda(q, x), w),$$

$$\mu(q, \Lambda) = \Lambda, \quad \mu(q, xw) = \mu(q, x)\mu(\lambda(q, x), w),$$

where $q \in Q$, $x \in \mathsf{X}$, $w \in \mathsf{X}^*$. Then at each state $q \in Q$ the automaton $\mathcal{A}$ defines a transformation μ_q on the set X^* by the following rule

$$\mu_q(w) = \mu(q, w), \quad w \in \mathsf{X}^*.$$

We say that a transformation f on X^* is defined by an automaton over X if $f = \mu_q$ for some automaton $\mathcal{A} = (Q, \lambda, \mu)$ and its state q. All transformations defined by automata over X form a semigroup $SA(\mathsf{X})$ under superposition. The semigroup $SA(\mathsf{X})$ is the semigroup of endomorphisms of the rooted tree $\mathcal{T}(\mathsf{X})$. All transformations on X^* defined by finite automata form a countable subsemigroup $FSA(\mathsf{X})$ of $SA(\mathsf{X})$.

For arbitrary transformation $f \in SA(\mathsf{X})$ the minimal automaton $\mathcal{A}_f = (Q_f, \lambda_f, \mu_f)$ such that $\mathcal{A}_f$ defines f is constructed as follows. Since f is a length preserving and prefix preserving transformation on X^* for each $w \in \mathsf{X}^*$ the transformation $f_w \in SA(\mathsf{X})$ such that

$$f(wu) = f(w) f_w(u), \quad u \in \mathsf{X}^*,$$

is well defined. The transformation f_w is called the state of f at w. In particular, $f_\Lambda = f$. Then the set of states Q_f of $\mathcal{A}_f$ is the set $\{f_w : w \in \mathsf{X}^*\}$. The set Q_f is finite iff $f \in FSA(\mathsf{X})$. The transition and the output functions λ_f, μ_f are defined by the equalities

$$\lambda_f(f_w, x) = f_{wx}, \quad \mu_f(f_w, x) = f_w(x), \quad w \in \mathsf{X}^*, x \in \mathsf{X}.$$

Then f is defined by $\mathcal{A}_f$ at its state f_Λ.

3. Semigroups defined by automata

We say that a semigroup S is defined by (finite) automata over alphabet X if S is isomorphic to a subsemigroup of $SA(\mathsf{X})$ (of $FSA(\mathsf{X})$).

Assume that S is a subsemigroup of $SA(\mathsf{X})$. For each transformation $f \in S$ consider the minimal automaton $\mathcal{A}_f = (Q_f, \lambda_f, \mu_f)$ over X such that $f = \mu_{f_q}$ for some state $q \in Q_f$. For each state $q \in Q_f$ define a transformation $t_q : \mathsf{X} \to \mathsf{X}$ as follows

$$t_q(x) = \mu_f(q, x), x \in \mathsf{X}.$$

The set of transformations $\{t_q \mid q \in Q_f, f \in S\}$ generates a subsemigroup in the symmetric transformation semigroup on X. We call this semigroup the base transformation semigroup associated with S and denote it by T_S. The following theorem allows for to reduce the size of the alphabet X.

Theorem 1. *Let* X, Y *be finite alphabets,* S *be a semigroup defined by (finite) automata over alphabet* X *and* T_S *be the base semigroup associated with* S. *If there exists* $m \geq 1$ *such that* T_S *is isomorphic as a transformation semigroup on* X *to a subsemigroup of the* m*th wreath power of the symmetric transformation semigroup* $T(\mathsf{Y})$ *acting on* Y^m *then the semigroup* S *is defined by (finite) automata over alphabet* Y.

Proof. Denote by φ an isomorphic embedding of T_S into the mth wreath power $\wr_{i=1}^{m} T(\mathsf{Y})^{(i)}$, $T(\mathsf{Y})^{(i)} \simeq T(\mathsf{Y})$, $1 \leq i \leq m$, and by ψ an injection from X to Y^m such that

$$\psi(x^t) = (\psi(x))^{\varphi(t)}, \quad x \in \mathsf{X}, t \in T_S.$$

Let Y_1 be the image of X under ψ. Then (T_S, X) is isomorphic as a transformation semigroup to $(\varphi(T_S), \mathsf{Y}_1)$. Recall that the wreath product $\wr_{i=1}^{m} T(\mathsf{Y})^{(i)}$ acts on $\mathsf{Y}^{(m)}$. This action is length preserving and prefix preserving. Hence, $\varphi(T_S)$ acts on the set Y_2 that consists of all prefixes of all words from Y_1.

Let $f \in S$ and $\mathcal{A}_f = (Q, \lambda, \mu)$ be the minimal automaton over X such that $f = \mu_{q_f}$ for some state $q_f \in Q$. We define the automaton $\mathcal{B} = (Q_1, \lambda_1, \mu_1)$ over Y. The set of sates Q_1 of $\mathcal{B}$ is the Cartesian product $Q \times \mathsf{Y}^{(m-1)}$. The transition function λ_1 is defined by the equality

$$\lambda_1((q, w), y) = \begin{cases} (q, wy), & \text{if } |w| < m-1 \text{ and } wy \in \mathsf{Y}_2, \\ (\lambda(q, \psi^{-1}(wy)), \Lambda), & \text{if } |w| = m-1 \text{ and } wy \in \mathsf{Y}_1, \\ (q, w), & \text{otherwise.} \end{cases}$$

Since ψ is injective the definition is correct. The output function μ_1 is defined by the equality

$$\mu_1((q, w), y) = \begin{cases} y_1, & \text{if } wy \in \mathsf{Y}_2 \text{ and } (wy)^{\varphi(\mu_q)} = (w)^{\varphi(\mu_q)} y_1, \\ y, & \text{otherwise.} \end{cases}$$

Since $\varphi(\mu_q)$, $q \in Q$, is length preserving and prefix preserving on Y_2 the definition is correct.

To complete the proof we show that the mapping $\mu_{q_f} \mapsto \mu_{1(q_f, \Lambda)}$, $f \in S$, defines an isomorphism.

Consider the monoid monomorphism $\Psi : \mathsf{X}^* \to \mathsf{Y}^*$ that extends injection ψ. Then the image $\Psi(\mathsf{X}^*)$ is a free monoid with basis Y_1. Hence, $\Psi(\mathsf{X}^*) = \mathsf{Y}_1^*$.

For arbitrary $q \in Q$, $x \in \mathsf{X}$ and $w \in \mathsf{X}^*$ we have the equalities

$$\Psi(xw)^{\mu_{1(q,\Lambda)}} = (\psi(x)\Psi(w))^{\mu_{1(q,\Lambda)}} = \\ (\psi(x))^{\mu_{1(q,\Lambda)}}\Psi(w)^{\mu_{1(\lambda(q,x),\Lambda)}} = \psi(x^{\mu_q})\Psi(w)^{\mu_{1(\lambda(q,x),\Lambda)}}.$$

Since $\psi(x) \in \mathsf{Y}_1$ the last equality follows from the definition of the output function μ_1. Therefore, transformation semigroup (S, X^*) is isomorphic to a transformation semigroup on Y_1^*.

Consider an arbitrary $w \in \mathsf{Y}^*$. Then there exist unique $w_1 \in \mathsf{Y}_1^*$, $w_2 \in \mathsf{Y}_2 \setminus \mathsf{Y}_1$, $w_3 \in \mathsf{Y}^*$ such that $w = w_1w_2w_3$ and the word w_2 is the longest prefix of w_2w_3 from Y_2. Let $q \in Q$. Then

$$w^{\mu_{1(q,\Lambda)}} = w_1^{\mu_{1(q,\Lambda)}}w_4w_3$$

for some $w_4 \in \mathsf{Y}_2 \setminus \mathsf{Y}_1$ such that for arbitrary $w_1w_2u \in \mathsf{Y}_1^*$, $u \in \mathsf{Y}^*$, the word $w_1^{\mu_{1(q,\Lambda)}}w_4$ is a prefix of $(w_1w_2u)^{\mu_{1(q,\Lambda)}}$. Hence, the action of $\mu_{1(q,\Lambda)}$ on Y^* is completely defined by its action on Y_1^*. The proof is complete. □

4. Actions of free products

Theorem 2. *Let X be a finite alphabets, $|\mathsf{X}| > 1$, S_1, S_2 be semigroups defined by (finite) automata over X. Then the free product $S_1 * S_2$ is defined by (finite) automata over X.*

Proof. Let $\mathsf{X} = \{x_0, x_1, \ldots, x_n\}$, $n \geq 1$. Consider a disjoint alphabet $\mathsf{Y} = \{y_0, y_1, \ldots, y_n\}$ of the same size $n + 1$. Assume that S_1 and S_2 are defined by (finite) automata over X and Y correspondingly. For the alphabet

$$\mathsf{Z} = \mathsf{X} \cup \mathsf{Y} \cup (\{0, \ldots, n\} \times \{0, \ldots, n\})$$

of size $n^2 + 4n + 3$ we construct a subsemigroup S of $SA(\mathsf{Z})$ (of $FSA(\mathsf{Z})$) isomorphic to the free product $S_1 * S_2$. Then we apply Theorem 1 to reduce the size of the alphabet and complete the proof.

For each $s \in S_i$, $i = 1, 2$, consider the minimal automaton $\mathcal{A}_s = (Q, \lambda, \mu)$ over X or Y correspondingly such that s is defined by $\mathcal{A}_s$ at some state $q_s \in Q$. Define an automaton $\mathcal{B}_s = (Q_1, \lambda_1, \mu_1)$ over Z. The set of states Q_1 is obtained from Q by adding 4 new states $q_{sI}, q_{sT}, q_{s0}, q_{s1}$. Without loss of generality we assume that $i = 1$, i.e. $\mathcal{A}_s$ is an automaton over X. Then the functions λ_1, μ_1 are extentions of the functions λ, μ from

$Q \times \mathsf{X}$ to $Q_1 \times \mathsf{Z}$ defined for $q \in Q_1$, $z \in \mathsf{Z}$ by the following equations:

$$\lambda_1(q,z) = \begin{cases} q_s, & q = q_{sI}, z = x_0, \\ q_{sI}, & q = q_{sI}, z \in \mathsf{X}, z \neq x_0 \text{ or } q = q_{s0}, q_{s1}, z \in \mathsf{Y}, \\ q_{s0}, & q \in Q, z = (k,l), 0 \leq k,l \leq n, \text{ and } \mu(q, x_k) \neq x_l, \\ q_{s1}, & q \in Q, z = (k,l), 0 \leq k,l \leq n, \text{ and } \mu(q, x_k) = x_l, \\ q_{sT}, & \text{otherwise,} \end{cases}$$

$$\mu_1(q,z) = \begin{cases} y_0, & q = q_{s1}, z = y_1, \\ z, & \text{otherwise.} \end{cases}$$

In case $i = 2$, i.e. $\mathcal{A}_s$ is an automaton over Y in this definition letters from X are replaced by letters from Y and vice versa. Denote by $\hat{s}$ the transformation of Z^* defined by B_s at q_{sI}.

Let S be the semigroup generated by the set $\{\hat{s} : s \in S_1 \cup S_2\}$. We will show that S splits into the free product $S_1 * S_2$.

Each non-empty word w over Z can be uniquely decomposed as a product

$$w = w_1 w_2 \ldots w_k$$

such that each word $w_1, w_2, \ldots, w_k$ is a word over one of the alphabets X, Y or $\{0, \ldots, n\} \times \{0, \ldots, n\}$. Then for arbitrary $s \in S_1$ the image $w^{\hat{s}}$ has the form

$$w^{\hat{s}} = w_1^{s_1} w_2^{s_2} \ldots w_k^{s_k},$$

where for each i, $1 \leq i \leq k$, the transformation s_i is the identity except the following two cases. Let $w_0 = \Lambda$.

1. If $\lambda_1(q_{sI}, w_0 w_1 \ldots w_{i-1}) = q_s$ and $w_i = x_0 u$ for some $u \in \mathsf{X}^*$ then $w_i^{s_i} = x_0 u^s$.

2. If $i > 2$, $\lambda_1(q_{sI}, w_0 w_1 \ldots w_{i-3}) = q_{sI}$, $w_{i-2} = x_0 u$ for some $u \in \mathsf{X}^*$, $w_{i-1} = (k,l)$ for some k, l, $0 \leq k, l \leq n$, such that $\mu(\lambda(q_s, u), x_k) = x_l$, $w_i = y_1 v$ for some $v \in \mathsf{Y}^*$, then $w_i^{s_i} = y_0 v$.

It immediately implies that the mapping $s \mapsto \hat{s}$, $s \in S_1$, is an isomorphism. Similarly, the mapping $s \mapsto \hat{s}$, $s \in S_2$, is an isomorphism as well.

Now consider two distinct words

$$W_1 = s_{11} \ldots s_{1m_1}, W_2 = s_{21} \ldots s_{2m_2}, 1 \leq m_1 \leq m_2,$$

such that s_{ij}, s_{ij+1}, $1 \leq j < m_i$, $i = 1, 2$, belong to different semigroups S_1, S_2. We need to show that W_1 and W_2 define distinct transformations on Z^*. It is sufficient to find a word over Z with distinct images under W_1 and W_2.

Without loss of generality we may assume that $s_{21} \in S_1$. For each k, $1 \leq k \leq m_2$, let

$$z_k = (k, l_k),$$

where $\mu_{s_{2k}}(q_{s_{2k}}, t_k) = t_{l_k}$ and (t_k, t_{l_k}) equals (x_k, x_{l_k}) or (y_k, y_{l_k}) depending on $s_{2k} \in S_2$ or $s_{2k} \in S_1$. Define a word

$$u = x_0 z_1 y_1 z_2 x_1 \ldots z_{m_2} t_1$$

of length $2m_2 + 1$. Consider 3 cases.

Case 1. The length of the word W_1 is less than the length of the word W_2, i.e. $m_1 < m_2$. Then W_2 transforms the word u to the word with the last symbol x_0 or y_0 correspondingly. On the other hand W_1 preserves the last symbol of the word u.

Case 2. Words W_1 and W_2 have the same length, i.e. $m_1 = m_2$, and the first letters of W_1 and W_2 belong to different semigroups S_1, S_2. Under our assumptions it means that $s_{11} \in S_2$. Then W_1 preserves the last symbol x_1 of the word u as above.

Case 3. Words W_1 and W_2 have the same length m_2 and $s_{11} \in S_1$. Let j be the smallest index such that $s_{1j} \neq s_{2j}$. Then $1 \leq j \leq m_2$ and both elements s_{1j}, s_{2j} belong to the same semigroup S_1, S_2. Without loss of generality we suppose that they belong to S_1. Choose a word $v \in \mathsf{X}^*$ of the shortest length such that the images of v under s_{1j}, s_{2j} are distinct. Then the last letters of these images are distinct. Let the last letters of v and it image under s_{2j} be x_{j_1}, x_{j_2}, $0 \leq j_1, j_2 \leq n$, correspondingly. Consider $z = (j_1, j_2)$. Then

$$\lambda_1(q_{s_1}, vz) = q_{s_1 0}, \lambda_1(q_{s_2}, vz) = q_{s_2 1}.$$

Define a word

$$u_1 = x_0 z_1 y_1 z_2 x_1 \ldots z_{j-1} x_{j-1} v z y_1 z_{j+1} \ldots z_{m_2} t_1$$

of length $2m_2 + |v| + 1$. Then the last symbol of u_1 under the action of W_2 is x_0 or y_0 but W_1 preserves the last symbol of u_1.

Hence, the semigroup S is isomorphic to the free product $S_1 * S_2$.

The base semigroup T_S associated with S is a subsemigroup of the direct sum of $T(\mathsf{X})$, $T(\mathsf{Y})$ and the trivial semigroups. Hence T_S is isomorphic as a transformation semigroup to the third wreath power of $T(\mathsf{X})$. Applying Theorem 1 we complete the proof. $\square$

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