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Existence of Dynkin scanning trees for non-serial posets with positive Tits quadratic form

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Dedicated to Professor Yu. A. Drozd
on the occasion of his 80th birthday

Abstract. Let G be a finite undirected connected graph. The minimum number of edges that must be removed to make the graph acyclic is called the circuit rank of G. If such edges are fixed, the graph that remains is called a spanning tree of G. In this paper we study scanning trees of the Hasse diagrams of connected posets with positive Tits quadratic form.

Introduction

This paper is related to the posets with positive Tits quadratic forms, which are analogues of the Dynkin diagrams.

The Tits quadratic form was first introduced by P. Gabriel [18] for finite quivers (directed graphs). Namely, if $Q = (Q_0, Q_1)$ is a quiver with the set of vertices Q_0 and the set of arrows Q_1, then its *Tits quadratic form* $q_Q : \mathbb{Z}^n \to \mathbb{Z}$, $n = |Q_0|$, is given by the equality

$$q_Q(z) = \sum_{i \in Q_0} z_i^2 - \sum_{\{i \to j\} \in Q_1} z_i z_j.$$

The Tits quadratic form of posets, the closest structure to quivers, was first considered by Yu. A. Drozd in [17]. By definition, the *Tits quadratic*

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form $q_S : \mathbb{Z}^{1+n} \to \mathbb{Z}$ *of a poset* $S \not\ni 0$ of order n has the form

$$q_S(z) = z_0^2 + \sum_{i \in S} z_i^2 + \sum_{i<j, i,j \in S} z_i z_j - z_0 \sum_{i \in S} z_i.$$

The main results of the mentioned papers inspired the study of posets with positive Tits quadratic form as analogues of the Dynkin diagrams (in more details, see [12, Introduction]). Such posets, which are simply called *positive*, were studied by the authors from different points of view in many papers (see, e.g. [4–7, 10, 11, 14, 15]).

In particular, in [5] the authors described all positive posets, using analogous result on posets of width 2 obtained a little earlier in [4] (see also [12, Theorem 1] and [14, Section 4] for serial and non-serial positive posets, respectively). In the same paper, the authors also described all minimal posets with non-positive Tits form [5], calling them *P-critical* (see also [13] and [15]).

Note that the paper [5] has been often cited, but is today virtually inaccessible. The main ideas and many results of this paper are published (in a translation from Russian) in [3, Sections 1–3].

The main method in the above-mentioned papers of the authors is a method based on the notion of (min, max)-equivalence (later called minimax equivalence) of posets, which was introduced by the first author in [2]. This equivalence preserves $\mathbb{Z}$-equivalence of the corresponding Tits quadratic forms. The minimax equivalence method has been used to solving many other problems (see, e.g. [8, 9, 16, 19]).

In this paper we study the Hasse diagrams of connected positive posets with respect to the circuit rank, scanning trees, etc.

1. The main results

Through the paper, by a graph we mean any finite undirected graph and by a poset any finite poset without an element denoted as 0. By a subgraph of a graph $G = (G_0, G_1)$, with the set of vertices G_0 and the set of edges G_1, we mean any graph $G' = (G'_0, G'_1)$ with $G'_0 \subseteq G_0$ and $G'_1 \subseteq G_1$. By a subposet S' of a poset S we always mean a full one (i.e. with the order relation induced by a given order relation on S).

A poset $S = (A, \leq)$ is usually represented by a quiver $Q = (Q_0, Q_1)$, where $Q_0 = A$ and Q_1 consists of the arrows $(x, y) : x \to y$ with $x < y$ and y to be neighboring (i.e. there is no z satisfying $x < z < y$). We denote the quiver Q by $\overrightarrow{H}(S)$, but mean by the Hasse diagram of S instead of this quiver its underlying undirected graph $H(S)$. In this case, the Hasse diagram in the plane is represented in such way that an edge (x, y) with $x < y$ always goes upward from x to y. For a subgraph F of $H(S)$, we denote by $F^{\leq}$ the corresponding subposet of S (then $H(F^{\leq}) = F$).

We call a graph G *positive* if so is the Tits quadratic form of a quiver whose underlying undirected graph coincides with G. As already said above, a poset is called *positive* if so is its Tits quadratic form. Such a poset S is called *serial* if there is an infinite increasing sequence $S \subset S^{(1)} \subset S^{(2)} \subset \ldots$ with positive terms, and *non-serial* if otherwise.

Let G be a connected graph (for us it is enough to consider only simple graphs, i.e. without loops and multiple edges). The minimum number $cr(G)$ of edges that must be removed to break all its cycles is called the *circuit* or *cycle rank* of G. If such edges are fixed, the graph that remains is called a *spanning tree* of G. The circuit rank can be easily computed by the formula $cr(G) = m - n + 1$ with m and n the number of edges and vertices of G, respectively, and is equal to the number of independent cycles in G. For more details on this topic see [1, 20].

Let us proceed directly to the formulation of the main theorems.

A poset S is called *connected* if so is its Hasse diagram. An element of a poset S is said to be *nodal* if it is comparable with all other elements. For simplicity,we write $\overline{cr}(S)$ instead of $cr[H(S)]$.

Theorem 1. *For any connected non-serial positive poset S, $\overline{cr}(S) < 3$. If $\overline{cr}(S) = 2$, then S has such a nodal element x that $\overline{cr}(S \setminus x) = 1$.*

Theorem 2. *Let S be a connected non-serial positive poset. Then the Hasse graph $H(S)$ has such scanning trees F that*

(1) *the graph F is positive;*

(2) *the poset $F^{\leq}$ is positive.*

By a result of P. Gabriel [18], (1) is eqquivalent to the following condition: F is a simply faced Dynkin diagram (i.e. A_n, D_m, E_6, E_7, E_8).

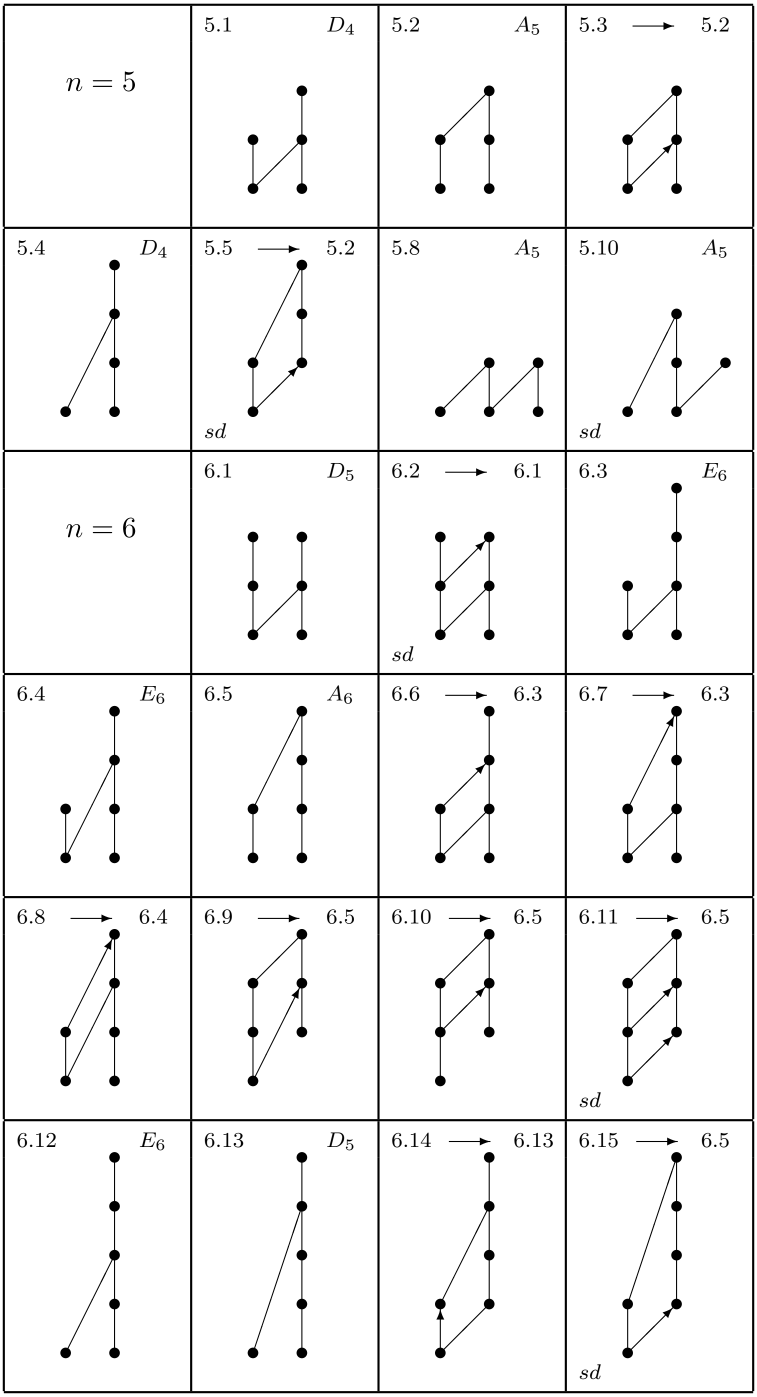
It is easy to see that the theorems follow from the classification Table A attached below (as a part of the general classification Tables 4.1, 4.2 and 4.3 [14]) together with additional information for each Hasse diagram numbered by $p.q$ in the upper left corner. Namely,

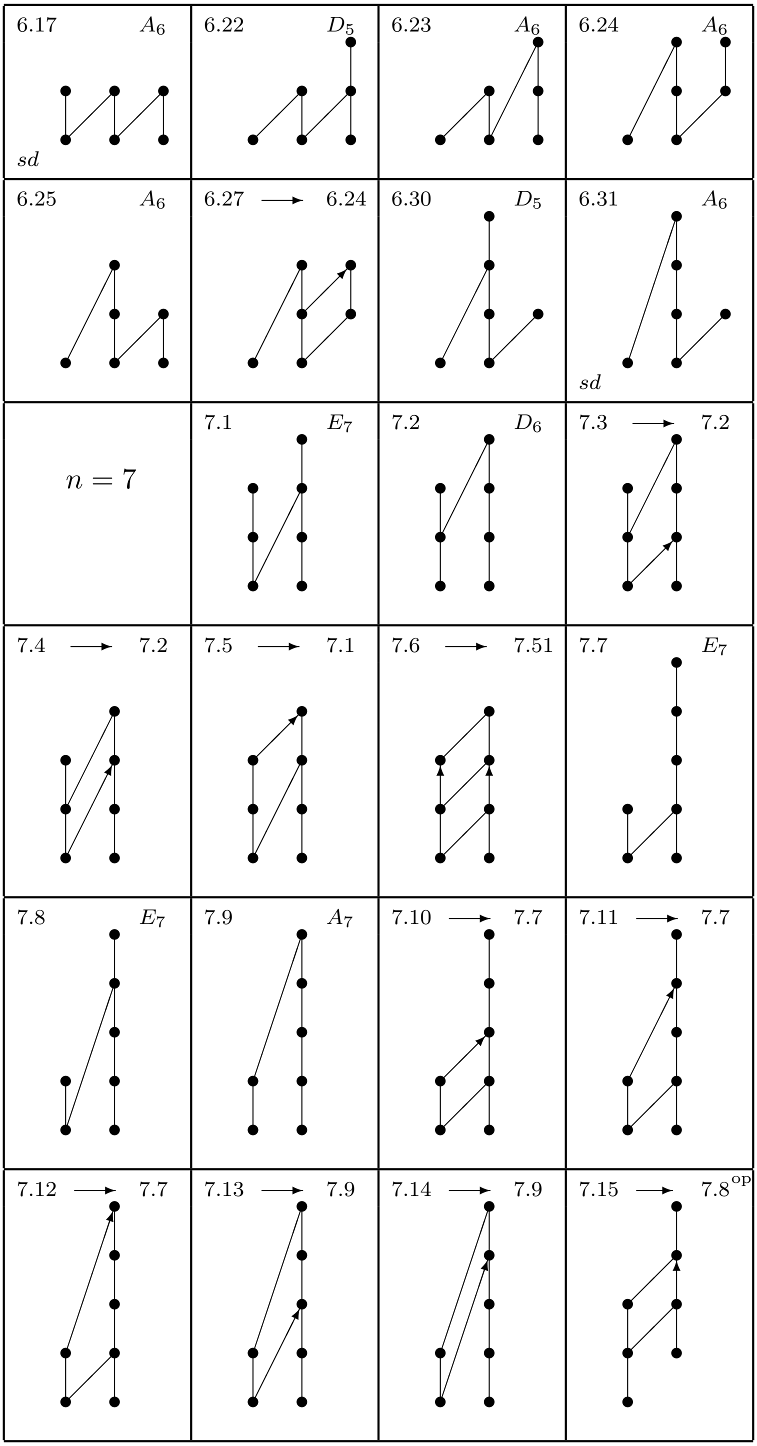
(a) if a graph $p.q$ is acyclic, it is isomorphic to a Dynkin diagram which is indicated in the upper right corner;

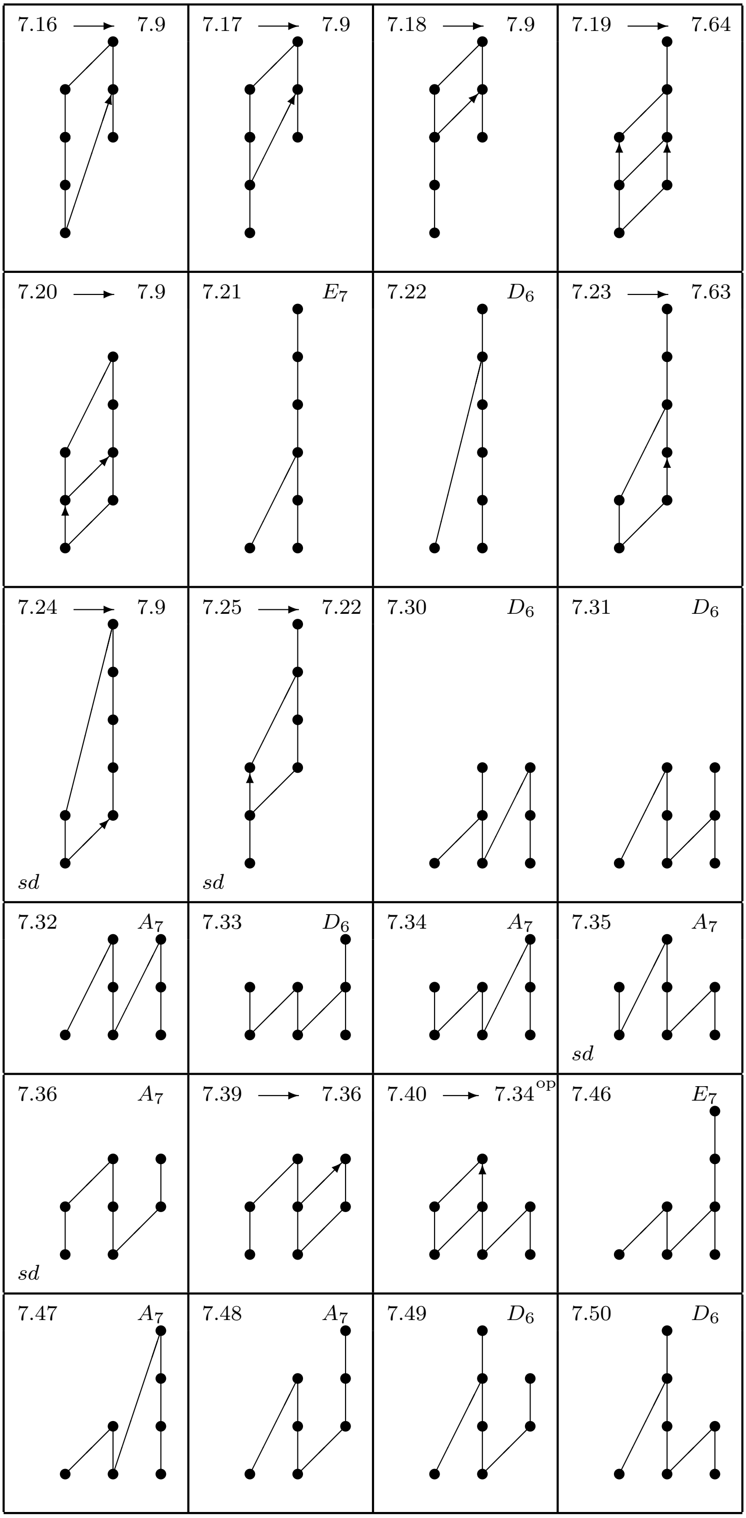
(b) if a graph $p.q$ has a cycle, the top row $p.q \to s.t$ means that the graph $s.t$ is a scanning tree for the graph $p.q$ after removing the edges highlighted by arrows (the directed edges).

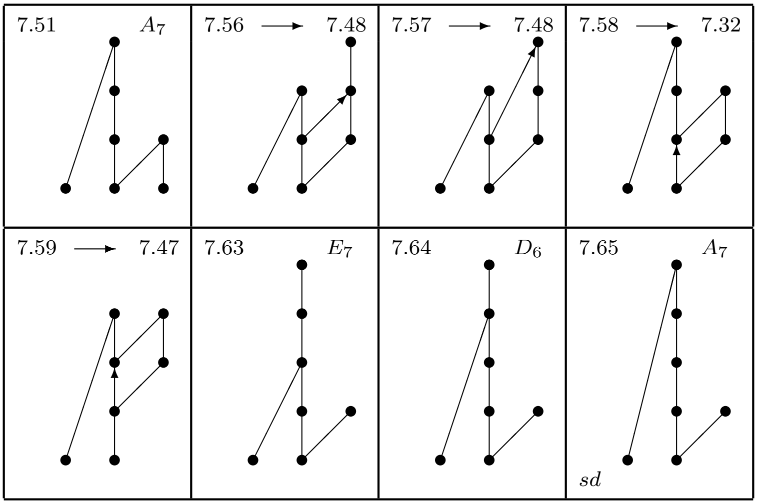
One must also use the mentioned result of P. Gabriel.

Table A of the connected non-serial positive posets up to isomorphism and duality (with additional information).




6.17 A6
sd
6.22 D5
6.23 A6
6.24 A6
6.25 A6
6.27 6.24
6.30 D5
6.31 A6
sd
n = 7
7.1 E7
7.2 D6
7.3 7.2
7.4 7.2
7.5 7.1
7.6 7.51
7.7 E7
7.8 E7
7.9 A7
7.10 7.7
7.11 7.7
7.12 7.7
7.13 7.9
7.14 7.9
7.15 7.8op


7.16 → 7.9
7.17 → 7.9
7.18 → 7.9
7.19 → 7.64
7.20 → 7.9
7.21 E_7
7.22 D_6
7.23 → 7.63
7.24 → 7.9
sd
7.25 → 7.22
sd
7.30 D_6
7.31 D_6
7.32 A_7
7.33 D_6
7.34 A_7
7.35 A_7
sd
7.36 A_7
sd
7.39 → 7.36
7.40 → 7.34^{op}
7.46 E_7
7.47 A_7
7.48 A_7
7.49 D_6
7.50 D_6



Note that in proving we do not consider the posets dual to those located in the table (except for self-dual market as sd), since all the properties of posets that we study are closed under duality.

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