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RESEARCH ARTICLE

Exceptional hereditary curves and real curve orbifolds

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Dedicated to Professor Yuriy Drozd
on the occasion of his 80th birthday

Abstract. In this paper, we elaborate the theory of exceptional hereditary curves over arbitrary fields. In particular, we study the category of equivariant coherent sheaves on a regular projective curve whose quotient curve has genus zero and prove existence of a tilting object in this case. We also establish a link between wallpaper groups and real hereditary curves, providing details to an old observation made by Helmut Lenzing.

1. Introduction

Let $\Bbbk$ be an arbitrary field. The categories $\mathsf{Coh}(\mathbb{X})$ of coherent sheaves on a non-commutative projective hereditary curve $\mathbb{X} = (X, \mathcal{H})$ (where $X = (X, \mathcal{O})$ is a commutative regular projective curve over $\Bbbk$ and $\mathcal{H}$ is a sheaf of hereditary $\mathcal{O}$-orders) provide an important class of $\Bbbk$-linear Ext-finite hereditary categories. In the case when $X = \mathbb{P}^1_{\Bbbk}$ and $\Bbbk = \bar{\Bbbk}$ is algebraically closed, $\mathsf{Coh}(\mathbb{X})$ is equivalent to the category of coherent sheaves on an appropriate weighted projective line of Geigle and Lenzing and admits a tilting object [16]. In particular, there exist a finite dimensional $\Bbbk$-algebra Σ (which belongs to the class of so-called canonical

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algebras [43]) and an exact equivalence of derived categories

$$D^b\bigl(\mathsf{Coh}(\mathbb{X})\bigr) \longrightarrow D^b(\Sigma\text{−}\mathsf{mod}). \qquad (1)$$

In the case of an arbitrary base field $\Bbbk$, a hereditary projective curve $\mathbb{X}$ is called *exceptional* if its derived category $D^b\bigl(\mathsf{Coh}(\mathbb{X})\bigr)$ admits a tilting object. Dropping the assumption $\Bbbk = \bar{\Bbbk}$ makes the theory of such curves significantly richer. Firstly, the underlying commutative curve X can be an arbitrary Brauer–Severi curve. Another reason for complications is caused by the fact that the Brauer group $\mathsf{Br}\bigl(\Bbbk(X)\bigr)$ of the function field $\Bbbk(X)$ of X is no longer zero and arithmetic phenomena start to play an important role in the study of the category $\mathsf{Coh}(\mathbb{X})$. At this point let us mention that the Brauer class $\eta_{\mathbb{X}} = \bigl[\Gamma(X, \mathcal{K} \otimes_{\mathcal{O}} \mathcal{H})\bigr] \in \mathsf{Br}\bigl(\Bbbk(X)\bigr)$ of an exceptional hereditary curve $\mathbb{X}$ can not take arbitrary values. Moreover, $\mathbb{X}$ is a weighted projective line if and only if $X = \mathbb{P}^1_{\Bbbk}$ and $\eta_{\mathbb{X}} = 0$.

The study of exceptional hereditary curves over arbitrary base fields was initiated by Lenzing in [31]. However, the underlying hereditary abelian category $\mathsf{Coh}(\mathbb{X})$ was defined in an implicit way, without an involvement of sheaves of orders. Quoting for example [19, page 415]: "Since there is at present no "geometric" definition available for coherent sheaves on a weighted projective line over an arbitrary field, the formulation of our main result will be somewhat different from the formulation for an algebraically closed field $\Bbbk$."

A classification of $\Bbbk$-linear Ext-finite hereditary abelian categories (see [18, 42] for the case $\Bbbk = \bar{\Bbbk}$ and [19, 35] for an arbitrary $\Bbbk$) allowed one to define exceptional hereditary curves in an "axiomatic way" by providing a list of characterizing properties of the category $\mathsf{Coh}(\mathbb{X})$. In this work, we give a further elaboration of this theory, starting with a ringed space $\mathbb{X} = (X, \mathcal{H})$ itself as a primary object.

The first main result of this paper is Theorem 3.12 which gives a straightforward construction of a tilting complex in the derived category $D^b\bigl(\mathsf{Coh}(\mathbb{X})\bigr)$ for a complete hereditary curve $\mathbb{X}$ of a special type. This allows one to prove a generalization of the equivalence (1) in the case of an arbitrary field $\Bbbk$.

A natural class of examples of exceptional heredirary curves arise from finite group actions. Let Y be a complete regular curve over $\Bbbk$ and $G \subset \mathsf{Aut}_{\Bbbk}(Y)$ be a finite group such that $\mathsf{gcd}\bigl(|G|, \mathsf{char}(\Bbbk)\bigr) = 1$ and the quotient $X = Y/G$ is a curve of genus zero. Then there exists a hereditary curve $\mathbb{X} = Y /\!\!/ G = (X, \mathcal{H})$ such that $\mathsf{Coh}^G(Y) \simeq \mathsf{Coh}(\mathbb{X})$, where $\mathsf{Coh}^G(Y)$ is the category of G-equivariant coherent sheaves on Y. This result is well-known but we elaborate its proof in Proposition 5.3. Then we show that all such $\mathbb{X}$ are exceptional with $\eta_{\mathbb{X}} = 0$ (see Theorem 5.5), extending results of [39] on the case of an arbitrary base field $\Bbbk$; see also [16,23,31].

Wallpaper groups lead to a very interesting class of finite group actions *over* $\mathbb{R}$ on *complex* elliptic curves, what allows one to make a link to the so-called real tubular curves. This striking observation was made by Lenzing many years ago [33], although the underlying details were never published. This gap in the literature is filled by Theorem 6.11. Namely, to any wallpaper group W one can attach a hereditary curve $\mathbb{X}_W$ and in 13 cases out of 17 the corresponding derived category $D^b\big(\mathsf{Coh}(\mathbb{X}_W)\big)$ admits a tilting object, whose endomorphism algebra Σ_W is a tubular canonical algebra and whose type can be read off the orbifold description of the group W; see also Remark 6.12 for a different approach.

2. Hereditary orders

We begin by recalling the notion of a classical order and its properties.

Definition 2.1. Let O be an excellent reduced equidimensional ring of Krull dimension one and $K := \mathsf{Quot}(O)$ be the corresponding total ring of fractions. An O-algebra A is an *O-order* if the following conditions are fulfilled:

- A is a finitely generated torsion free O-module.
- $A_K := K \otimes_O A$ is a semi-simple K-algebra, having finite length as a K-module.

Let O be as above, $O' \subseteq O$ be a subring such that the corresponding ring extension is finite and A be an O-algebra. Then A is an O-order if and only if A is an O'-order. Moreover, if $K' := \mathsf{Quot}(O')$ then we have: $A_K \cong A_{K'}$; see for instance [8, Lemma 2.8].

Definition 2.2. Let A be a ring.

- A is a *classical order* (or just an *order*) provided its center $O = Z(A)$ is a reduced excellent equidimensional ring of Krull dimension one, and A is an O-order.
- Let $K := \mathsf{Quot}(O)$. Then $A_K := K \otimes_O A$ is called the *rational envelope* of A.

- A ring $\widetilde{A}$ is called an *overorder* of A if $A \subseteq \widetilde{A} \subset A_K$ and $\widetilde{A}$ is finitely generated as (a left) A-module.

- An order A is called *maximal* if it has no proper overorders.

Note that for any overorder $\widetilde{A}$ of A, the map $K \otimes_O \widetilde{A} \longrightarrow A_K$ is automatically an isomorphism. Hence, $A_K = \widetilde{A}_K$ and $\widetilde{A}$ is an order over O.

Lemma 2.3. *Let H be an order and $O = Z(H)$ be its center. Then the following results are true.*

(a) Assume that H is hereditary *(i.e.* $\mathsf{gl.dim}(H) = 1$*). Then $O \cong O_1 \times \cdots \times O_r$, where O_i is a Dedekind domain for all $1 \le i \le r$.*

(b) Suppose that O is semilocal. Let J be the Jacobson radical of H and $\widehat{H} = \varprojlim_k \bigl(H/J^k\bigr)$ be the completions of H. Then H is hereditary if and only if $\widehat{H}$ is hereditary.

Proofs of all these results can be for instance found in [40]. □

Let O be a complete discrete valuation ring, A be a maximal order with center O and J be the Jacobson radical of A. We chose an element $w \in J$ such that $J = Aw = wA$; see [40, Theorem 18.7] for the existence of such w. For any sequence of natural numbers $\vec{p} = \bigl(p_1, \dots, p_r\bigr)$, consider the O-algebra

$$H(A,\vec{p}) := \left[\begin{array}{ccc|ccc|c|ccc}
A & \dots & A & J & \dots & J & \dots & J & \dots & J \\
\vdots & \ddots & \vdots & \vdots & & \vdots & & \vdots & & \vdots \\
A & \dots & A & J & \dots & J & \dots & J & \dots & J \\
\hline
A & \dots & A & A & \dots & A & \dots & J & \dots & J \\
\vdots & & \vdots & \vdots & \ddots & \vdots & & \vdots & & \vdots \\
A & \dots & A & A & \dots & A & \dots & J & \dots & J \\
\hline
\vdots & & \vdots & \vdots & & \vdots & \ddots & \vdots & & \vdots \\
\hline
A & \dots & A & A & \dots & A & \dots & A & \dots & A \\
\vdots & & \vdots & \vdots & & \vdots & & \vdots & \ddots & \vdots \\
A & \dots & A & A & \dots & A & \dots & A & \dots & A
\end{array}\right] = \begin{bmatrix}
A & J & \dots & J \\
A & A & \dots & J \\
\vdots & \vdots & \ddots & \vdots \\
A & A & \dots & A
\end{bmatrix}^{(p_1,\dots,p_r)}$$

where the size of the i-th diagonal block is $(p_i \times p_i)$ for each $1 \le i \le r$.

Theorem 2.4. *The following results are true.*

(i) $H(A,\vec{p})$ is a hereditary order, whose center is isomorphic to O.

(ii) Let A' be another maximal order and $\vec{p}\,' \in \mathbb{N}^s$. Then $H(A,\vec{p}) \cong H(A',\vec{p}\,')$ if and only if $A \cong A'$, $r = s$ and $\vec{p}\,'$ is a cyclic shift of $\vec{p}$.

(iii) *Let H be a hereditary order, whose center is isomorphic to O. Then $H \cong H(A, \vec{p})$ for some maximal order A and a vector $\vec{p} \in \mathbb{N}^r$ for some $r \in \mathbb{N}$.*

(iv) *We have the following description of the Jacobson radical of $H = H(A, \vec{p})$:*

$$\mathsf{rad}(H) = \begin{bmatrix} J & J & \dots & J \\ A & J & \dots & J \\ \vdots & \vdots & \ddots & \vdots \\ A & A & \dots & J \end{bmatrix}^{(p_1,\dots,p_r)}.$$

In particular, we have:

$$H/\mathsf{rad}(H) \cong M_{p_1}(D) \times \cdots \times M_{p_r}(D),$$

where $D = A/J$ is the residue skew field of A.

(v) *Let $\vec{e} := (1, \dots, 1) \in \mathbb{N}^r$. Then the orders $H(A, \vec{p})$ and*

$$H_r(A) := H(A, \vec{e}) = \begin{bmatrix} A & J & \dots & J \\ A & A & \dots & J \\ \vdots & \vdots & \ddots & \vdots \\ A & A & \dots & A \end{bmatrix}$$

are Morita equivalent.

Proofs of all these results can be for instance found in [20, 21] as well as in [40]. □

Remark 2.5. In what follows, the hereditary order $H = H(A, \vec{p})$ will be called *standard*. Moreover, the following statements are true.

(i) There are precisely r pairwise non-isomorphic indecomposable projective left H-modules:

$$P_1 = \begin{bmatrix} A \\ A \\ \vdots \\ A \end{bmatrix}^{(p_1,\dots,p_r)} \quad P_2 = \begin{bmatrix} J \\ A \\ \vdots \\ A \end{bmatrix}^{(p_1,\dots,p_r)} \quad \dots \quad P_r = \begin{bmatrix} J \\ J \\ \vdots \\ A \end{bmatrix}^{(p_1,\dots,p_r)}. \tag{2}$$

(ii) Next, there are exactly r pairwise non-isomorphic simple left H-modules $S_1, \dots, S_r$, whose minimal projective resolutions are

$$0 \longrightarrow P_{j+1} \xrightarrow{\varepsilon_j} P_j \longrightarrow S_j \longrightarrow 0 \quad \text{for} \quad 1 \le j \le r. \tag{3}$$

For $1 \le j < r$ the morphism ε_j is just the natural inclusion, whereas $P_{r+1} = P_1$ and ε_r is given by the right multiplication with the chosen generator $w \in J$.

(iii) Let $1 \le i, j \le r$. It is clear that

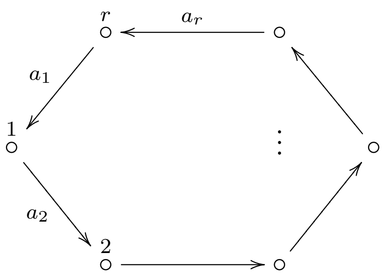
$$\mathsf{Hom}_H(S_i, S_j) \cong \begin{cases} D^\circ & \text{if } i = j, \\ 0 & \text{otherwise,} \end{cases}$$

where D° is the opposite ring of D. Moreover,

$$\mathsf{Ext}^1_H(S_i, S_j) \cong \begin{cases} D^\circ & \text{if } j = i+1, \\ 0 & \text{otherwise,} \end{cases} \tag{4}$$

where $S_{r+1} = S_1$.

(iv) Let $A = \Bbbk[\![z]\!]$. Then $H_r(A)$ is isomorphic to the arrow completion $\widehat{\Bbbk[\vec{C}_r]}$ of the path algebra of the cyclic quiver $\vec{C}_r$:



(5)

Let $A \times A \xrightarrow{\kappa} O$ be the pairing induced by the so-called *reduced trace map* $A \xrightarrow{tr} O$; see [40, Section 9]. It is symmetric and invariant (i.e. $\kappa(a,b) = \kappa(b,a)$ and $\kappa(ab,c) = \kappa(a,bc)$ for any $a,b,c \in A$). Moreover, it defines an isomorphism of $(A$–$A)$-bimodules

$$A \longrightarrow \Omega_A := \mathsf{Hom}_O(A, O),\ a \mapsto \kappa(a, -).$$

As a consequence, we have the following isomorphisms of $(H$–$H)$-bimodules:

$$\Omega = \Omega_H := \mathsf{Hom}_O(H, O) \cong \mathsf{Hom}_A\big(H, \mathsf{Hom}_O(A, O)\big) \cong \mathsf{Hom}_A(H, A).$$

It follows that

$$\Omega \cong \left[\begin{array}{cccc} A & A & \dots & A \\ J^{-1} & A & \dots & A \\ \vdots & \vdots & \ddots & \vdots \\ J^{-1} & J^{-1} & \dots & A \end{array}\right]^{\underline{(p_1,\dots,p_r)}},$$

where $J^{-1} = Aw^{-1} = w^{-1}A$ viewed as a subset of the rational hull of A.

Consider the functor $\tau := \Omega \otimes_H - : H-\mathsf{mod} \longrightarrow H-\mathsf{mod}$. It is clear that

$$\tau(P_1) \cong \begin{bmatrix} A \\ J^{-1} \\ \vdots \\ J^{-1} \end{bmatrix}^{\underline{(p_1,\ldots,p_r)}} \cong \begin{bmatrix} J \\ A \\ \vdots \\ A \end{bmatrix}^{\underline{(p_1,\ldots,p_r)}} = P_2,$$

where the last isomorphism is given by the right multiplication with w. In the same vein, we have: $\tau(P_i) \cong P_{i+1}$ for all $1 \le i \le r$. Note that Ω is projective (hence flat) viewed as a right H-module. It follows that τ is an exact functor. Actually, τ is an auto-equivalence of $H-\mathsf{mod}$; see the discussion below. It follows from (3) that $\tau(S_i) \cong S_{i+1}$ for all $1 \le i \le r$.

3. Exceptional hereditary curves

Let $\Bbbk$ be any field and X be a reduced quasi-projective equidimensional scheme of finite type over $\Bbbk$ of Krull dimension one. Let $X_\circ$ be the set of closed points of X, $\mathcal{O}$ be the structure sheaf of X, $\mathcal{K}$ be its sheaf of rational functions and $\mathbb{K} := \mathcal{K}(X)$ be the ring of rational functions on X. We follow the terminology introduced in [9, Section 7].

Definition 3.1. A non-commutative curve over $\Bbbk$ is a ringed space $\mathbb{X} = (X, \mathcal{R})$, where X is a commutative curve as above and $\mathcal{R}$ is a sheaf of $\mathcal{O}_X$-orders (i.e. $\mathcal{R}(U)$ is an $\mathcal{O}(U)$-order for any open affine subset $U \subseteq X$), which is coherent as a sheaf of $\mathcal{O}_X$-modules. Such $\mathbb{X}$ is called

(a) *central* if $\mathcal{O}_x$ is the center of $\mathcal{R}_x$,

(b) *homogeneous* (also called *regular* in [9]) if the order $\mathcal{R}_x$ is maximal,

(c) *hereditary* if the order $\mathcal{R}_x$ is hereditary

for any $x \in X_\circ$.

Remark 3.2. Without loss of generality one may assume $\mathbb{X} = (X, \mathcal{R})$ to be central; see [9, Remark 2.14]. We call such $\mathbb{X}$ *complete* if X is complete (i.e. integral and proper (hence, projective)) over $\Bbbk$. Then $\mathbb{K}$ is a field and $\mathbb{F}_{\mathbb{X}} := \Gamma(X, \mathcal{K} \otimes_{\mathcal{O}} \mathcal{R})$ is a central simple algebra over $\mathbb{K}$. Let $\eta := [\mathbb{F}_{\mathbb{X}}]$ be the corresponding class in the Brauer group $\mathsf{Br}(\mathbb{K})$ of $\mathbb{K}$.

We shall denote by $g(X)$ the genus of X. From now on, if not otherwise stated, all non-commutative curves over $\Bbbk$ are assumed to be *central* *and complete* and we shall frequently omit the term "non-commutative" when speaking about such $\mathbb{X}$.

If $\mathbb{X} = (X, \mathcal{R})$ is hereditary then X is regular; see Lemma 2.3. Recall the following easy but fundamental fact due to Artin and de Jong [3, Proposition 1.9.1] (see also [46, Proposition 2.9] and [9, Corollary 7.9]).

Theorem 3.3. *Let X be a complete regular curve over $\Bbbk$. Then for any $\eta \in \mathsf{Br}(\mathbb{K})$ there exists a homogeneous curve $\mathbb{X} = (X, \mathcal{R})$ such that $\big[\mathbb{F}_{\mathbb{X}}\big] = \eta$. If $\mathbb{X}' = (X', \mathcal{R}')$ is another homogeneous curve then the following statements are equivalent:*

- *(a) The categories of coherent sheaves $\mathsf{Coh}(\mathbb{X})$ and $\mathsf{Coh}(\mathbb{X}')$ are equivalent.*
- *(b) There exists an isomorphism $X \xrightarrow{f} X'$ such that $[\mathbb{F}_{\mathbb{X}}] = f^*\big([\mathbb{F}_{\mathbb{X}'}]\big) \in \mathsf{Br}(\mathbb{K})$.*

Remark 3.4. In the above theorem, the ringed spaces $\mathbb{X}$ and $\mathbb{X}'$ need not be isomorphic even if we assume $\mathbb{F}$ and $\mathbb{F}'$ to be skew fields; see [9, Remark 7.11] and references therein.

Let $\mathbb{X} = (X, \mathcal{H})$ be a hereditary curve. The full subcategory of finite length objects of $\mathsf{Coh}(\mathbb{X})$ is denoted by $\mathsf{Tor}(\mathbb{X})$. Clearly, it splits into a union of blocks:

$$\mathsf{Tor}(\mathbb{X}) = \bigvee_{x \in X_\circ} \mathsf{Tor}_x(\mathbb{X}), \tag{6}$$

where $\mathsf{Tor}_x(\mathbb{X})$ is equivalent to the category of finite length modules over the hereditary order $\mathcal{H}_x$ for any $x \in X_\circ$.

We denote by $\mathsf{VB}(\mathbb{X})$ the full subcategory of the category $\mathsf{Coh}(\mathbb{X})$ consisting of locally projective objects, i.e. those $\mathcal{E} \in \mathsf{Coh}(\mathbb{X})$ for which each stalk $\mathcal{E}_x$ is projective over $\mathcal{H}_x$ for any $x \in X_\circ$. Similarly to the case of regular commutative curves, one can show that for any $\mathcal{F} \in \mathsf{Coh}(\mathbb{X})$ there exist unique $\mathcal{E} \in \mathsf{VB}(\mathbb{X})$ and $\mathcal{Z} \in \mathsf{Tor}(\mathbb{X})$ such that $\mathcal{F} \cong \mathcal{E} \oplus \mathcal{Z}$.

Consider the Serre quotient category $\mathsf{Coh}(\mathbb{X})/\mathsf{Tor}(\mathbb{X})$. Then the functor

$$\Gamma(X, \mathcal{K} \otimes_{\mathcal{H}} -\,) : \mathsf{Coh}(\mathbb{X})/\mathsf{Tor}(\mathbb{X}) \longrightarrow \mathbb{F}_{\mathbb{X}}-\mathsf{mod}$$

is an equivalence of categories. For any $\mathcal{F} \in \mathsf{Coh}(\mathbb{X})$ we define its rank by the formula

$$\mathsf{rk}(\mathcal{F}) := \mathsf{length}_{\mathbb{F}_{\mathbb{X}}}\big(\Gamma(X, \mathcal{K} \otimes_{\mathcal{H}} \mathcal{F})\big).$$

Objects of $\mathsf{VB}(\mathbb{X})$ of rank one are called *line bundles*, the corresponding full subcategory of $\mathsf{VB}(\mathbb{X})$ is denoted by $\mathsf{Pic}(\mathbb{X})$.

Theorem 3.5. *Let $\mathbb{X} = (X, \mathcal{H})$ be a hereditary curve. Then the following results are true.*

(a) $\mathsf{Coh}(\mathbb{X})$ *is an* Ext*-finite noetherian hereditary* $\Bbbk$*-linear abelian category.*

(b) Let $\Omega = \Omega_{\mathbb{X}} := Hom_X\big(\mathcal{H}, \Omega_X\big)$*, where* Ω_X *is the dualizing sheaf of* X*. Then*

$$\tau := \Omega \otimes_{\mathcal{H}} - : \mathsf{Coh}(\mathbb{X}) \longrightarrow \mathsf{Coh}(\mathbb{X}) \tag{7}$$

is an auto-equivalence of $\mathsf{Coh}(\mathbb{X})$*. It restricts to auto-equivalences of its full subcategories* $\mathsf{VB}(\mathbb{X})$*,* $\mathsf{Tor}(\mathbb{X})$ *as well as* $\mathsf{Tor}_x(\mathbb{X})$ *for any* $x \in X$*.*

(c) Moreover, for any $\mathcal{F}, \mathcal{G} \in \mathsf{Coh}(\mathbb{X})$ *there are bifunctorial isomorphisms*

$$\mathsf{Hom}_{\mathbb{X}}(\mathcal{F}, \mathcal{G}) \cong \mathsf{Ext}^1_{\mathbb{X}}\big(\mathcal{G}, \tau(\mathcal{F})\big)^*. \tag{8}$$

Comment to the proof. Properties of the functor τ follow from much more general results about dualizing complexes and Serre functors; see for example [38, Theorem A.4] and [47, Proposition 6.14].

Remark 3.6. The category of coherent sheaves $\mathsf{Coh}(\mathbb{X})$ on a hereditary curve $\mathbb{X}$ is essentially characterized by the properties listed in Theorem 3.5 above; see [42, Theorem IV.5.2] for the case of an algebraically closed field $\Bbbk$ and [28, 35] for further elaborations in the case of an arbitrary $\Bbbk$.

Definition 3.7. Let X be a complete regular curve over $\Bbbk$. We say that $X_\circ \xrightarrow{\rho} \mathbb{N}$ is a *weight function* if $\rho(x) = 1$ for all but finitely many points $x \in X_\circ$.

Theorem 3.8. *Let X be a complete regular curve over $\Bbbk$,* $\eta \in \mathsf{Br}(\mathbb{K})$ *be any Brauer class and* $X_\circ \xrightarrow{\rho} \mathbb{N}$ *be any weight function. Consider a homogeneous curve* $\mathbb{X} = (X, \mathcal{R})$ *defined by* η *(see Theorem 3.3). Then there exists a hereditary curve* $\mathbb{E} = \mathbb{E}(X, \eta, \rho) = (X, \mathcal{H})$ *having the following properties.*

(a) For any $x \in X_\circ$*, the order* $\widehat{\mathcal{H}}_x$ *is Morita equivalent to the order* $H_{\rho(x)}(\mathcal{R}_x)$*.*

(b) We have: $[\mathbb{F}_{\mathbb{X}}] = \eta$*.*

Let (X', η', ρ') be another datum as above and $\mathbb{E}'$ be a hereditary curve attached to it. Then the categories $\mathsf{Coh}(\mathbb{E})$ *and* $\mathsf{Coh}(\mathbb{E}')$ *are equivalent if and only if there exists an isomorphism $X \xrightarrow{f} X'$ such that $f^*(\eta') = \eta \in$* $\mathsf{Br}(\mathbb{K})$ *and $\rho' f = \rho$.*

Proof can be found in [46, Proposition 2.9]; see also [9, Corollary 7.9]. □

Definition 3.9. A complete non-commutative curve $\mathbb{X}$ over a field $\Bbbk$ is called *exceptional* if its bounded derived category of coherent sheaves $D^b\big(\mathsf{Coh}(\mathbb{X})\big)$ admits a tilting object. Equivalently, there exists a finite-dimensional $\Bbbk$-algebra T and an exact equivalence of triangulated categories $D^b\big(\mathsf{Coh}(\mathbb{X})\big) \longrightarrow D^b(T-\mathsf{mod})$.

Remark 3.10. The concept of an exceptional hereditary non-commutative curve was introduced for the first time by Lenzing in [32, Section 2.5], following an axiomatic characterization of such categories. At this place let us mention that there are various classes of exceptional non-commutative curves which are not hereditary; see for instance [7, 8, 12].

Theorem 3.11. *Let $\mathbb{X} = (X, \mathcal{R})$ be an exceptional homogeneous curve. Then there exists a tilting object $\mathcal{F} \in \mathsf{VB}(\mathbb{X})$ such that*

$$\Lambda := \big(\mathsf{End}_{\mathbb{X}}(\mathcal{F})\big)^{\circ} \cong \begin{pmatrix} \mathbb{f} & \mathbb{w} \\ 0 & \mathbb{g} \end{pmatrix}, \tag{9}$$

where $\mathbb{f}$ and $\mathbb{g}$ are finite dimensional division algebras over $\Bbbk$ and $\mathbb{w}$ is a tame *$(\mathbb{f}$–$\mathbb{g})$-bimodule (this means that $\dim_{\mathbb{f}}(\mathbb{w}) \cdot \dim_{\mathbb{g}}(\mathbb{w}) = 4$; see [13]). Moreover, $g(X) = 0$.*

Comment to the proof. The first part of this theorem is due to Lenzing [32, Theorem 4.5]. The statement $g(X) = 0$ can be deduced from results of [3, Section 4.1]; see also [27]. □

Let (X, η, ρ) be a datum as in Theorem 3.8 with $g(X) = 0$ and $\eta \in \mathsf{Br}(\mathbb{K})$ be *exceptional*. The latter condition means that homogeneous curve $\mathbb{X} = (X, \mathcal{R})$ determined by η is exceptional. Let $\mathcal{F} \in \mathsf{VB}(\mathbb{X})$ be a tilting object from Theorem 3.11 and T be the corresponding tilted algebra (9). Then we have an exact equivalence

$$\mathsf{T} := \mathsf{RHom}_{\mathbb{X}}(\mathcal{F}, -\,) : D^b\big(\mathsf{Coh}(\mathbb{X})\big) \longrightarrow D^b(\Lambda-\mathsf{mod}).$$

Let $\mathfrak{E}_\rho := \big\{x \in X_\circ \,\big|\, \rho(x) \geq 2\big\} = \big\{x_1, \dots, x_t\big\}$ be the *special locus* of ρ. For any $1 \leq i \leq t$, let $\mathcal{S}_i$ be the unique (up to isomorphisms) simple object of the category $\mathsf{Tor}_{x_i}(\mathbb{X})$ and $U_i := \mathsf{Hom}_{\mathbb{X}}(\mathcal{F}, \mathcal{S}_i) \in \Lambda-\mathsf{mod}$ be the corresponding regular left Λ-module. Of course, we have: $\mathsf{T}\big(\mathcal{S}_i[0]\big) \cong U_i[0]$, where

$$\Lambda-\mathsf{mod} \longrightarrow D^b(\Lambda-\mathsf{mod}), M \mapsto M[0] = \big(\dots \longrightarrow 0 \longrightarrow M \longrightarrow 0 \longrightarrow \dots\big)$$

is the standard embedding. For any $1 \le i \le t$, let $A_i := \widehat{\mathcal{R}}_{x_i}$ and $D_i = A_i/\mathsf{rad}(A_i)$. Then

$$D_i^\circ \cong \mathsf{End}_{\mathbb{X}}(\mathcal{S}_i) \cong \mathsf{End}_\Lambda(U_i).$$

Recall that the duality functor $\mathsf{Hom}_{\Bbbk}(-\,,\Bbbk) : \Lambda-\mathsf{mod} \longrightarrow \mathsf{mod}-\Lambda$ is a contravariant equivalence of categories. For any $1 \le i \le t$, consider a $(D_i\text{–}\Lambda)$-bimodule $V_i := \mathsf{Hom}_{\Bbbk}(U_i, \Bbbk)$. In this notation, we put:

$$\Pi := \left[\begin{array}{ccc|c|ccc|c}
D_1 & \dots & D_1 & & 0 & \dots & 0 & V_1 \\
\vdots & \ddots & \vdots & & \vdots & \ddots & \vdots & \vdots \\
0 & \dots & D_1 & & 0 & \dots & 0 & V_1 \\
\hline
\vdots & & \vdots & \ddots & \vdots & & \vdots & \vdots \\
\hline
0 & \dots & 0 & & D_t & \dots & D_t & V_t \\
\vdots & \ddots & \vdots & & \vdots & \ddots & \vdots & \vdots \\
0 & \dots & 0 & & 0 & \dots & D_t & V_t \\
\hline
0 & \dots & 0 & & 0 & \dots & 0 & \Lambda
\end{array}\right], \qquad (10)$$

where each D_i occurs precisely $m_i := \rho(x_i) - 1$ times on the diagonal.

Theorem 3.12. *Let X be a complete regular curve over $\Bbbk$ of genus zero, $\eta \in \mathsf{Br}(\mathbb{K})$ be an exceptional class, $X_\circ \xrightarrow{\rho} \mathbb{N}$ a weight function and $\mathbb{E} = \mathbb{E}(X, \eta, \rho) = (X, \mathcal{H})$ be a hereditary curve attached to this datum (see Theorem 3.8). Then there exists an exact equivalence*

$$D^b\big(\mathsf{Coh}(\mathbb{E})\big) \simeq D^b(\Pi-\mathsf{mod}), \qquad (11)$$

where Π is the $\Bbbk$-algebra given by (10). In other words, the curve $\mathbb{E}$ is exceptional.

Proof. Consider a homogeneous curve $\mathbb{X} = (X, \mathcal{R})$ determined by $\eta \in \mathsf{Br}(\mathbb{K})$. Without loss of generality one may assume that $\mathbb{F} := \Gamma(X, \mathcal{K} \otimes_{\mathcal{O}} \mathcal{R})$ is a skew field. Then there exists $m \in \mathbb{N}$ such that $\Gamma(X, \mathcal{K} \otimes_{\mathcal{O}} \mathcal{H}) \cong M_m(\mathbb{F})$.

For any $1 \le i \le t$ we have an isomorphism of $\Bbbk$-algebras $H_i := \widehat{\mathcal{H}}_{x_i} \cong H(A_i, \vec{p}_i)$, where $A_i = \widehat{\mathcal{R}}_{x_i}$ and $\vec{p}_i \in \mathbb{N}^{\rho(x_i)}$ is some vector. In particular, there are precisely $\rho(x_i) = m_i + 1$ pairwise non-isomorphic simple left H_i–modules $S_i^{(0)}, S_i^{(1)}, \dots, S_i^{(m_i)}$ with a cyclic ordering such that

$$\tau(S_i^{(j)}) \cong S_i^{(j+1)} \text{ for all } 1 \le i \le t \text{ and } 0 \le j \le m_i. \qquad (12)$$

Let $P_i^{(j)}$ be an indecomposable projective left H_i–module defined by (2) such that

$$\mathsf{Hom}_{H_i}\big(P_i^{(j)}, S_i^{(j)}\big) \ne 0.$$

According to [9, Theorem 6.2] there exists $\mathcal{P} \in \mathsf{Pic}(\mathbb{E})$ such that $\widehat{\mathcal{P}}_{x_i} \cong P_i^{(0)}$ for all $1 \le i \le t$. Let $\mathcal{A} := \big(\mathit{End}_{\mathbb{X}}(\mathcal{P})\big)^{\circ}$. It is clear that $\widehat{\mathcal{A}}_x \cong \widehat{\mathcal{R}}_x$ for all $x \in X$ and $\Gamma(X, \mathcal{K} \otimes_{\mathcal{O}} \mathcal{A}) \cong \mathbb{F}$. It follows that $\mathbb{Y} := (X, \mathcal{A})$ is a complete homogeneous curve over $\Bbbk$ and by Theorem 3.3 we have: $\mathsf{Coh}(\mathbb{Y}) \simeq \mathsf{Coh}(\mathbb{X})$. In particular, the curve $\mathbb{Y}$ is exceptional.

Following the terminology of [11, Definition 4.1], the homogeneous curve $\mathbb{Y}$ is a *minor* of the hereditary curve $\mathbb{E}$. We have the following functors:

- $\mathsf{G} := \mathit{Hom}_{\mathcal{H}}(\mathcal{P}, -)$ from $\mathsf{Coh}(\mathbb{E})$ to $\mathsf{Coh}(\mathbb{Y})$.
- $\mathsf{F} := \mathcal{P} \otimes_{\mathcal{A}} -$ from $\mathsf{Coh}(\mathbb{Y})$ to $\mathsf{Coh}(\mathbb{E})$.

Note that (F, G) is an adjoint pair and both functors F and G are exact. The general theory of minors developed in [11, Section 4] leads to the following results.

First note that F is fully faithful. Next, denote by DG and DF the corresponding derived functors between the bounded derived categories of coherent sheaves $D^b\big(\mathsf{Coh}(\mathbb{E})\big)$ and $D^b\big(\mathsf{Coh}(\mathbb{Y})\big)$. Then $(\mathsf{DF}, \mathsf{DG})$ is again an adjoint pair and DF is fully faithful.

Consider the sheaf $\mathcal{I} = \mathcal{I}_{\mathcal{P}}$ of two-sided ideals in $\mathcal{H}$ defined as follows:

$$\mathcal{I} := \mathit{Im}\big(\mathcal{P} \otimes_{\mathcal{A}} \mathcal{P}^{\vee} \xrightarrow{ev} \mathcal{H}\big),$$

where ev is the evaluation morphism. It is clear that $\mathcal{I}_x = \mathcal{H}_x$ for all $x \in X_\circ \setminus \mathfrak{E}_\rho$ and $\overline{\mathcal{H}} := \mathcal{H}/\mathcal{I}$ is supported at $\mathfrak{E}_\rho$. One can check that for any $1 \le i \le t$ we have:

$$\widehat{\mathcal{I}}_{x_i} = \left[\begin{array}{cccc} A_i & J_i & \dots & J_i \\ A_i & J_i & \dots & J_i \\ \vdots & \vdots & \ddots & \vdots \\ A_i & J_i & \dots & J_i \end{array}\right]^{\underline{(p_0^{(i)}, \dots, p_{m_i}^{(i)})}},$$

where $\bigl(p_0^{(i)}, \dots, p_{m_i}^{(i)}\bigr) = \vec{p}_i$. Let $L := \Gamma\bigl(X, \overline{\mathcal{H}}\bigr)$. Then we have: $L \cong L_1 \times \dots \times L_t$, where

$$L_i \cong \overline{\mathcal{H}}_{x_i} \cong \left[\begin{array}{cccc} D_i & 0 & \dots & 0 \\ D_i & D_i & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ D_i & D_i & \dots & D_i \end{array}\right]^{\underline{(p_1^{(i)},\dots,p_{m_i}^{(i)})}}$$

for all $1 \le i \le t$. It is clear, that L_i is Morita equivalent to the algebra

$$\left[\begin{array}{cccc} D_i & 0 & \dots & 0 \\ D_i & D_i & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ D_i & D_i & \dots & D_i \end{array}\right] \subset M_{m_i}(D_i).$$

For any $\mathcal{E}^\bullet \in D^b\bigl(\mathsf{Coh}(\mathbb{E})\bigr)$ we have a distinguished triangle

$$(\mathsf{DF} \circ \mathsf{DG})(\mathcal{E}^\bullet) \xrightarrow{\xi_{\mathcal{E}^\bullet}} \mathcal{E}^\bullet \longrightarrow \mathcal{C}^\bullet \longrightarrow (\mathsf{DF} \circ \mathsf{DG})(\mathcal{E}^\bullet)[1],$$

where $\mathsf{DF} \circ \mathsf{DG} \xrightarrow{\xi} \mathsf{Id}$ is the counit of the adjoint pair $(\mathsf{DF}, \mathsf{DG})$. Since DF is fully faithful, the morphism $\mathsf{DG}(\xi_{\mathcal{E}^\bullet})$ is an isomorphism and, as a consequence, $\mathsf{DG}(\mathcal{C}^\bullet) = 0$. The kernel $\mathsf{Ker}(\mathsf{DG})$ of the functor DG consists of those complexes, whose cohomology is annihilated by the sheaf of ideals $\mathcal{I}$. Note that for any $1 \le i \le t$ the ideal $\widehat{\mathcal{I}}_{x_i}$ is projective (hence, flat), viewed as a right $\widehat{\mathcal{H}}_{x_i}$-module. It implies that $\mathsf{Ker}(\mathsf{DG})$ can be identified with the derived category $D^b(L-\mathsf{mod})$; see [11, Theorem 4.6]. Let $D^b\bigl(L-\mathsf{mod}\bigr) \xrightarrow{\mathsf{I}} D^b\bigl(\mathsf{Coh}(\mathbb{E})\bigr)$ be the corresponding fully faithful embedding, whose essential image is $\mathsf{Ker}(\mathsf{DG})$. Then we get a semi-orthogonal decomposition

$$D^b\bigl(\mathsf{Coh}(\mathbb{E})\bigr) = \bigl\langle \mathsf{Im}(\mathsf{I}), \mathsf{Im}(\mathsf{DF})\bigr\rangle = \Bigl\langle D^b\bigl(L-\mathsf{mod}\bigr), D^b\bigl(\mathsf{Coh}(\mathbb{Y})\bigr)\Bigr\rangle, \quad (13)$$

see [11, Theorem 4.5]. For any $1 \le i \le t$ and $1 \le j \le m_i$ consider the following L_i-modules $Z_i^{(j)}$ given in terms of their projective resolutions

$$\begin{cases} 0 \longrightarrow P_i^{(0)} \longrightarrow P_i^{(1)} \longrightarrow Z_i^{(1)} \longrightarrow 0, \\ 0 \longrightarrow P_i^{(m_i)} \longrightarrow P_i^{(1)} \longrightarrow Z_i^{(2)} \longrightarrow 0, \\ \vdots \\ 0 \longrightarrow P_i^{(2)} \longrightarrow P_i^{(1)} \longrightarrow Z_i^{(m_i)} \longrightarrow 0. \end{cases}$$

Note that $Z_i := \bigoplus\limits_{j=1}^{m_i} Z_i^{(j)}$ is an injective cogenerator of the category $L_i-\mathsf{mod}$. Let $Z := \bigoplus\limits_{i=1}^{t} Z_i$ and $\mathcal{Z}[0] := \mathsf{I}(Z)$, then we have: $\mathcal{Z} \in \mathsf{Tor}(\mathbb{X})$. Next, we set $\widetilde{\mathcal{F}} := \mathsf{F}(\mathcal{F}) \in \mathsf{VB}(\mathbb{E})$, where $\mathcal{F} \in \mathsf{VB}(\mathbb{Y})$ is a tilting object from Theorem 3.11. We claim that

$$\mathcal{X}^\bullet := \mathcal{Z}[-1] \oplus \widetilde{\mathcal{F}}[0] \tag{14}$$

is a tilting object in the derived category $D^b\bigl(\mathsf{Coh}(\mathbb{E})\bigr)$.

The statement that $\mathcal{X}^\bullet$ generates $D^b\bigl(\mathsf{Coh}(\mathbb{E})\bigr)$ follows from existence of a semi-orthogonal decomposition (13) and the facts that Z generates $D^b\bigl(L-\mathsf{mod}\bigr)$ and $\mathcal{F}$ generates $D^b\bigl(\mathsf{Coh}(\mathbb{Y})\bigr)$. Since both functors I and DF are fully faithful and Z and $\mathcal{F}$ are tilting objects in the corresponding derived categories, we have:

$$\mathsf{Ext}^i_{\mathbb{E}}(\mathcal{Z}, \mathcal{Z}) = 0 = \mathsf{Ext}^i_{\mathbb{E}}(\widetilde{\mathcal{F}}, \widetilde{\mathcal{F}})$$

for all $i \ge 1$. Since the functor DF is left adjoint to DG and $\mathsf{DG}(\mathcal{Z}) = 0$, we have:

$$\mathsf{Ext}^i_{\mathbb{E}}(\widetilde{\mathcal{F}}, \mathcal{Z}) \cong \mathsf{Hom}_{D^b(\mathbb{E})}\bigl(\mathsf{DF}(\mathcal{F}), \mathcal{Z}[i]\bigr) \cong \mathsf{Hom}_{D^b(\mathbb{Y})}\bigl(\mathcal{F}, \mathsf{DG}(\mathcal{Z})[i]\bigr) = 0$$

for all $i \in \mathbb{Z}$.

This vanishing is also a consequence of the semi-orthogonal decomposition (13). Finally, for any $i \in \mathbb{Z}$ we have: $\mathsf{Ext}^i_{\mathbb{E}}(\mathcal{Z}, \widetilde{\mathcal{F}}) \cong \Gamma\bigl(X, \mathit{Ext}^i_{\mathcal{H}}(\mathcal{Z}, \widetilde{\mathcal{F}})\bigr)$. Since $\mathcal{Z}$ is torsion and $\widetilde{\mathcal{F}}$ is locally projective, we have: $\mathit{Hom}_{\mathcal{H}}(\mathcal{Z}, \widetilde{\mathcal{F}}) = 0$. As $\mathbb{E}$ is hereditary, we also have: $\mathit{Ext}^i_{\mathcal{H}}(\mathcal{Z}, \widetilde{\mathcal{F}}) = 0$ for all $i \ge 2$. Therefore, $\mathsf{Hom}_{D^b(\mathbb{E})}\bigl(\mathcal{X}^\bullet, \mathcal{X}^\bullet[i]\bigr) = 0$ for $i \ne 0$. We have shown that $\mathcal{X}^\bullet$ is a tilting object in $D^b\bigl(\mathsf{Coh}(\mathbb{E})\bigr)$. Put

$$\Pi := \bigl(\mathsf{End}_{D^b(\mathbb{E})}(\mathcal{X}^\bullet)\bigr)^\circ \cong \begin{pmatrix} \bigl(\mathsf{End}_{\mathbb{E}}(\mathcal{Z})\bigr)^\circ & \mathsf{Ext}^1_{\mathbb{E}}(\mathcal{Z}, \widetilde{\mathcal{F}}) \\ 0 & \bigl(\mathsf{End}_{\mathbb{E}}(\widetilde{\mathcal{F}})\bigr)^\circ \end{pmatrix}. \tag{15}$$

Then the triangulated categories $D^b\bigl(\mathsf{Coh}(\mathbb{E})\bigr)$ and $D^b(\Pi-\mathsf{mod})$ are equivalent; see [22].

Note that $\bigl(\mathsf{End}_{\mathbb{E}}(\widetilde{\mathcal{F}})\bigr)^\circ \cong \bigl(\mathsf{End}_{\mathbb{Y}}(\mathcal{F})\bigr)^\circ = \Lambda$ and $\mathsf{End}_{\mathbb{E}}(\mathcal{Z}) \cong \mathsf{End}_L(Z) \cong \prod\limits_{i=1}^{t} \mathsf{End}_{L_i}(Z_i)$. An easy computation shows that

$$\bigl(\mathsf{End}_{L_i}(Z_i)\bigr)^\circ \cong \begin{bmatrix} D_i & D_i & \dots & D_i \\ 0 & D_i & \dots & D_i \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & D_i \end{bmatrix} \subset M_{m_i}(D_i).$$

Finally, using the Auslander–Reiten duality formula (8) and the fact that (F, G) is an adjoint pair, we get binatural isomorphisms

$$\mathsf{Ext}^1_{\mathbb{E}}\big(\mathcal{Z}, \widetilde{\mathcal{F}}\big) \cong \mathsf{Hom}_{\mathbb{E}}\big(\mathsf{F}(\mathcal{F}), \tau^{-1}(\mathcal{Z})\big)^* \cong \mathsf{Hom}_{\mathbb{Y}}\big(\mathcal{F}, \mathsf{G}\big(\tau^{-1}(\mathcal{Z})\big)\big)^*.$$

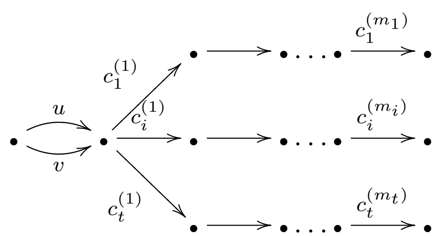
Next, we have: $\mathsf{G}\big(\tau^{-1}(\mathcal{Z})\big) \cong \bigoplus_{i=1}^{t} \mathcal{S}_i^{\oplus m_i}$ where $\mathcal{S}_i$ is the unique (up to isomorphism) simple object of the category $\mathsf{Tor}_{x_i}(\mathbb{Y})$. Hence, we get isomorphisms

$$\mathsf{Hom}_{\mathbb{Y}}\big(\mathcal{F}, \mathsf{G}\big(\tau^{-1}(\mathcal{Z})\big)\big) \cong \bigoplus_{i=1}^{t} \mathsf{Hom}_{\mathbb{Y}}\big(\mathcal{F}, \mathcal{S}_i\big)^{\oplus m_i} \cong \bigoplus_{i=1}^{t} U_i^{\oplus m_i}.$$

Taking the duals over $\Bbbk$, we get a bimodule isomorphism $\mathsf{Ext}^1_{\mathbb{E}}\big(\mathcal{Z}, \widetilde{\mathcal{F}}\big) \cong \bigoplus_{i=1}^{t} V_i^{\oplus m_i}$. This implies that the $\Bbbk$-algebras given by (10) and (15) are isomorphic. □

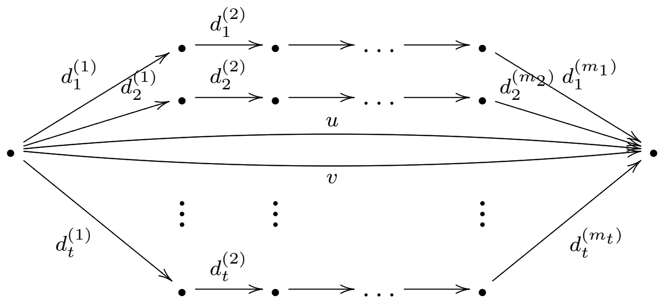
Remark 3.13. Let $\Bbbk$ be an algebraically closed field and $X = \mathbb{P}^1_{\Bbbk}$. We chose homogeneous coordinates $(u : v)$ on X. Then $\mathcal{F} := \mathcal{O}(-1) \oplus \mathcal{O} \in \mathsf{VB}(X)$ is a tilting bundle u and v can be viewed as elements of a distinguished basis of $\mathsf{Hom}_X\big(\mathcal{O}(-1), \mathcal{O}\big)$. Hence, $\Lambda := \big(\mathsf{End}_X(\mathcal{F})\big)^{\circ}$ can be identified with the path algebra of the Kronecker quiver $\bullet \underset{v}{\overset{u}{\rightrightarrows}} \bullet$ and we have an exact equivalence $\mathsf{T} := \mathsf{RHom}(\mathcal{F}, -\,) : D^b\big(\mathsf{Coh}(X)\big) \longrightarrow D^b(\Lambda{-}\mathsf{mod})$.

Let $X_{\circ} \xrightarrow{\rho} \mathbb{N}$ be any weight function and $\mathfrak{E}_{\rho} = \{x_1, \dots, x_t\}$ be the corresponding special locus. We write $x_i = (\alpha_i : \beta_i)$ for all $1 \le i \le t$. Let $\mathcal{S}_i \in \mathsf{Tor}_{x_i}(X)$ be the simple object and $U_i \in \Lambda{-}\mathsf{mod}$ be its image under the equivalence T (i.e. $\mathsf{T}\big(\mathcal{S}_i[0]\big) \cong U_i[0]$). Then $U_i = \Bbbk \underset{\beta_i}{\overset{\alpha_i}{\rightrightarrows}} \Bbbk$ and $\mathsf{End}_{\Lambda}(U_i) \cong \Bbbk$ for all $1 \le i \le t$. Let $\rho(x_i) = m_i + 1$. Then the algebra Π defined by (10) is isomorphic to the path algebra of the following quiver



(16)

subject to the relations $c_i^{(1)}(\beta_i u - \alpha_i v) = 0$ for all $1 \le i \le t$. This is a so-called *squid algebra* (see [4, Section IV.6] and [44, Section 4]). The *canonical algebra* Σ attached to the same datum $\big((x_1, m_1+1), \dots, (x_t, m_t+1)\big)$ is the path algebra of the quiver



(17)

modulo the relations

$$d_i^{(m_i)} \dots d_i^{(1)} = \beta_i u - \alpha_i v \quad \text{for} \quad 1 \le i \le t, \tag{18}$$

see [43]. Then there exists an exact equivalence of triangulated categories

$$D^b\big(\Pi{-}\mathsf{mod}\big) \simeq D^b\big(\Sigma{-}\mathsf{mod}\big), \tag{19}$$

see [43, 44]. For $t \ge 3$ one may without loss of generality assume that $x_1 = (1:0)$, $x_2 = (0:1)$ and $x_3 = (1:1)$. Suppose that $t = 3$. If $l_i = \rho(x_i)$ then we use the notation $\Pi_{(l_1,l_2,l_3)}$ and $\Sigma_{(l_1,l_2,l_3)}$ for the corresponding squid and canonical algebras, respectively.

In the case of an arbitrary field $\Bbbk$, the algebra Π given by (10) is a variation of a squid algebra introduced by Ringel in [44, Section 4].

Remark 3.14. Let us mention that Theorem 3.12 is not entirely original; see e.g. [19, Theorem 2.8 and Theorem 3.4] as well as [27, 34]. However, that works are based on the "axiomatic approach" to non-commutative hereditary curves and analogues of the derived equivalence (11) serve rather as a definition of $\mathbb{E}$ than as its property.

4. Generalities on skew group products

Let A be a ring, G be a finite group and $G \xrightarrow{\phi} \mathsf{Aut}(A)$ be a group homomorphism. For any $g \in G$, let $A \xrightarrow{\phi_g} A$ be the corresponding ring automorphism of A. The associated skew group ring $A\big[G,\phi\big]$ is a free left A–module of rank $|G|$

$$A\big[G,\phi\big] = \Big\{\sum_{g\in G} a_g[g]\,\big|a_g \in A\Big\} \tag{20}$$

equipped with the product given by the rule

$$a[f]\cdot b[g] := a\phi_f(b)[fg] \text{ for any } a,b\in A \text{ and } f,g\in G.$$

Then $A\big[G,\phi\big]$ is a unital ring, whose multiplicative unit element is $1[e]$, where 1 is the unit in A and e is the neutral element of G. Let

$$A^G := \big\{a\in A\,\big|\,\phi_g(a) = a \text{ for all}\, g\in G\big\}$$

be the ring of invariants. If A is commutative then A^G is the center of $A\big[G,\phi\big]$. In what follows, we put $n = |G|$.

Lemma 4.1. *Let L be a field, $G \xrightarrow{\phi} \mathsf{Aut}(A)$ be injective and $K = L^G$. Then we have an isomorphism of K-algebras*

$$L\big[G,\phi\big] \cong M_n(K). \tag{21}$$

Proof. By Artin's Theorem (see e.g. [30, Theorem VI.1.8]) L/K is a finite Galois extension and $G \cong \mathsf{Gal}(L/K)$. Next, we have a group isomorphism

$$H^2(G,L^*) \xrightarrow{\cong} \mathsf{Br}(L/K), [\omega]\mapsto L[G,(\phi,\omega)] \tag{22}$$

see e.g. [14, Theorem 5.6.6]. Here, $L[G,(\phi,\omega)]$ is the crossed product of L and G with respect to the two-cocycle $G\times G \xrightarrow{\omega} L^*$; see [41]. If ω is the trivial cocycle then $L[G,(\phi,\omega)] = L[G,\phi]$. Hence, we have an isomorphism of K-algebras $L[G,\phi] \cong M_m(K)$ for some $m\in\mathbb{N}$. From the dimension reasons it follows that $m = n$. □

Lemma 4.2. *Let $A = A_1\times\cdots\times A_t$, where A_i is a connected ring for all $1\le i\le t$. Let $e_i := (0,\ldots,0,1,0,\ldots,0)$ be the i-th central idempotent of A. Assume that G acts transitively on the set $\big\{e_1,\ldots,e_t\big\}$. Let $A_\diamond = A_1$, $G_\diamond$ be the stabilizer of e_1 and $G_\diamond \xrightarrow{\phi_\diamond} \mathsf{Aut}(A_\diamond)$ be the restricted action. Then the skew group rings $A\big[G,\phi\big]$ and $A_\diamond\big[G_\diamond,\phi_\diamond\big]$ are Morita equivalent.*

Proof. By the transitivity assumption, for any $1 \le i, j \le t$ there exists $g \in G$ such that $\phi_g(e_i) = e_j$. Then we have: $e_j = [g]e_i[g]^{-1}$. Since $1_{A[G,\phi]} = e_1 + \dots + e_t$ and the idempotents $\{e_1, \dots, e_t\}$ are orthogonal and pairwise conjugate, the rings $A[G,\phi]$ and $e_1 A[G,\phi] e_1$ are Morita equivalent. Now we prove that $e_1 A[G,\phi]e_1 \cong A_\diamond\bigl[G_\diamond, \phi_\diamond\bigr]$. Let $\{g_1, \dots, g_s\} \subset G$ be such that $g_1 = e$ and $G = g_1 G_\diamond \sqcup \dots \sqcup g_s G_\diamond$. Consider an arbitrary element $A \ni a = (a_1, \dots, a_t) = a_1 + \dots + a_t$, where $a_i \in A_i$ for $1 \le i \le t$ as well as an arbitrary element $g \in G$. First note that $e_1 a = a_1$. Next, there exist unique $1 \le j \le s$ and $h \in G_\diamond$ such that $g = g_j h$. Then we have: $\phi_h(e_1) = e_1$ and

$$e_1 \cdot a[g]e_1 = a_1[g_j h]e_1 = a_1 \phi_{g_j}(e_1)[g_j h] = \begin{cases} a_1[h] & \text{if } j = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Hence, $e_1 A[G,\phi]e_1 \cong A_\diamond\bigl[G_\diamond, \phi_\diamond\bigr]$, as asserted. □

From now on, let $\Bbbk$ be a field such that $\mathsf{gcd}\bigl(n, \mathsf{char}(\Bbbk)\bigr) = 1$, A be a $\Bbbk$-algebra and $G \xrightarrow{\phi} \mathsf{Aut}_\Bbbk(A)$ be a group homomorphism. Then the skew product $A[G,\phi]$ is a $\Bbbk$-algebra.

Theorem 4.3. *Let A be a commutative connected Dedekind $\Bbbk$-algebra, $O = A^G$ and $H = A[G,\phi]$. Then the following statements are true.*

(i) O is again a Dedekind $\Bbbk$-algebra and $O \subseteq A$ is a finite extension.

(ii) H is a hereditary order, whose center is O and whose rational hull is $M_n(K)$, where K is the quotient field of O.

Proof. For the first statement, see for instance [6, Theorem 4.1]. We conclude that $O \subseteq H$ is finite and H is a torsion free module over O. It follows from Lemma 4.1 that the rational hull of H is $M_n(K)$. Hence H is an order, whose center is O. Finally, it follows from [41, Theorem 1.3] that H is hereditary; see also [10, Corollary 2.7]. □

Lemma 4.4. *Let $\Bbbk$ be algebraically closed, $A = \Bbbk[\![z]\!]$ and $G \xrightarrow{\phi} \mathsf{Aut}_\Bbbk(A)$ be an* injective *group homomorphism. Then the following statement are true:*

(i) The group G is cyclic, i.e. $G \cong \mathbb{Z}_n$.

(ii) We have: $A[G,\phi] \cong H_n(O)$, where $O = \Bbbk[\![z^n]\!]$.

Proof. Let $\mathfrak{m} = (z)$ be the maximal ideal in A. For any $g \in G$, let $\mathfrak{m} / \mathfrak{m}^2 \xrightarrow{\bar{\phi}_g} \mathfrak{m} / \mathfrak{m}^2$ be the induced automorphism. We identify $\mathbb{k}$ with $\mathfrak{m} / \mathfrak{m}^2$ sending 1 to $[z]$. Then $\bar{\phi}_g([z]) = \xi_g[z]$ for some $\xi_g \in \mathbb{k}^*$. Clearly, $\xi_e = 1$ and $\xi_{g_1 g_2} = \xi_{g_1}\xi_{g_2}$ for all $g_1, g_2 \in G$. Next, for any $g \in G$ consider the automorphism of $\mathbb{k}$-algebras

$$A \xrightarrow{\psi_g} A, \; f(z) \mapsto f(\xi_g z).$$

We define $\tau \in \mathsf{Aut}_{\mathbb{k}}(A)$ by the rule $\tau(z) = \frac{1}{n} \sum\limits_{g \in G} \psi_g^{-1}\phi_g(z)$. It is easy to see that $\psi_e = \mathsf{id}$, $\psi_{g_1 g_2} = \psi_{g_1}\psi_{g_2}$ and $\psi_g \tau = \tau\phi_g$ for all $g_1, g_2, g \in G$. Hence, τ can be extended to an isomorphism of $\mathbb{k}$-algebras $A[G, \phi] \xrightarrow{\tau} A[G, \psi]$.

Since ϕ is injective, $G \xrightarrow{\psi} \mathsf{Aut}_{\mathbb{k}}(A)$ is injective, too. It follows that $G \longrightarrow \mathbb{k}^*, g \mapsto \xi_g$ is an injective group homomorphism. Moreover, $\xi_g^n = 1$ for all $g \in G$. This implies that G is a cyclic group of order n.

Let h be a generator of G. Then $\xi = \xi_h$ is a primitive n-th root of 1 in $\mathbb{k}$. For $1 \le k \le n$, let $\zeta_k := \xi^k$ and

$$\varepsilon_k := \frac{1}{n} \sum_{j=0}^{n-1} \zeta_k^j [h]^j \in A[G, \psi]. \tag{23}$$

Then we have:

$$\left\{ \begin{array}{ccc} 1 & = & \varepsilon_1 + \cdots + \varepsilon_n, \\ \varepsilon_k \cdot \varepsilon_l & = & \delta_{kl}\varepsilon_k, \; 1 \le k, l \le n. \end{array} \right.$$

In other words, $\{\varepsilon_1, \dots, \varepsilon_n\}$ is a complete set of primitive idempotents of $A[G, \psi]$. An isomorphism $A[G, \psi] \xrightarrow{\mu} \widehat{\mathbb{k}[\vec{C}_n]}$ is given by the rule:

$$\left\{ \begin{array}{ccc} \varepsilon_k & \overset{\mu}{\mapsto} & e_k, \\ \varepsilon_{k+1} z \varepsilon_k & \overset{\mu}{\mapsto} & a_k, \end{array} \right. \tag{24}$$

where $\widehat{\mathbb{k}[\vec{C}_n]}$ is the complete path algebra of a cyclic quiver $\vec{C}_n$ (see (5)) and $e_k \in \widehat{\mathbb{k}[\vec{C}_n]}$ is the idempotent corresponding to the vertex $1 \le k \le r$. This gives us the desired isomorphisms $A[G, \phi] \cong A[G, \psi] \cong \widehat{\mathbb{k}[\vec{C}_n]} \cong H_n(O)$. □

5. Equivariant coherent sheaves on regular curves and hereditary non-commutative curves

As in the previous section, let G be a finite group of order n and $\Bbbk$ be a field such that $\mathsf{gcd}\big(n, \mathsf{char}(\Bbbk)\big) = 1$. Let Y be a quasi-projective variety over $\Bbbk$ and $G \xrightarrow{\gamma} \mathsf{Aut}_{\Bbbk}(Y)$ be a group homomorphism, which we assume to be injective. Then we have a quasi-projective variety $X := Y/G$ and a canonical projection $Y \xrightarrow{\pi} X$. Let us now recall the corresponding constructions, following [17] (see also [37, Appendix 1]).

We can always find an open affine G-invariant covering $Y = Y_1 \cup \dots \cup Y_m$. For any $1 \le i \le m$ let $A_i = \mathcal{O}_Y(Y_i)$. Then for any $g \in G$ we have a $\Bbbk$-algebra automorphism $A_i \xrightarrow{\gamma^\sharp_{i,g}} A_i$. Moreover, $\gamma^\sharp_{i,e} = \mathrm{id}$ and $\gamma^\sharp_{i,g_1 g_2} = \gamma^\sharp_{i,g_2}\gamma^\sharp_{i,g_1}$ for all $g_1, g_2 \in G$. For any $g \in G$ we put $\widehat{\gamma}^{(i)}_g := \gamma^\sharp_{i,g^{-1}}$. In this way, for any $1 \le i \le m$ we get a group homomorphism $G \xrightarrow{\widehat{\gamma}^{(i)}} \mathsf{Aut}_{\Bbbk}(A_i)$. Let $O_i := A_i^G$ and $X_i = \mathsf{Spec}(O_i)$. By the construction of $X = Y/G$, we have an open affine covering $X = X_1 \cup \dots \cup X_m$ with $Y_i = \pi^{-1}(X_i)$. Moreover, the morphism $Y_i \xrightarrow{\pi_i} X_i$ is dual to the inclusion $O_i \subseteq A_i$. Next, we put $H_i := A_i[G, \widehat{\gamma}^{(i)}]$. In this way we construct a coherent sheaf of $\mathcal{O}_X$-algebras $\mathcal{H}$ on X such that $\mathcal{H}(X_i) = H_i$ for all $1 \le i \le m$.

Proposition 5.1. *The following results are true.*

- *(a) Assume Y is integral. Then for any $1 \le i \le m$, the homomorphism $G \xrightarrow{\widehat{\gamma}^{(i)}} \mathsf{Aut}_{\Bbbk}(A_i)$ is injective.*
- *(b) Let $\mathcal{K}$ be the sheaf of rational functions on X, $\mathbb{K} = \Gamma(X, \mathcal{K})$ be the field of rational functions on X and $\mathbb{F} = \Gamma(X, \mathcal{K} \otimes_{\mathcal{O}} \mathcal{H})$. Then we have an isomorphism of $\mathbb{K}$-algebras $\mathbb{F} \cong M_n(\mathbb{K})$.*
- *(c) Suppose furthermore that Y is a regular curve. Then X is regular as well and $\mathbb{X} = Y /\!\!/ G = (X, \mathcal{H})$ is a non-commutative hereditary curve.*
- *(d) Let Y be as above, $y \in Y_\circ$, $G_\diamond \subseteq G$ be its stabilizer group, $r = |G_\diamond|$, $x := \pi(y) \in X$, $O = \widehat{\mathcal{O}}_x$ and $H = \widehat{\mathcal{H}}_x$. If $\Bbbk$ is algebraically closed then H is Morita equivalent to the standard hereditary order $H_r(O)$.*

Proof. (a) Let $\mathbb{L}$ be the field of rational functions on Y. Then for any

$1 \le i \le m$ we have a commutative diagram

$$\begin{array}{ccc} G & \xrightarrow{\widehat{\gamma}^{(i)}} & \mathsf{Aut}_{\Bbbk}(A_i) \\ {\scriptstyle\gamma}\big\downarrow & & \big\downarrow \\ \mathsf{Aut}_{\Bbbk}(Y) & \hookrightarrow & \mathsf{Aut}_{\Bbbk}(\mathbb{L}) \end{array}$$

where three out four group homomorphisms are known to be injective. Hence, $\widehat{\gamma}^{(i)}$ is injective, too.

(b) Since $\mathbb{K} = \mathbb{L}^G$, this result is a consequence of Lemma 4.1.

(c) This statement follows from Theorem 4.3.

(d) Let $\pi^{-1}(x) = \{y_1, \dots, y_t\}$ with $y = y_1$. For $1 \le i \le t$ we put $B_i = \widehat{\mathcal{O}}_{y_i}$ and $B := B_1 \times \dots \times B_t$. Then we have an injective group homomorphism $G \xrightarrow{\widehat{\gamma}} \mathsf{Aut}_{\Bbbk}(B)$. Moreover, we have an isomorphism of $\Bbbk$-algebras $H \cong B[G, \widehat{\gamma}]$. By Lemma 4.2, the $\Bbbk$-algebras H and $B_\diamond[G_\diamond, \widehat{\gamma}_\diamond]$ are Morita equivalent, where $B_\diamond = B_1$ and $G_\diamond \xrightarrow{\widehat{\gamma}_\diamond} \mathsf{Aut}_{\Bbbk}(B_\diamond)$ is the restricted action. Since γ is injective, $\widehat{\gamma}_\diamond$ is injective, too. If $\Bbbk$ is algebraically closed, then by Lemma 4.4 we have: $G_\diamond \cong \mathbb{Z}_r$ and $B_\diamond[G_\diamond, \widehat{\gamma}_\diamond] \cong H_r(O)$. □

For any $g \in G$ the automorphism $Y \xrightarrow{\gamma_g} Y$ induces a pair of $\Bbbk$-linear auto-equivalences γ_g^* and $\gamma_{g*} : \mathsf{Coh}(Y) \longrightarrow \mathsf{Coh}(Y)$, which assign to a coherent sheaf on Y its inverse (respectively, direct) image sheaf. We have: $\gamma_{g_1 g_2 *} = \gamma_{g_1 *}\gamma_{g_2 *}$ and $\gamma_{g*} = \gamma^*_{g^{-1}}$ for all $g, g_1, g_2 \in G$. Hence, in what follows we shall assume that the canonical isomorphisms of functors $\gamma^*_{g_1 g_2} \xrightarrow{\cong} \gamma^*_{g_2}\gamma^*_{g_1}$ are trivial for all $g_1, g_2 \in G$.

Definition 5.2. The category $\mathsf{Coh}^G(Y)$ of G-equivariant coherent sheaves on Y is defined as follows.

(a) Its objects are tuples $\big(\mathcal{F}, (\alpha_g)_{g\in G}\big)$, where $\mathcal{F} \in \mathsf{Coh}(Y)$ and $\mathcal{F} \xrightarrow{\alpha_g} \gamma_g^*(\mathcal{F})$ is an isomorphism in $\mathsf{Coh}(Y)$ for any $g \in G$ such that $\alpha_e = \mathrm{id}$ and
$$\alpha_{g_2 g_1} = \gamma^*_{g_1}(\alpha_{g_2})\alpha_{g_1} \in \mathsf{Hom}_Y\big(\mathcal{F}, \gamma^*_{g_2 g_1}(\mathcal{F})\big) \tag{25}$$
for any $g_1, g_2 \in G$.

(b) A morphism $\big(\mathcal{F}, (\alpha_g)_{g\in G}\big) \longrightarrow \big(\mathcal{F}', (\alpha'_g)_{g\in G}\big)$ of G-equivariant coherent sheaves is given by a morphism $f \in \mathsf{Hom}_Y(\mathcal{F}, \mathcal{F}')$ such that the diagram

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{\alpha_g} & \gamma_g^*(\mathcal{F}) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \gamma_g^*(f)} \\ \mathcal{F}' & \xrightarrow{\alpha'_g} & \gamma_g^*(\mathcal{F}') \end{array} \tag{26}$$

is commutative for all $g \in G$.

The following result is well-known to the experts. For the reader's convenience, we give below its proof.

Proposition 5.3. *The categories* $\mathsf{Coh}^G(Y)$ *and* $\mathsf{Coh}(\mathbb{X})$ *are equivalent.*

Proof. We first prove the local statement. Let A be a (commutative) $\Bbbk$-algebra and $G \xrightarrow{\phi} \mathsf{Aut}_{\Bbbk}(A)$ be a group homomorphism. Consider a left $A[G,\phi]$-module M. Then M is also a left A-module and for any $g \in G$ we have a $\Bbbk$-linear automorphism

$$M \xrightarrow{\alpha_g} M, x \mapsto [g]x.$$

We have: $\alpha_e = \mathrm{id}$ and $\alpha_{g_1}\alpha_{g_2} = \alpha_{g_1 g_2}$ for all $g_1, g_2 \in G$. Moreover,

$$\alpha_g(ax) = [g]ax = \phi_g(a)[g]x = \phi_g(a)\alpha_g(x)$$

for all $a \in A$ and $x \in M$. Conversely, let M be a left A-module and $\Big(M \xrightarrow{\alpha_g} M\Big)_{g\in G}$ be a family of $\Bbbk$-linear automorphisms such that $\alpha_g(ax) = \phi_g(a)\alpha_g(x)$ for any $a \in A$ and $x \in M$ and such that $\alpha_e = \mathrm{id}$ and $\alpha_{g_1}\alpha_{g_2} = \alpha_{g_1 g_2}$ for all $g_1, g_2 \in G$. Then M can be equipped with a unique structure of a left $A[G,\phi]$-module such that $[g]x = \alpha_g(x)$. In these terms, a morphism $(M, (\alpha_g)_{g\in G}) \xrightarrow{f} (M', (\alpha'_g)_{g\in G})$ of $A[G,\phi]$-modules is a morphism of A-modules $M \xrightarrow{f} M'$ such that

$$\begin{array}{ccc} M & \xrightarrow{\alpha_g} & M \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ M' & \xrightarrow{\alpha'_g} & M' \end{array} \tag{27}$$

is commutative for all $g \in G$.

Let A' be another commutative $\Bbbk$-algebra and $A \xrightarrow{\vartheta} A'$ be a homomorphism of $\Bbbk$-algebras. Let $X' = \mathsf{Spec}(A') \xrightarrow{\nu} X = \mathsf{Spec}(A)$ be the morphism of schemes induced by ϑ. The functors of global sections give equivalences of categories $\mathsf{QCoh}(Y) \simeq A\!-\!\mathsf{Mod}$ and $\mathsf{QCoh}(Y') \simeq A'\!-\!\mathsf{Mod}$. In this identification, for $M \in A\!-\!\mathsf{Mod}$ we have: $\nu^*(M) = A' \otimes_A M$. For any $a \in A$ and $x \in M$ we have: $\vartheta(a) \otimes x = 1 \otimes ax$. Now, consider a special case when $A' = A$. Then we have mutually inverse isomorphisms of A-modules $M \to \nu^*(M), x \mapsto 1 \otimes x$ and $\nu^*(M) \to M$, $a \otimes x \mapsto \vartheta^{-1}(a)x$.

Now, let $\mathcal{F} \in \mathsf{Coh}(Y)$ and $Y = Y_1 \cup \dots \cup Y_m$ be a G-invariant open affine covering. For any $1 \le i \le m$ let $A_i = \mathcal{O}_Y(Y_i)$, $M_i = \mathcal{F}(Y_i)$ and $H_i = A_i[G, \widehat{\gamma}^{(i)}]$. Let $\bigl(\mathcal{F} \xrightarrow{\alpha_g} \gamma_g^*(\mathcal{F})\bigr)_{g \in G}$ be a family of isomorphisms in $\mathsf{Coh}(Y)$ making $\mathcal{F}$ to an G-equivariant sheaf. For each $1 \le i \le m$ $\alpha_g^{(i)} = \alpha_g\big|_{Y_i} : M_i \longrightarrow M_i$ is a $\Bbbk$-linear map satisfying the property $\alpha_g^{(i)}(ax) = \widehat{\gamma}^{(i)}(a)\alpha_g^{(i)}(x)$ for all $a \in A_i$ and $x \in M_i$. The above discussion allows one to equip M_i with a structure of a left H_i-module. Globalizing this correspondence, we equip $\mathcal{F}$ with a structure of a left $\mathcal{H}$-module. Comparing (26) with (27) we conclude that we get a functor $\mathsf{Coh}^G(Y) \xrightarrow{\mathsf{E}} \mathsf{Coh}(\mathbb{X})$. Moreover, the above discussion shows that E is fully faithful and dense, hence an equivalence of categories. □

Summary. Let Y be a complete regular curve over a field $\Bbbk$ and G be a finite group of order n such that $\mathsf{gcd}\bigl(n, \mathsf{char}(\Bbbk)\bigr) = 1$. Let $G \xrightarrow{\gamma} \mathsf{Aut}_{\Bbbk}(Y)$ be an injective group homomorphism, $X = Y/G$ and $\mathbb{X} = Y /\!\!/ G = (X, \mathcal{H})$ be the corresponding non-commutative hereditary curve. Then X is also complete and the following statements are true.

(i) Let $\mathbb{K}$ be the field of rational functions of $\mathbb{X}$. Then the class $[\mathbb{F}_{\mathbb{X}}]$ of $\mathbb{X}$ in the Brauer group $\mathsf{Br}(\mathbb{K})$ is trivial, where $\mathbb{F}_{\mathbb{X}} = \Gamma(X, \mathcal{K} \otimes_{\mathcal{O}} \mathcal{H})$.

(ii) Let $y \in Y_\circ$, $x = \pi(y) \in X$ and G_y be the stabilizer of y. Then $\widehat{H}_x$ is Morita equivalent to $\widehat{\mathcal{O}}_y[G_y, \widehat{\gamma}_y]$.

(iii) If $\Bbbk$ is algebraically closed then $\widehat{\mathcal{O}}_y[G_y, \widehat{\gamma}_y] \cong H_r(\widehat{\mathcal{O}}_x)$, where $r = \bigl|G_y\bigr|$. In particular, the special locus $\mathfrak{E}_{\mathbb{X}}$ of the hereditary curve $\mathbb{X}$ admits the following description. Let $y \in Y_\circ$ be such that $x = \pi(y)$. Then $x \in \mathfrak{E}_{\mathbb{X}}$ if and only if $G_y \ne \{e\}$. Moreover, $\rho(x) = \bigl|G_y\bigr|$.

Remark 5.4. In the case the field $\Bbbk$ is algebraically closed of characteristic zero, the theory of non-commutative hereditary curves was considered in [15] from the perspective of algebraic stacks.

Theorem 5.5. *Let $\Bbbk$ be a field of $\mathsf{char}(\Bbbk) \ne 2$, Y be a complete regular and geometrically integral curve over $\Bbbk$ and $G \subset \mathsf{Aut}_{\Bbbk}(Y)$ be a finite* *group of order n acting faithfully on Y. Assume that* $\mathsf{gcd}\big(n, \mathsf{char}(\Bbbk)\big) = 1$ *and $X = Y/G$ is a curve of genus zero. Then there exists a finite dimensional $\Bbbk$-algebra $\Pi_{(Y,G)}$ such that we have an exact equivalence*

$$D^b\big(\mathsf{Coh}^G(Y)\big) \simeq D^b\big(\Pi_{(Y,G)}\!-\!\mathsf{mod}\big). \tag{28}$$

Proof. Any geometrically integral regular projective curve X over $\Bbbk$ of genus zero is isomorphic to a plane conic

$$X_{(a,b)} := \mathsf{Proj}\big(\Bbbk[x,y,z]/(ax^2+by^2-z^2)\big) \tag{29}$$

for some $a, b \in \Bbbk^*$. Let

$$\Lambda_{(a,b)} = \big\langle i, j \,\big|\, i^2 = a, j^2 = b, ij = -ji\big\rangle_{\Bbbk}$$

be the corresponding generalized quaternion algebra. It was shown in [26] that there exists a tilting bundle $\mathcal{F} \in \mathsf{VB}(X_{(a,b)})$ such that $\big(\mathsf{End}_X(\mathcal{F})\big)^{\circ} \cong \Lambda_{(a,b)}$. The statement is therefore a consequence of Theorem 3.12 and Proposition 5.3. □

Example 5.6. Let $G \subset \mathsf{SL}_2(\mathbb{C})$ be a finite subgroup. Then G acts on the complex projective line $Y = \mathbb{P}^1$ by the fractional-linear transformations. Then $X = Y/G \cong \mathbb{P}^1$. Let $\mathbb{X} = Y /\!\!/ G$ be the corresponding noncommutative hereditary curve. Then there exists a finite-dimensional algebra $\Pi_{(\mathbb{P}^1,G)}$ of the form (16) such that

$$D^b\big(\mathsf{Coh}^G(Y)\big) \simeq D^b\big(\mathsf{Coh}(\mathbb{X})\big) \simeq D^b\big(\Pi_{(\mathbb{P}^1,G)}\!-\!\mathsf{mod}\big).$$

Up to a conjugation, a classification of finite subgroups of $\mathsf{SL}_2(\mathbb{C})$ is well-known; see for instance [25]. In all the cases, the cardinality of the exceptional set $\mathfrak{E}_{\mathbb{X}}$ is either two or three. The group $\mathsf{Aut}_{\mathbb{C}}(\mathbb{P}^1)$ acts transitively on the set of triples on distinct points of $\mathbb{P}^1$. In the case of two special points, we may assume that $\mathfrak{E}_{\mathbb{X}} = \big\{(0:1),(1:0)\big\}$. In the case of three special points, we may assume that $\mathfrak{E}_{\mathbb{X}} = \big\{(0:1),(1:0),(1:1)\big\}$. Therefore, to define $\mathbb{X}$, it is sufficient to specify the sequence (a,b,c) of orders on non-trivial stabilizers of the G-action on $\mathbb{P}^1$ (with $a \le b \le c$ and allowing $a = 1$ in the case there are only two special points). The corresponding hereditary curve $\mathbb{X}$ will be therefore denoted by $\mathbb{P}^1_{(a,b,c)}$. The following cases can occur.

(a) $G \cong \mathbb{Z}_n$ with $n \ge 2$. The corresponding weight sequence is (n,n).

(b) $G \cong \mathbb{D}_n$ is a binary dihedral group with $n \geq 2$. The corresponding weight sequence is $(2, 2, n)$.

(c) G is a binary tetrahedral, octahedral or icosahedral group. The corresponding weight sequences are $(2, 3, 3)$, $(2, 3, 4)$ and $(2, 3, 5)$, respectively.

On the other hand, the simply-laced Dynkin diagrams are parametrized by the triples $(a, b, c) \in \mathbb{N}^3$ such that

$$a \leq b \leq c \quad \text{and} \quad \frac{1}{a} + \frac{1}{b} + \frac{1}{c} > 1.$$

Hence, we may write $\Pi_{(\mathbb{P}^1, G)} = \Pi_{(a,b,c)}$. On the other hand, let $\Gamma_{(a,b,c)}$ be the path algebra of the corresponding Euclidean quiver. Then there exists an exact equivalence of triangulated categories $D^b\bigl(\Pi_{(a,b,c)}-\mathsf{mod}\bigr) \simeq D^b\bigl(\Gamma_{(a,b,c)}-\mathsf{mod}\bigr)$, see [43, Section 4.3] and [45, Section XII.1]. Hence, there exists an exact equivalence

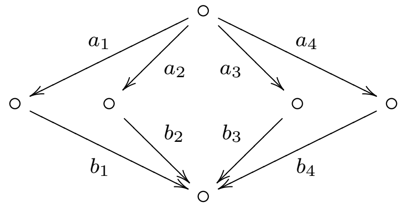
$$D^b\bigl(\mathsf{Coh}(\mathbb{P}^1_{(a,b,c)})\bigr) \simeq D^b\bigl(\Gamma_{(a,b,c)}-\mathsf{mod}\bigr).$$

This striking observation was made for the first time by Lenzing in [31]. Later it led to a development of the theory of weighted projective lines of Geigle and Lenzing in [16]. An elaboration of the equivalence $D^b\bigl(\mathsf{Coh}^G(\mathbb{P}^1)\bigr) \simeq D^b\bigl(\Pi_{(\mathbb{P}^1,G)}-\mathsf{mod}\bigr)$ in the framework of genuine equivariant coherent sheaves on $\mathbb{P}^1$ can be found in [23, 39].

Example 5.7. Let $\Bbbk$ be a field of $\mathsf{char}(\Bbbk) \neq 2$, $\lambda \in \Bbbk^* \setminus \{1\}$ and

$$Y_\lambda = \mathsf{Proj}\bigl(\Bbbk[x, y, z]/(zy^2 - x(x - z)(x - \lambda z))\bigr)$$

be an elliptic curve over $\Bbbk$. Then $G = \bigl\langle \imath \,\big|\, \imath^2 = e \bigr\rangle \cong \mathbb{Z}_2$ acts on Y_λ by the rule $(x : y : z) \stackrel{\imath}{\mapsto} (x : -y : z)$. There are precisely four points of Y_λ with non-trivial stabilizers: $(0 : 0 : 1)$, $(0 : 1 : 0)$, $(1 : 0 : 1)$ and $(\lambda : 0 : 1)$. Next, we have: $X = Y_\lambda/G \cong \mathbb{P}^1_{\Bbbk}$. Let $Y_\lambda \xrightarrow{\pi} X$ be the canonical projection. One can choose homogeneous coordinates on X so that the image of the set of four ramification points of π is $\mathfrak{E} = \bigl\{(0 : 1), (1 : 0), (1 : 1), (\lambda : 1)\bigr\}$. For any $x \in \mathfrak{E}$ we have $\rho(x) = 2$. Let Σ_λ be the tubular canonical algebra of type $((2, 2, 2, 2); \lambda)$ [44], i.e. the path algebra of the following quiver



(30)

modulo the relations $b_1a_1 - b_2a_2 = b_3a_3$ and $b_1a_1 - \lambda b_2a_2 = b_4a_4$. An exact equivalence of triangulated categories

$$D^b\big(\mathsf{Coh}^G(Y_\lambda)\big) \longrightarrow D^b(\Sigma_\lambda\text{−}\mathsf{mod}) \tag{31}$$

was for the first time discovered by Geigle and Lenzing; see [16, Example 5.8]. The algebra Σ_λ is derived-equivalent to the squid algebra (16) of the same type $((2,2,2,2);\lambda)$ (see [43, 44]), which is of course consistent with Theorem 5.5.

Example 5.8. Let $\Bbbk = \mathbb{C}$. Consider the following finite group actions on the following complex elliptic curves.
(I) Let $Y = \mathsf{Proj}\big(\Bbbk[x,y,z]/(zy^2 - x^3 - z^3)\big)$ and $G = \big\langle \varrho \,\big|\, \varrho^6 = e \big\rangle \cong \mathbb{Z}_6$. Then G acts on Y by the rule $\varrho(x : y : z) = (\xi x : -y : z)$, where $\xi = \exp\left(\dfrac{2\pi i}{3}\right)$ and $Y/G \cong \mathbb{P}^1$. Moreover,

(a) The stabilizer of $(-1 : 0 : 1)$ is $\mathbb{Z}_2$.

(b) The stabilizer of $(0 : 1 : 1)$ is $\mathbb{Z}_3$.

(c) The stabilizer of $(0 : 1 : 0)$ is $\mathbb{Z}_6$.

Combining the exact equivalences of triangulated categories (28) and (19) we get

$$D^b\big(\mathsf{Coh}^G(Y)\big) \simeq D^b\big(\Pi_{(2,3,6)}\text{−}\mathsf{mod}\big) \simeq D^b\big(\Sigma_{(2,3,6)}\text{−}\mathsf{mod}\big),$$

where $\Pi_{(2,3,6)}$ and $\Sigma_{(2,3,6)}$ are the squid and canonical algebras of type $(2,3,6)$, respectively.
(II) Next, let $\widetilde{\varrho} = \varrho^4$. Consider the subgroup $\mathbb{Z}_3 \cong N = \langle \widetilde{\varrho} \rangle \subset G$. Then N acts on Y by the rule $\widetilde{\varrho}(x : y : z) = (\xi x : y : z)$ Again, we have $Y/N \cong \mathbb{P}^1$. However, this time the stabilizer of the point $(-1 : 0 : 1)$ is trivial, whereas $(0 : 1 : 1)$ and $(0 : -1 : 1)$ belong to different orbits. The stabilizer of each point $(0 : 1 : 1)$, $(0 : -1 : 1)$ and $(0 : 1 : 0)$ is the

group N itself. Therefore, we have exact equivalences of triangulated categories

$$D^b\big(\mathsf{Coh}^N(Y)\big) \simeq D^b\big(\Pi_{(3,3,3)}-\mathsf{mod}\big) \simeq D^b\big(\Sigma_{(3,3,3)}-\mathsf{mod}\big).$$

(III) Now, let $Y = \mathsf{Proj}\big(\Bbbk[x,y,z]/(zy^2 - x^3 + xz^2)\big)$ and $G = \langle \varrho \,|\, \varrho^4 = e\rangle \cong \mathbb{Z}_4$. Then G acts on Y by the rule $\varrho(x:y:z) = (-x:iy:z)$ and $Y/G \cong \mathbb{P}^1$. The stabilizer of the point $(1:0:1)$ is $\mathbb{Z}_2$, whereas the stabilizer of $(0:0:1)$ and $(0:1:0)$ is the group G itself. Therefore, we have exact equivalences of triangulated categories

$$D^b\big(\mathsf{Coh}^G(Y)\big) \simeq D^b\big(\Pi_{(2,4,4)}-\mathsf{mod}\big) \simeq D^b\big(\Sigma_{(2,4,4)}-\mathsf{mod}\big).$$

6. Tilting on real curve orbifolds

In this section, we shall discuss some interesting and natural actions *over* $\mathbb{R}$ on *complex* projective curves. Do this, we begin with the local case.

Proposition 6.1. *Let G be a finite group, $A = \mathbb{C}[\![z]\!]$, $\mathfrak{m} = (z)$ and $G \xrightarrow{\phi} \mathsf{Aut}_{\mathbb{R}}(A)$ be an injective group homomorphism. Then the following two cases can occur.*

(a) For any $g \in G$ the homomorphism $A \xrightarrow{\phi_g} A$ is $\mathbb{C}$-linear. Then $G = \langle \varrho \,|\, \varrho^n = e\rangle$ is a cyclic group and there exists another choice of a local parameter $w \in \mathfrak{m}$ such that $\phi_\varrho(w) = \xi w$, where $\xi = \exp\left(\dfrac{2\pi i}{n}\right)$.

(b) Otherwise,

$$G \cong D_n = \big\langle \sigma, \varrho \,\big|\, \sigma^2 = e = \varrho^n, \sigma\varrho\sigma^{-1} = \varrho^{-1}\big\rangle \tag{32}$$

is a dihedral group for some $n \in \mathbb{N}$. Moreover, there exists a choice of a local parameter $w \in \mathfrak{m}$ such that

$$\begin{cases} \phi_\sigma(\alpha) = \bar{\alpha} \text{ for } \alpha \in \mathbb{C} \text{ and } \phi_\sigma(w) = w, \\ \phi_\varrho(\alpha) = \alpha \text{ for } \alpha \in \mathbb{C} \text{ and } \phi_\varrho(w) = \xi w, \end{cases} \tag{33}$$

where $\xi = \exp\left(\dfrac{2\pi i}{n}\right)$.

Proof. First note that $\{a \in A \,|\, a^2+1=0\} = \{i, -i\}$. Since for any $g \in G$ the map $A \xrightarrow{\phi_g} A$ is an automorphism of $\mathbb{R}$-algebras, we conclude that $\phi_g(i) = \pm i$. Hence, any ϕ_g is either $\mathbb{C}$-linear or $\mathbb{C}$-antilinear. We put

$$N := \{g \in G \,|\, \phi_g \text{ is } \mathbb{C}\text{-linear}\}.$$

By Lemma 4.4 we have: $N = \langle \varrho \,|\, \varrho^n = e\rangle \cong \mathbb{Z}_n$ for some $\varrho \in N$ and $n = |N|$. Moreover, there exists a local parameter $w \in \mathfrak{m}$ such that $\phi_\varrho(w) = \xi w$, where $\xi = \exp\left(\frac{2\pi i}{n}\right)$. The same proof allows one to construct $w \in \mathfrak{m}$ such that $\phi_g(w) = \xi_g w$ for any $g \in G$, where $\xi_g \in \mathbb{C}^*$.

If $N = G$ then we are done and have the case (a). Now assume that there exists $\sigma \in G \setminus N$. Then $\sigma^2 \in N$ and $\phi_\sigma(\alpha) = \bar{\alpha}$ for any $\alpha \in \mathbb{C}$. Moreover, for any $g \in G \setminus N$ we have: $g\sigma \in N$. Hence, the elements ϱ and σ generate the group G.

We know that $\phi_\sigma(w) = \alpha w$ for some $\alpha \in \mathbb{C}$ such that $|\alpha|^2 = 1$. Let $\zeta \in \mathbb{C}^*$ be such that $\zeta^2 = \alpha$. Then $\phi_\sigma(\zeta w) = \bar{\zeta}\alpha w = \zeta w$. Replacing w by ζw we obtain:

$$\begin{cases} \phi_\varrho(\alpha) = \alpha \text{ for } \alpha \in \mathbb{C} \text{ and } \phi_\varrho(w) = \xi w, \\ \phi_\sigma(\alpha) = \bar{\alpha} \text{ for } \alpha \in \mathbb{C} \text{ and } \phi_\sigma(w) = w. \end{cases}$$

The last formula implies that $\phi_{\sigma^2} = \mathrm{id}$. Since ϕ is injective, we conclude that $\sigma^2 = e$. Analogously, we have $\phi_{\sigma\varrho} = \phi_{\varrho^{-1}\sigma}$, hence $\sigma\varrho = \varrho^{-1}\sigma$ and G is a dihedral group. □

Lemma 6.2. *Let $G = D_n$ be the dihedral group given by the presentation (32), $N = \langle\varrho\rangle \cong \mathbb{Z}_n$ and $C = \langle\sigma\rangle \cong \mathbb{Z}_2$. Let A be a ring and $G \xrightarrow{\phi} \mathsf{Aut}(A)$ be a group homomorphism. Then the following results are true.*

(a) We have a group homomorphism $C \xrightarrow{\psi} \mathsf{Aut}\big(A[N,\phi]\big)$, where

$$\psi_\sigma\big(a[h]\big) = \phi_\sigma(a)[h^{-1}] \text{ for any } a \in A, h \in N. \tag{34}$$

(b) There is a ring isomorphism

$$\begin{gathered} \big(A[N,\phi]\big)[C,\psi] \cong A[G,\phi],\ (a[h])\{\sigma^m\} \mapsto a[h\sigma^m] \\ \text{for any } a \in A, h \in N \text{ and } m \in \mathbb{N}. \end{gathered} \tag{35}$$

Moreover, if $\Bbbk$ is a field and G acts on A by $\Bbbk$-algebra automorphisms then the action ψ is also $\Bbbk$-linear and (35) is an isomorphism of $\Bbbk$-algebras.

Comment to the proof. Both results can be verified by a straightforward computation and are therefore left to an interested reader as an exercise. □

Proposition 6.3. *For any $n \in \mathbb{N}$, let $G = D_n$ be the corresponding dihedral group acting on $A = \mathbb{C}[\![z]\!]$ by $\mathbb{R}$-algebra homomorphisms given by the formula (33). Then we have an isomorphism of $\mathbb{R}$-algebras*

$$A[G,\phi] \cong M_2\big(H_n(O)\big), \tag{36}$$

where $O = \mathbb{R}[\![t^n]\!]$.

Proof. By Lemma 6.2 we have: $A[G,\phi] \cong \big(A[N,\phi]\big)[C,\psi]$. Recall that we have an isomorphism of $\mathbb{C}$-algebras $A[N,\phi] \xrightarrow{\mu} \widehat{\mathbb{C}\big[\vec{C}_n\big]}$ given by the formula (24). For any $\zeta \in \mathbb{C}^*$ with $|\zeta| = 1$ we have: $\psi_\sigma(\zeta) = \bar{\zeta} = \zeta^{-1}$. For all $h \in N$ we have: $\psi_\sigma([h]) = \big[h^{-1}\big]$. Hence,

$$\psi_\sigma\big(\varepsilon_k\big) = \psi_\sigma\left(\frac{1}{n}\sum_{j=0}^{n-1}\zeta_k^j[\varrho^j]\right) = \sum_{j=0}^{n-1}\zeta_k^{-j}[\varrho^{-j}] = \varepsilon_k$$

for any $1 \le k \le n$. It follows that the induced action $\widehat{\mathbb{C}\big[\vec{C}_n\big]} \xrightarrow{\psi_\sigma} \widehat{\mathbb{C}\big[\vec{C}_n\big]}$ is given by the complex conjugation.

According to Lemma 4.1 we have: $\mathbb{C}[C,\psi] \cong M_2(\mathbb{R})$, where $\mathbb{C} \xrightarrow{\psi_\sigma} \mathbb{C}$, $\alpha \mapsto \bar{\alpha}$ is the complex conjugation. As a consequence, we get isomorphisms of $\mathbb{R}$-algebras

$$A[G,\phi] \cong \widehat{\mathbb{C}\big[\vec{C}_n\big]}\,[C,\psi] \cong M_2\big(H_n(O)\big),$$

what proves the statement. □

Definition 6.4. Let Y be a complete regular curve over $\mathbb{C}$, which we view as a scheme over $\mathbb{R}$. Let $G \subseteq \mathsf{Aut}_{\mathbb{R}}(Y)$ be a finite subgroup and $Y \xrightarrow{\pi} Y/G =: X$ be the canonical projection. For $y \in Y_\circ$ let $N \subseteq G$ be the corresponding stabilizer group, $A = \widehat{\mathcal{O}}_y$ and $x = \pi(y)$. We suppose that $|N| \ge 2$.

(a) Assume that N acts on A by $\mathbb{C}$-linear automorphisms. Then we say that $x \in X$ has type n for $n = |N|$ (note that according to Proposition 6.1 we have $G \cong \mathbb{Z}_n$).

(b) Assume that N contains an element which acts as the complex conjugation on A. Then $N \cong D_n$ for some $n \in N$ (see again Proposition 6.1) and we say that x has type $\bar{n}$ provided $n \geq 2$.

Remark 6.5. In the notation of Definition 6.4, let $\mathbb{X} = Y /\!\!/ G = (X, \mathcal{H})$ be the corresponding non-commutative hereditary curve. Then points of X of types n and $\bar{n}$ for $n \in \mathbb{N}_{\geq 2}$ are precisely those ones for which the order $\widehat{\mathcal{H}}_x$ is not maximal; see Proposition 6.3.

Remark 6.6. There are precisely three pairwise non-isomorphic real projective curves of genus zero:

(a) The real projective line $X_{\mathsf{re}} = \mathbb{P}^1_{\mathbb{R}}$. The corresponding tame bimodule Λ_{re} (see (9)) is the path algebra of the Kronecker quiver:

$$\Lambda_{\mathsf{re}} = \mathbb{R}\Big[\bullet \rightrightarrows \bullet\Big] \cong \begin{pmatrix} \mathbb{R} & \mathbb{R} \oplus \mathbb{R} \\ 0 & \mathbb{R} \end{pmatrix}.$$

(b) The complex projective line $X_{\mathsf{co}} = \mathbb{P}^1_{\mathbb{C}}$. The corresponding tame bimodule Λ_{co} is the path algebra of the Kronecker quiver over $\mathbb{C}$:

$$\Lambda_{\mathsf{co}} = \mathbb{C}\Big[\bullet \rightrightarrows \bullet\Big] \cong \begin{pmatrix} \mathbb{C} & \mathbb{C} \oplus \mathbb{C} \\ 0 & \mathbb{C} \end{pmatrix}.$$

(c) The real conic $X_{\mathsf{qt}} = \mathsf{Proj}\big(\mathbb{R}[x, y, z]/(x^2 + y^2 + z^2)\big)$. The corresponding tame bimodule is

$$\Lambda_{\mathsf{qt}} = \begin{pmatrix} \mathbb{R} & \mathbb{H} \\ 0 & \mathbb{H} \end{pmatrix},$$

see [32, Proposition 7.5].

Let Y' be a complete geometrically integral regular curve over $\mathbb{R}$ and

$$Y = \mathsf{Spec}\,(\mathbb{C}) \times_{\mathsf{Spec}(\mathbb{R})} Y'.$$

Then the Galois group $\mathsf{Gal}(\mathbb{C}/\mathbb{R}) = \langle \sigma \,\big|\, \sigma^2 = e \rangle$ canonically acts on Y viewed as a scheme over $\mathbb{R}$. In all examples below σ acts as the complex conjugation.

Analogously to Example 5.7 and Example 5.8, we can consider finite group actions on *complex* elliptic curves viewed as schemes over $\mathbb{R}$.

Example 6.7. Let $Y_\lambda = \mathsf{Proj}\big(\mathbb{C}[x,y,z]/(y^2 z - (x-\lambda z)(x^2+z^2))\big)$ for some $\lambda \in \mathbb{R}$. Then the dihedral group $G = D_2 = \langle \sigma, \varrho \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ acts on Y_λ by the rule $(x:y:z) \overset{\varrho}{\mapsto} (x:-y:z)$. The fixed points of this action are $(\lambda:0:1)$, $(0:i:1)$ and $(0:1:0)$ (note that σ permutes $(0:i:1)$ and $(0:-i:1)$). The stabilizer of $(\lambda:0:1)$ and $(0:1:0)$ is the group G itself, whereas the stabilizer of $(0:i:1)$ is $\langle \varrho \rangle \cong \mathbb{Z}_2$.

We have: $Y_\lambda/G \cong X_{\mathsf{re}}$. Moreover, one can naturally choose homogeneous coordinates $(u:v)$ on $X_{\mathsf{re}} = \mathsf{Proj}\big(\mathbb{R}[u,v]\big)$ such that for the canonical projection $Y_\lambda \xrightarrow{\pi} X_{\mathsf{re}}$ we have: $\pi(\lambda:0:1) = (\lambda:1)$ and $\pi(0:1:0) = (1:0)$. The point $o = \pi(i:0:1) \in X_{\mathsf{re}}$ corresponds to the homogeneous ideal $u^2+v^2 \in \mathbb{R}[u,v]$.

The above discussion shows that the corresponding non-commutative hereditary curve $\mathbb{X}$ is of type $\big(X_{\mathsf{re}}, (2, \bar{2}, \bar{2})\big)$. More precisely, $\mathbb{X}$ has

(a) one special complex point o of weight 2;

(b) two special real points $(\lambda:1)$ and $(1:0)$ of weight 2.

We have an exact equivalence of triangulated categories

$$D^b\big(\mathsf{Coh}^G(Y_\lambda)\big) \longrightarrow D^b(\Pi_{Y_\lambda,G}-\mathsf{mod})$$

for an appropriate squid algebra $\Pi_{Y_\lambda,G}$ of the form (10).

Example 6.8. Let $Y = \mathsf{Proj}\,\big(\mathbb{C}[x,y,z]/(zy^2 - x^3 - z^3)\big)$ and $G = \langle \sigma, \varrho \rangle \cong D_6$. Then G acts on Y by the rule $\varrho(x:y:z) = (\xi x : -y : z)$, where $\xi = \exp\left(\dfrac{2\pi i}{3}\right)$. The special orbits of the G-action are those of

(a) the point $(-1:0:1)$, whose stabilizer is D_2;

(b) the point $(0:1:1)$, whose stabilizer is D_3;

(c) the point $(0:1:0)$, whose stabilizer is D_6.

The corresponding hereditary curve $\mathbb{X}$ has type $\big(X_{\mathsf{re}}, (\bar{2}, \bar{3}, \bar{6})\big)$. Since the group $\mathsf{Aut}_{\mathbb{R}}(X_{\mathsf{re}})$ acts transitively on triples of distinct closed real points of X_{re}, we may assume that the special points of $\mathbb{X}$ are $(0:1)$, $(1:0)$ and $(1:1)$, respectively.

Now, let $\widetilde{\varrho} = \varrho^4$. Consider the subgroup $D_3 \cong N = \langle \sigma, \widetilde{\varrho} \rangle \subset G$. Then N acts on Y by the rule $\widetilde{\varrho}(x:y:z) = (\xi x : y : z)$. Again, we have $Y/N \cong X_{\mathsf{re}}$. The points $(0:1:1)$, $(0:-1:1)$ and $(0:1:0)$ are stabilized by N. Hence, the corresponding hereditary curve $\mathbb{X}$ hat type $\big(X_{\mathsf{re}}, (\bar{3}, \bar{3}, \bar{3})\big)$.

Example 6.9. Consider now $Y = \mathsf{Proj}\big(\mathbb{C}[x,y,z]/(zy^2 - x^3 + z^3)\big)$ and $D_3 \cong N = \langle \sigma, \widetilde{\varrho}\rangle$, where $\widetilde{\varrho}(x:y:z) = (\xi x : y : z)$ for $\xi = \exp\left(\dfrac{2\pi i}{3}\right)$. Again, we have $Y/N \cong X_{\mathsf{re}}$. However, this time $\sigma(0:i:1) = (0:-i:1)$. As a consequence, we now have only two special orbits of the N-action on Y:

(a) those of $(0:i:1)$ whose stabilizer is $\mathbb{Z}_3$;

(b) those of $(0:1:0)$ whose stabilizer is D_3.

As a consequence, the corresponding hereditary curve $\mathbb{X}$ has type $\big(X_{\mathsf{re}}, (3,\bar{3})\big)$.

Example 6.10. Let $A = \mathbb{C}[x,y]/(y^2 + (x^2+\lambda)^2 + 1)$ for some $\lambda \in \mathbb{R}$ and $Y = Y_\lambda$ be the smooth regular projective curve over $\mathbb{C}$ with is the completion of $\breve{Y} = \mathsf{Spec}(A) \subset \mathbb{A}^2_{\mathbb{C}}$. The dihedral group $D_2 = \langle \sigma, \varrho\rangle$ operates on A by the rule $x \overset{\varrho}{\mapsto} -x$, $y \overset{\varrho}{\mapsto} y$. It is clear that this action on $\mathsf{Spec}(A)$ can be extended to an action on Y. Since $A^G = \mathbb{R}[w,y]/(z^2 + y^2 + 1)$ for $w = x^2 + \lambda$, we may conclude that $Y/G \cong X_{\mathsf{qt}}$.

The action of G on Y has two special orbits. The first one is the orbit of the point $(0, i\sqrt{1+\lambda^2}) \in \breve{Y}$. The corresponding stabilizer is $\langle\varrho\rangle \cong \mathbb{Z}_2$. To describe the second orbit, consider the closure $\bar{Y}$ of Y in $\mathbb{P}^2_{\mathbb{C}}$. We have: $\bar{Y} = \mathsf{Proj}\big(\mathbb{C}[x,y,z]/(y^2z^2 + (x^2+\lambda z^2)^2 + z^4)\big)$. Note that the point $o = (0:1:0) \in \bar{Y}$ is singular. The curve Y is the normalization of $\bar{Y}$. Let $Y \xrightarrow{\nu} \bar{Y}$ be the normalization map. Then $\nu^{-1}(o) = \{o_+, o_-\}$ and $\sigma(o_\pm) = o_\mp$. A straightforward local computation shows that the stabilizer of o_+ is $\langle\varrho\rangle \cong \mathbb{Z}_2$. It follows that the corresponding hereditary curve $\mathbb{X}$ has type $\big(X_{\mathsf{qt}}, (2,2)\big)$.

A systematic way to construct finite group actions on complex elliptic curves viewed as real algebraic schemes comes from wallpaper groups. To explain this construction, recall that a *Klein surface* $\mathfrak{X}$ is a *dianalytic manifold* (possibly, with non-empty boundary) of complex dimension one; see [1,2,5] for the details. Klein surfaces naturally form a category. An important result due to Alling and Greenleaf asserts that the category of compact Klein surfaces is equivalent to the category of regular complete curves over $\mathbb{R}$; see [1, Theorem 3], [2, Section II.3] as well as [5, Appendix A] for further elaborations. The key point is the following: the set $\mathbb{M}(\mathfrak{X})$ of all meromorphic functions on a connected Klein surface $\mathfrak{X}$ is an algebraic function field of one variable over $\mathbb{R}$ (i.e. a finitely generated field extension of $\mathbb{R}$ of transcendence degree one); see [1, Theorem 1] as well as [2]. The field $\mathbb{M}(\mathfrak{X})$ defines a uniquely determined (up to isomorphisms) regular projective curve X over $\mathbb{R}$. The main point is to prove that the correspondence $\mathfrak{X} \mapsto \mathbb{M}(\mathfrak{X})$ defines a contravariant equivalence between the category of connected Klein surfaces and the category of real algebraic function fields in one variable.

In particular, in genus zero we have:

(a) the closed disc $\mathfrak{D} = \left\{z \in \mathbb{C} \,\middle|\, |z| \le 1\right\}$ has the function field $\mathbb{R}(z)$ and corresponds to the curve X_{re};

(b) the Riemann sphere $\mathfrak{S}$ has the function field $\mathbb{C}(z)$ and corresponds to X_{co};

(c) the real projective plane $\mathfrak{P}$ has the function field $\mathbb{R}(y)[x]/(x^2 + y^2 + 1)$ and corresponds to the curve X_{qt}.

Recall that the Euclidean group $\mathsf{E}_2 = \mathsf{O}_2(\mathbb{R}) \ltimes \mathbb{R}^2$ is the group of isometries of the Euclidean plane $\mathbb{R}^2 = \mathbb{C}$. For any $(A, \vec{v}) \in \mathsf{E}_2$ we have the corresponding automorphism

$$\mathbb{R}^2 \longrightarrow \mathbb{R}^2, \vec{x} \mapsto A\vec{x} + \vec{v},$$

which is either analytic (if $\det(A) = 1$) or anti-analytic (if $\det(A) = -1$) with respect to the standard complex structure on $\mathbb{R}^2 = \mathbb{C}$.

A *wallpaper group* W (also called plane crystallographic group) is a discrete cocompact subgroup of E_2; see for example [24,36]. Let T be the subgroup of W consisting of all translations. Bieberbach's Theorem asserts that $T \lhd W$ is a normal subgroup, $T \cong \mathbb{Z}^2$ and $G := W/T \subset \mathsf{O}_2(\mathbb{R})$ is a finite group (called *point group* of W). Obviously, $\mathfrak{Y} = \mathbb{C}/T$ is a complex torus and the point group G acts on $\mathfrak{Y}$ by dianalytic automorphisms. The quotient $\mathfrak{X}_W = \mathbb{R}^2/W = \mathfrak{Y}/G$ is a compact flat surface orbifold; see [36, Appendix A.3].

Let $\mathfrak{Z}$ be a surface orbifold and $p \in \mathfrak{Z}$ be its singular point. Then p belongs to a one of the following three classes:

(a) Mirror point, if it admits a neighbourhood isomorphic to $\mathbb{R}^2/\mathbb{Z}_2$, where the generator of $\mathbb{Z}_2$ acts by a reflection (say, with respect to the x-axis).

(b) Elliptic point of order $n \in \mathbb{N}_{\ge 2}$ (denoted by n), if it admits a neighbourhood isomorphic to $\mathbb{R}^2/\mathbb{Z}_n$, where $\mathbb{Z}_n$ acts on $\mathbb{R}^2$ by rotations.

(c) Corner reflector point of order $n \in \mathbb{N}_{\geq 2}$ (denoted by $\bar{n}$), if it admits a neighbourhood isomorphic to $\mathbb{R}^2/D_n$ with respect to the natural action of the dihedral group on $\mathbb{R}^2$.

If $p \in \mathfrak{X}_W$ is a mirror point then it is just an ordinary point of the boundary of $\mathfrak{X}_W$. An essential information about $\mathfrak{X}_W$ (viewed as an surface orbifold) is governed by its diffeomorphism type and by the number/position of its elliptic and corner reflector points.

Let $\mathbb{M}$ be the field of meromorphic functions on $\mathfrak{Y}$. Then we have a natural group embedding $G \subset \mathsf{Aut}_{\mathbb{R}}(\mathbb{M})$ induced by the action of G on $\mathfrak{Y}$ (viewed as a Klein surface). Let Y be the complex elliptic curve corresponding to $\mathfrak{Y}$. Then we have a group embedding $G \subset \mathsf{Aut}_{\mathbb{R}}(Y)$. Let $X = Y/G$ and $\mathbb{X} = \mathbb{X}_W = Y /\!\!/ G$ be the corresponding hereditary curve. The key Proposition 6.1 as well as the aforementioned Alling–Greenleaf equivalence of categories allows one to relate the datum (X, ρ) defining $\mathbb{X}$ with the orbifold notation of the underlying wallpaper group W.

Theorem 6.11. *Let W be a wallpaper group for which $g(X) = 0$. Then there exists a real squid algebra Π_W of* tubular type *and an exact equivalence of triangulated categories*

$$D^b\big(\mathsf{Coh}(\mathbb{X}_W)\big) \simeq D^b\big(\Pi_W\text{--}\mathsf{mod}\big). \tag{37}$$

Proof. Since $g(X) = 0$, Theorem 3.12 implies that there exists a squid algebra Π_W such that $D^b\big(\mathsf{Coh}(\mathbb{X}_W)\big) \simeq D^b\big(\Pi_W\text{--}\mathsf{mod}\big)$. Recall (see [24,36]) the classification of the isomorphism classes of wallpaper groups and the corresponding flat surface orbifolds:

№	Wallpaper group	Orbifold type	hereditary curve type
1	hexatrope group	$\mathfrak{S}(2,3,6)$	$X_{\mathsf{co}}(2,3,6)$
2	tetratrope group	$\mathfrak{S}(2,4,4)$	$X_{\mathsf{co}}(2,4,4)$
3	tritrope group	$\mathfrak{S}(3,3,3)$	$X_{\mathsf{co}}(3,3,3)$
4	ditrope group	$\mathfrak{S}(2,2,2,2)$	$X_{\mathsf{co}}(2,2,2,2)$
5	hexascope group	$\mathfrak{D}(\bar{2},\bar{3},\bar{6})$	$X_{\mathsf{re}}(\bar{2},\bar{3},\bar{6})$
6	tetrascope group	$\mathfrak{D}(\bar{2},\bar{4},\bar{4})$	$X_{\mathsf{re}}(\bar{2},\bar{4},\bar{4})$
7	triscope group	$\mathfrak{D}(\bar{3},\bar{3},\bar{3})$	$X_{\mathsf{re}}(\bar{3},\bar{3},\bar{3})$
8	discope group	$\mathfrak{D}(\bar{2},\bar{2},\bar{2},\bar{2})$	$X_{\mathsf{re}}(\bar{2},\bar{2},\bar{2},\bar{2})$
9	tetragyro group	$\mathfrak{D}(4,\bar{2})$	$X_{\mathsf{re}}(4,\bar{2})$
10	trigyro group	$\mathfrak{D}(3,\bar{3})$	$X_{\mathsf{re}}(3,\bar{3})$
11	digyro group	$\mathfrak{D}(2,2)$	$X_{\mathsf{re}}(2,2)$
12	dirhomb group	$\mathfrak{D}(2,\bar{2},\bar{2})$	$X_{\mathsf{re}}(2,\bar{2},\bar{2})$
13	diglide group	$\mathfrak{P}(2,2)$	$X_{\mathsf{qt}}(2,2)$
14	monotrope group	torus	$\mathsf{Proj}\big(\mathbb{C}[x,y,z]/(zy^2 - x^3 + xz^2)\big)$
15	monoglide group	Klein bottle	$\mathsf{Proj}\big(\mathbb{R}[x,y,z]/(z^2y^2 + (x^2+z^2)(x^2+z^2))\big)$
16	monorhomb group	Möbius band	$\mathsf{Proj}\big(\mathbb{R}[x,y,z]/(zy^2 - x^3 - xz^2)\big)$
17	monoscope group	annulus	$\mathsf{Proj}\big(\mathbb{R}[x,y,z]/(zy^2 - x^3 + xz^2)\big)$

The last four types of the above table correspond to real projective curve of genus one, the stated correspondence is taken from [2, Example 1]. The corresponding derived category $D^b(\mathsf{Coh}(\mathbb{X}))$ does not have tilting objects. In the first thirteen cases, it follows from the stated classification, that the squid algebra Π_W has a tubular type. □

Remark 6.12. The correspondence between wallpaper groups and real hereditary curves of tubular type was for the first time observed by Lenzing many years ago [33]. Kussin in [28, Corollary 13.23] gave a classification of all hereditary curves of tubular type. From this classification it became apparent that the curves of type $\mathbb{X}_W$ are precisely those ones, for which $[\eta_{\mathbb{X}}] = 0$: indeed, the corresponding numerical patterns are the same. Kussin informed me about another approach to establish a more concrete correspondence between wallpaper groups and exceptional hereditary curves of tubular type [29]. However, the works [28, 29] are heavily based on the "axiomatic approach" to non-commutative hereditary curves and the corresponding proofs are technically different from the ones given in this paper.

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References

[1] Alling, N.L., Greenleaf, N.: Klein surfaces and real algebraic function fields. Bull. Amer. Math. Soc. **75**, 869–872 (1969). https://doi.org/10.1090/S0002-9904-1969-12332-3

[2] Alling, N.L., Greenleaf, N.: Foundations of the theory of Klein surfaces. Lecture Notes in Math., vol. **219**. Springer, Berlin–New York (1971)

[3] Artin, M., de Jong, A.: Stable orders over surfaces (unpublished manuscript). https://www.math.lsa.umich.edu/courses/711/

[4] Brenner, S., Butler, M.C.R.: Generalizations of the Bernstein–Gelfand–Ponomarev reflection functors. In: Dlab, V., Gabriel, P. (eds) Representation theory II. Lecture Notes in Mathematics, vol. **832**, pp. 103–169. Springer, Berlin (1980). https://doi.org/10.1007/BFb0088461

[5] Bujalance, E., Etayo, J., Gamboa J., Gromadzki, G.: Automorphism groups of compact bordered Klein surfaces. A combinatorial approach. Lecture Notes in

Mathematics, vol. **1439**. Springer-Verlag, Berlin (1990). https://doi.org/10.1007/BFb0084977

[6] Burban, I., Drozd, Yu.: Maximal Cohen–Macaulay modules over surface singularities. In: Skowroński, A. (ed.) Trends in representation theory of algebras and related topics, pp. 101–166. EMS Ser. Congr. Rep. Eur. Math. Soc., Zürich (2008). https://doi.org/10.4171/062-1/3

[7] Burban, I., Drozd, Yu.: Tilting on non-commutative rational projective curves. Math. Ann. **351**, 665–709 (2011). https://doi.org/10.1007/s00208-010-0585-4

[8] Burban, I., Drozd, Yu.: Non-commutative nodal curves and derived tame algebras. Preprint at https://arxiv.org/abs/1805.05174 (2019)

[9] Burban, I., Drozd, Yu.: Morita theory for non-commutative noetherian schemes. Adv. Math. **399**, 108273 (2022). https://doi.org/10.1016/j.aim.2022.108273

[10] Burban, I., Drozd, Yu.: Some aspects of the theory of nodal orders. Preprint at https://arxiv.org/abs/2406.02375 (2024)

[11] Burban, I., Drozd, Yu., Gavran, V.: Minors and resolutions of non-commutative schemes. Eur. J. Math. **3**(2), 311–341 (2017). https://doi.org/10.1007/s40879-017-0128-6

[12] Burban, I., Drozd, Yu., Gavran, V.: Singular curves and quasi-hereditary algebras. Int. Math. Res. Not. **2017**(3), 895–920 (2017). https://doi.org/10.1093/imrn/rnw045

[13] Dlab, V., Ringel, C.-M.: Indecomposable representations of graphs and algebras. Mem. Amer. Math. Soc. **6**(173) (1976)

[14] Drozd, Yu., Kirichenko, V.: Finite dimensional algebras. Translated from the 1980 Russian original and with an appendix by V. Dlab. Springer (1994). https://doi.org/10.1007/978-3-642-76244-4

[15] Chan, D., Ingalls, C.: Non-commutative coordinate rings and stacks. Proc. London Math. Soc. **88**(1), 63–88 (2004). https://doi.org/10.1112/S0024611503014278

[16] Geigle, W., Lenzing, H.: A class of weighted projective curves arising in representation theory of finite-dimensional algebras. Singularities, representation of algebras, and vector bundles. Lecture Notes in Mathematics, vol. **1273**, pp. 265–297. Springer, Berlin (1987). https://doi.org/10.1007/BFb0078849

[17] Grothendieck, A.: Revêtements étales et groupe fondamental. Fasc. I: Exposés 1 à 5. Séminaire de Géométrie Algébrique, 1960/61. Institut des Hautes Études Scientifiques, Paris (1963)

[18] Happel, D.: A characterization of hereditary categories with tilting object. Invent. Math. **144**(2), 381–398 (2001). https://doi.org/10.1007/s002220100135

[19] Happel, D., Reiten, I.: Hereditary abelian categories with tilting object over arbitrary base fields. J. Algebra. **256**(2), 414–432 (2002). https://doi.org/10.1016/S0021-8693(02)00088-1

[20] Harada, M.: Hereditary orders. Trans. Amer. Math. Soc., vol. **107**, pp. 273–290 (1963). https://doi.org/10.1090/S0002-9947-1963-0151489-9

[21] Harada, M.: Structure of hereditary orders over local rings. J. Math. Osaka City Univ. **14**, 1–22 (1963)

[22] Keller, B.: Deriving DG categories. Ann. Sci. École Norm. Sup. (4). **27**(1), 63–102 (1994). https://doi.org/10.24033/asens.1689

[23] Kirillov, A.: McKay correspondence and equivariant sheaves on $\mathbb{P}^1$. Mosc. Math. J. **6**(3), 505–529, 587–588 (2006)

[24] Klemm, M.: Symmetrien von Ornamenten und Kristallen. Hochschultext. Springer Berlin, Heidelberg (1982). https://doi.org/10.1007/978-3-642-68625-2

[25] Kostrikin, A.: Introduction to algebra. Springer-Verlag, Berlin–New York (1982). https://doi.org/10.1017/S0013091500004557

[26] Kussin, D.: Factorial algebras, quaternions and preprojective algebras. Algebras and modules, II (Geiranger, 1996). CMS Conf. Proc., vol. **24**, pp. 393–402. Amer. Math. Soc., Providence, RI (1998)

[27] Kussin, D.: Noncommutative curves of genus zero: related to finite dimensional algebras. Mem. Amer. Math. Soc. **201**(942) (2009). https://doi.org/10.1090/memo/0942

[28] Kussin, D.: Weighted noncommutative regular projective curves. J. Noncommut. Geom. **10**(4), 1465–1540 (2016). https://doi.org/10.4171/jncg/264

[29] Kussin, D.: Weighted noncommutative regular projective curves II (work in preparation)

[30] Lang, S.: Algebra. In: Axler, S., Gehring, F.W., Ribet, K.A. (eds) Graduate Texts in Mathematics, vol. **211**. Springer, New York (2002)

[31] Lenzing, H.: Curve singularities arising from the representation theory of tame hereditary algebras. In: Dlab, V., Gabriel, P., Michler, G. (eds) Representation Theory I Finite Dimensional Algebras. Lecture Notes in Mathematics, vol. **1177**, pp. 199–231. Springer Berlin, Heidelberg (1986). https://doi.org/10.1007/BFb0075266

[32] Lenzing, H.: Representations of finite dimensional algebras and singularity theory. Trends in ring theory (Miskolc, 1996). CMS Conf. Proc. **22**, pp. 71–97. Amer. Math. Soc., Providence, RI (1998)

[33] Lenzing, H.: Klein bottle, Möbius band and their non-commutative siblings: Talk in the Oberseminar "Algebra and Algebraic Geometry", Paderborn University, 26 June 2019. A Colloquium talk at Erlangen University, 13 November 2001

[34] Lenzing, H., de la Peña, J.A.: Concealed-canonical algebras and separating tubular families. Proc. London Math. Soc. (3). **78**(3), 513–540 (1999). https://doi.org/10.1112/S0024611599001872

[35] Lenzing, H., Reiten, I.: Hereditary Noetherian categories of positive Euler characteristic. Math. Z. **254**(1), 133–171 (2006). https://doi.org/10.1007/s00209-006-0938-6

[36] Montesinos, J.M.: Classical tessellations and three-manifolds. Universitext. Springer Berlin, Heidelberg (1987). https://doi.org/10.1007/978-3-642-61572-6

[37] Mustata, M.: Zeta functions in algebraic geometry (unpublished manuscript)

[38] de Naeghel, K., van den Bergh, M.: Ideal classes of three-dimensional Sklyanin algebras. J. Algebra. **276**(2), 515–551 (2004). https://doi.org/10.1016/j.jalgebra.2003.09.023

[39] Polishchuk, A.: Holomorphic bundles on 2-dimensional noncommutative toric orbifolds. In: Consani, C., Marcolli, M. (eds) Noncommutative Geometry and Number Theory. Aspects of Mathematics. Vieweg. https://doi.org/10.1007/978-3-8348-0352-8_16

[40] Reiner, I.: Maximal orders. London Math. Soc. Monographs. New Series, vol. **28**. The Clarendon Press. Oxford University Press, Oxford (2003)

[41] Reiten, I., Riedtmann, Ch.: Skew group algebras in the representation theory of Artin algebras. J. Algebra. **92**(1), 224–282 (1985). https://doi.org/10.1016/0021-8693(85)90156-5

[42] Reiten, I., Van den Bergh, M.: Noetherian hereditary abelian categories satisfying Serre duality. J. Amer. Math. Soc. **15**(2), 295–366 (2002). https://doi.org/10.1090/S0894-0347-02-00387-9

[43] Ringel, C.-M.: Tame algebras and integral quadratic forms. Lecture Notes in Math., vol. **1099**. Springer Berlin, Heidelberg (1984). https://doi.org/10.1007/BFb0072870

[44] Ringel, C.-M.: The canonical algebras. Topics in Algebra. Part 1. Banach Center Publ., vol. **26**, pp. 407–432. PWN. With an appendix by W. Crawley-Boevey (1990).

[45] Simson, D., Skowroński, A.: Elements of the representation theory of associative algebras: tubes and concealed algebras of Euclidean type, vol. **2**. LMS Student Texts; 71. Cambridge University Press, Cambridge (2007)

[46] Spieß, M.: Twists of Drinfeld–Stuhler modular varieties. A collection of manuscripts written in Honour of Andrei A. Suslin on the occasion of his sixtieth birthday, pp. 595–654 (2010). https://doi.org/10.4171/dms/5/17

[47] Yekutieli, A., Zhang, J.J.: Dualizing complexes and perverse sheaves on noncommutative ringed schemes. Sel. math., New ser. **12**, 137–177 (2006). https://doi.org/10.1007/s00029-006-0022-4

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