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On embedding groups into digroups

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Dedicated to Professor Yu. A. Drozd on the occasion of his 80th birthday

Abstract. An idea of the notion of a digroup which generalizes groups and has close relationships with the dimonoids, trioids, Leibniz algebras and other structures was proposed by J.-L. Loday. In terms of digroups, Kinyon obtained an analogue of Lie's third theorem for the class of so-called split Leibniz algebras. In this paper, we use group operations (semigroup operations) to construct new digroups (dimonoids) and show that any group (semigroup) can be embedded into a suitable non-trivial digroup (dimonoid). We present a universal extension for an arbitrary dimonoid, give a construction of the free abelian generalized digroup and characterize the least group congruence on it. We also describe the least abelian digroup congruence on the free generalized digroup.

1. Introduction

The digroups first implicitly appeared in the paper of Loday [8] and then they have been also proposed independently in [2, 4, 7]. The notion of a digroup generalizes the notion of a group. A digroup is a nonempty set equipped with two binary associative operations, a unary operation and a nullary operation satisfying additional axioms relating these operations. A digroup is a group iff these both binary operations coincide. Digroups are closely related to dimonoids and trioids which play an important role in problems from the theory of Leibniz algebras and algebraic topology

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and they have been studied, for example, in [8, 14, 16, 17, 20, 21, 24]. Cayley's theorem for digroups was given in [7]. Kinyon showed that every digroup is a product of a group and a trivial digroup [5]. Linear representations of digroups were considered by Felipe [3]. A more simple basis of independent axioms for the variety of digroups was obtained by Phillips [10]. Different examples of digroups can be found in [19] and some analogues of known structural results of group theory were obtained in [9]. The free digroup of an arbitrary rank was constructed in [15], but a clearer description of the free monogenic digroup was given in [26]. The properties of generalized digroups and generalized dimonoids were investigated in [12, 22]. For other recent works on (generalized) digroups see, for instance, [13, 18, 25].

This paper is organised as follows. In Section 2, we define a new class of dimonoids and describe some representations of semigroups and dimonoids into suitable dimonoids (Theorems 1 and 2). In Section 3, we construct a new class of digroups from groups and show that an arbitrary group can be embedded into some non-trivial digroup of this class (Theorem 4). In Section 4, we give a construction of the free abelian generalized digroup of an arbitrary rank (Theorem 6) and characterize the least group congruence on it. In addition, we present the least abelian digroup congruence on the free generalized digroup.

2. Dimonoids

Definition 1. A nonempty set D equipped with two binary associative operations $\dashv$ and $\vdash$ is called a *dimonoid* [8] if for all $x, y, z \in D$,

$$(D_1)\ \ (x \dashv y) \dashv z = x \dashv (y \vdash z),$$
$$(D_2)\ \ (x \vdash y) \dashv z = x \vdash (y \dashv z),$$
$$(D_3)\ \ (x \dashv y) \vdash z = x \vdash (y \vdash z).$$

These algebraic systems under the name "dimonoids" first appeared in Loday's paper [8]. It should be noted that for the notion of a dimonoid some authors use a termin "disemigroup" (see, e.g. [15]).

If operations of a dimonoid coincide, then it becomes a semigroup.

Example 1. A dimonoid is any (finite dimensional) vector space V with the operations on V defined by $x \dashv y := x + y\varphi$ and $x \vdash y := x\varphi + y$, where φ is an idempotent linear operator on V (it follows from [2, Example 3.2]).

Definition 2. For a semigroup S, a nonempty subset with the property that every element commutes with any other element of the semigroup is called *the center* of S and it is denoted by $Z(S)$. For a dimonoid $\mathcal{D} = (D, \dashv, \vdash)$, the intersection $Z((D, \dashv)) \cap Z((D, \vdash))$ we call *the center* of $\mathcal{D}$. We will denote by $Z(\mathcal{D})$ the center of the dimonoid $\mathcal{D}$.

Let $\mathcal{G} = (G, \prec, \succ)$ be an arbitrary dimonoid with a nonempty center, $\xi \in Z(\mathcal{G})$ and $D(G) = G \cup (G \times G)$. We extend the dimonoid operations on G to the binary operations $\dashv$ and $\vdash$ on $D(G)$ as follows:

$$a \dashv (b, c) = a \prec b \prec c \prec \xi,$$

$$(b, c) \dashv a = (b, c \prec a),$$

$$(a, b) \dashv (c, d) = (a, b \prec c \prec d \prec \xi),$$

$$a \vdash (b, c) = (a \succ b, c),$$

$$(b, c) \vdash a = b \succ c \succ a \succ \xi,$$

$$(a, b) \vdash (c, d) = (a \succ b \succ c \succ \xi, d)$$

for all $a, b, c, d \in G$. The algebra $(D(G), \dashv, \vdash)$ is denoted by $\mathcal{D}_\xi(\mathcal{G})$.

Proposition 1. *For any dimonoid $\mathcal{G}$ with a nonempty center and every $\xi \in Z(\mathcal{G})$, the algebraic system $\mathcal{D}_\xi(\mathcal{G})$ is a dimonoid.*

Proof. Similar as in [26], one can prove the associativity of operations $\dashv$ and $\vdash$ on $D(G)$. Now we will check the axiom (D_3) for $\mathcal{D}_\xi(\mathcal{G})$.

Let a, b, $c \in D(G)$. The case $a, b, c \in G$ is trivial. If $a \in G \times G$, $a = (a_1, a_2)$, and $b, c \in G$, then using (D_3) for $\mathcal{G}$,

$$(a \dashv b) \vdash c = (a_1, a_2 \prec b) \vdash c =$$

$$= a_1 \succ ((a_2 \prec b) \succ c) \succ \xi = a_1 \succ a_2 \succ b \succ c \succ \xi =$$

$$= (a_1, a_2) \vdash (b \succ c) = a \vdash (b \vdash c).$$

Let $a, c \in G$ and $b = (b_1, b_2) \in G \times G$. Using the axiom (D_3) for $\mathcal{G}$ three times and fact $\xi \in Z(G)$, we obtain that

$$(a \dashv b) \vdash c = (a \prec b_1 \prec b_2 \prec \xi) \succ c =$$

$$= a \succ b_1 \succ b_2 \succ \xi \succ c = a \succ b_1 \succ b_2 \succ c \succ \xi =$$

$$= a \vdash (b \vdash c).$$

For $a, b \in G$ and $c = (c_1, c_2) \in G \times G$, we have

$$(a \dashv b) \vdash c = (a \prec b) \vdash (c_1, c_2) =$$

$$= ((a \prec b) \succ c_1, c_2) = (a \succ b \succ c_1, c_2) =$$

$$= a \vdash (b \succ c_1, c_2) = a \vdash (b \vdash c).$$

Assume $a = (a_1, a_2)$, $b = (b_1, b_2) \in G \times G$ and $c \in G$. Applying (D_3) to $\mathcal{G}$ three times and taking into acount that $\xi \in Z(G)$,

$$(a \dashv b) \vdash c = (a_1, a_2 \prec b_1 \prec b_2 \prec \xi) \vdash c =$$

$$= a_1 \succ (a_2 \prec b_1 \prec b_2 \prec \xi) \succ c \succ \xi =$$

$$= a_1 \succ a_2 \succ b_1 \succ b_2 \succ c \succ \xi \succ \xi =$$

$$= (a_1, a_2) \vdash b_1 \succ b_2 \succ c \succ \xi = a \vdash (b \vdash c).$$

Let $a = (a_1, a_2), c = (c_1, c_2) \in G \times G$ and $b \in G$. Then

$$(a \dashv b) \vdash c = (a_1, a_2 \prec b) \vdash (c_1, c_2) =$$

$$= (a_1 \succ (a_2 \prec b) \succ c_1 \succ \xi, c_2) =$$

$$= (a_1, a_2) \vdash (b \succ c_1, c_2) = a \vdash (b \vdash c).$$

Take $a \in G$ and $b = (b_1, b_2), c = (c_1, c_2) \in G \times G$. Similarly as above,

$$(a \dashv b) \vdash c = (a \prec b_1 \prec b_2 \prec \xi) \vdash (c_1, c_2) =$$

$$= ((a \prec b_1 \prec b_2 \prec \xi) \succ c_1, c_2) =$$

$$= (a \succ b_1 \succ b_2 \succ c_1 \succ \xi, c_2) =$$

$$= a \vdash (b_1 \succ b_2 \succ c_1 \succ \xi, c_2) = a \vdash (b \vdash c).$$

Finally, for $a = (a_1, a_2), b = (b_1, b_2), c = (c_1, c_2) \in G \times G$,

$$(a \dashv b) \vdash c = (a_1, a_2 \prec b_1 \prec b_2 \prec \xi) \vdash (c_1, c_2) =$$

$$= (a_1 \succ (a_2 \prec b_1 \prec b_2 \prec \xi) \succ c_1 \succ \xi, c_2) =$$

$$= (a_1 \succ a_2 \succ b_1 \succ b_2 \succ c_1 \succ \xi \succ \xi, c_2) =$$

$$= (a_1, a_2) \vdash (b_1 \succ b_2 \succ c_1 \succ \xi, c_2) = a \vdash (b \vdash c).$$

Therefore, the axiom (D_3) holds for $\mathcal{D}_\xi(\mathcal{G})$. Analogously, one can prove that (D_1) holds, too. Furthermore, we prove the axiom (D_2).

The case $a, b, c \in G$ is clear. In the case $a = (a_1, a_2) \in G \times G$ and $b, c \in G$, we have

$$(a \vdash b) \dashv c = (a_1 \succ a_2 \succ b \succ \xi) \prec c =$$

$$= a_1 \succ a_2 \succ b \succ (\xi \prec c) = a_1 \succ a_2 \succ ((b \succ \xi) \prec c) =$$
$$= a_1 \succ a_2 \succ ((\xi \succ b) \prec c) = a_1 \succ a_2 \succ (\xi \succ (b \prec c)) =$$
$$= a_1 \succ a_2 \succ (b \prec c) \succ \xi = (a_1, a_2) \vdash (b \prec c) = a \vdash (b \dashv c).$$

If $a, c \in G$ and $b = (b_1, b_2) \in G \times G$, then

$$(a \vdash b) \dashv c = (a \succ b_1, b_2) \dashv c =$$

$$= (a \succ b_1, b_2 \prec c) = a \vdash (b_1, b_2 \prec c) = a \vdash (b \dashv c).$$

For $a, b \in G$ and $c = (c_1, c_2) \in G \times G$,

$$(a \vdash b) \dashv c = (a \succ b) \dashv (c_1, c_2) =$$

$$= (a \succ b) \prec c_1 \prec c_2 \prec \xi =$$
$$= a \succ (b \prec c_1 \prec c_2 \prec \xi) = a \vdash (b \dashv c).$$

Take $a = (a_1, a_2)$, $b = (b_1, b_2) \in G \times G$ and $c \in G$. Then

$$(a \vdash b) \dashv c = (a_1 \succ a_2 \succ b_1 \succ \xi, b_2) \dashv c =$$

$$= (a_1 \succ a_2 \succ b_1 \succ \xi, b_2 \prec c) =$$
$$= (a_1, a_2) \vdash (b_1, b_2 \prec c) = a \vdash (b \dashv c).$$

For the case $a = (a_1, a_2), c = (c_1, c_2) \in G \times G$ and $b \in G$,

$$(a \vdash b) \dashv c = (a_1 \succ a_2 \succ b \succ \xi) \dashv (c_1, c_2) =$$

$$= (a_1 \succ a_2 \succ b \succ \xi) \prec c_1 \prec c_2 \prec \xi =$$
$$= (a_1 \succ a_2 \succ \xi \succ b) \prec c_1 \prec c_2 \prec \xi =$$
$$= a_1 \succ a_2 \succ \xi \succ (b \prec c_1 \prec c_2 \prec \xi) =$$
$$= a_1 \succ a_2 \succ (b \prec c_1 \prec c_2 \prec \xi) \succ \xi =$$
$$= (a_1, a_2) \vdash (b \prec c_1 \prec c_2 \prec \xi) = a \vdash (b \dashv c).$$

Supposing $a \in G$ and $b = (b_1, b_2)\,, c = (c_1, c_2) \in G \times G$, we obtain

$$(a \vdash b) \dashv c = (a \succ b_1, b_2) \dashv (c_1, c_2) =$$

$$= (a \succ b_1, b_2 \prec c_1 \prec c_2 \prec \xi) =$$

$$= a \vdash (b_1, b_2 \prec c_1 \prec c_2 \prec \xi) = a \vdash (b \dashv c).$$

Let $a = (a_1, a_2), b = (b_1, b_2), c = (c_1, c_2) \in G \times G$, then

$$(a \vdash b) \dashv c = (a_1 \succ a_2 \succ b_1 \succ \xi, b_2) \dashv (c_1, c_2) =$$

$$= (a_1 \succ a_2 \succ b_1 \succ \xi, b_2 \prec c_1 \prec c_2 \prec \xi) =$$

$$= (a_1, a_2) \vdash (b_1, b_2 \prec c_1 \prec c_2 \prec \xi) = a \vdash (b \dashv c).$$

Thus, $\mathcal{D}_\xi(\mathcal{G})$ is a dimonoid. □

Proposition 1 gives a new class of dimonoids (both dimonoid operations are not commutative) which are defined by arbitrary dimonoids with a nonempty center.

Theorem 1. *For any semigroup S there exists a monoid $(T, \cdot)$ with unity e such that the dimonoid $\mathcal{D}_e((T, \cdot, \cdot))$ contains S as a subsemigroup.*

Proof. Let S be an arbitrary semigroup (with the operation $*$). Assume that S does not have a unity. We take an arbitrary symbol $1 \notin S$ and define on the set $S^1 = S \cup \{1\}$ a new binary operation $\cdot$ as follows: $1 \cdot x = x = x \cdot 1$ and $y \cdot z = y * z$ for all $x \in S^1$ and $y, z \in S$. It is well-known (see, e.g., [6]) that $(S^1, \cdot)$ is a monoid. Then we put

$$\mathcal{T} = \begin{cases} S, & \text{if } S \text{ has a unity,} \\ (S^1, \cdot) & \text{otherwise} \end{cases}$$

for any semigroup S. Therefore, $\mathcal{T}$ is a monoid (say, with the unity e). Clearly, the monoid $\mathcal{T}$ can be considered as a dimonoid in which operations coincide. In this case $Z(\mathcal{T}) \neq \emptyset$ and by Proposition 1, $\mathcal{D}_e(\mathcal{T})$ is a dimonoid. By construction of $\mathcal{D}_e(\mathcal{T})$, we conclude that $\mathcal{D}_e(\mathcal{T})$ contains S as a subsemigroup. □

Let $(G, \prec, \succ)$ be an arbitrary dimonoid and $D(G) = G \cup (G \times G)$. We extend the dimonoid operations on G to the binary operations $\dashv$ and $\vdash$ on $D(G)$ as follows:

$$a \dashv (b, c) = a \prec b \prec c,$$

$$(b, c) \dashv a = (b, c \prec a),$$

$$(a, b) \dashv (c, d) = (a, b \prec c \prec d),$$

$$a \vdash (b,c) = (a \succ b, c),$$

$$(b,c) \vdash a = b \succ c \succ a,$$

$$(a,b) \vdash (c,d) = (a \succ b \succ c, d).$$

The algebra $(D(G), \dashv, \vdash)$ is denoted by $\mathcal{D}(\mathcal{G})$.

Theorem 2. *For any dimonoid $\mathcal{G} = (G, \prec, \succ)$, the algebraic system $\mathcal{D}(\mathcal{G}) = (D(G), \dashv, \vdash)$ is a dimonoid containing $\mathcal{G}$ as a subdimonoid.*

Proof. Similarly as in Proposition 1 one can show that the algebra $\mathcal{D}(\mathcal{G})$ is a dimonoid. By construction, $\mathcal{G} \subset \mathcal{D}(\mathcal{G})$. □

We call the obtained dimonoid $\mathcal{D}(\mathcal{G})$ the *universal extension of the dimonoid $\mathcal{G}$*.

3. Digroups

There are several definitions of a digroup, so we consider them.

Definition 3. Following [5], a dimonoid $(D, \dashv, \vdash)$ is called a *digroup* if

(D_4) there exists an element $e \in D$ such that for all $g \in D$,

$$e \vdash g = g = g \dashv e,$$

(D_5) for any $g \in D$ there exists an element $g^{-1} \in D$ such that

$$g \vdash g^{-1} = e = g^{-1} \dashv g.$$

An element e is called a *bar-unit* of the digroup $(D, \dashv, \vdash)$ and g^{-1} is said to be *inverse* to g with respect to e.

A *bar-unit* of a dimonoid is defined in analogous way as for digroups (see, e.g., [8]). Axiom (D_4) asserts that a bar-unit exists in a digroup, but it is not assumed to be unique.

Example 2. Let $(V, \dashv, \vdash)$ be the dimonoid from Example 1. Then $(V, \dashv, \vdash)$ is a digroup if we take the zero vector as a bar-unit and if we put $g^{-1} = -g$ for all $g \in V$ [2, Example 3.2].

Recall that a non-empty class K of algebraic systems is a variety if the Cartesian product of any sequence of K-systems is a K-system, every subsystem of an arbitrary K-system is a K-system and any homomorphic image of an arbitrary K-system is a K-system [1]. Equivalently, a variety

K is an equational class, that is, a class of algebras defined by some set of identities of the given type.

Observe that the class of digroups according to Definition 3 is not a variety; in particular, it is not the variety of dimonoids similarly as the class of groups is not the variety of semigroups.

Definition 4. According to [10, 15], a nonempty set G equipped with two binary operations $\dashv$ and $\vdash$, a unary operation $^{-1}$, and a nullary operation 1, is called a *digroup* if the following conditions hold:

$$(D_0)\ (G, \dashv)\ \textit{and}\ (G, \vdash)\ \textit{are semigroups},$$
$$(D_1)\ (x \dashv y) \dashv z = x \dashv (y \vdash z),$$
$$(D_2)\ x \vdash (y \dashv z) = (x \vdash y) \dashv z,$$
$$(D_3)\ (x \dashv y) \vdash z = x \vdash (y \vdash z),$$
$$(D_4)\ 1 \vdash x = x = x \dashv 1,$$
$$(D_5)\ x \vdash x^{-1} = 1 = x^{-1} \dashv x.$$

An element 1 is called the *bar-unit* of the digroup and x^{-1} is said to be *inverse* of x.

We gave both definitions of digroups (Definitions 3 and 4), because both of them are currently in use. Observe that the class of digroups according to Definition 4 is a variety, in contrast to Definition 3. We will give a summary of these two classes of digroups in a forthcoming paper.

Definition 5. Let $\mathcal{D} = (D, \dashv, \vdash, ^{-1}, 1)$ be an arbitrary digroup (in the sense of Definition 4). The set of all bar-units of $\mathcal{D}$ is called the *halo part* and denoted by $E(\mathcal{D})$; the set of all inverse elements of $\mathcal{D}$ is called the *group part* and denoted by $J(\mathcal{D})$ (see, e.g., [15]).

Remark 1. The group part of any digroup $(D, \dashv, \vdash, ^{-1}, 1)$ is a group in which the binary operations $\dashv$ and $\vdash$ coincide [5, Lemma 4.5 (3)].

Remark 2. Note that $(D, \vdash, \dashv, ^{-1}, 1)$ does not yield a digroup if $(D, \dashv, \vdash, ^{-1}, 1)$ is a digroup. So, the digroup axioms are not "self-dual".

Example 3. Let G be an arbitrary set with $|G| > 1$. Define $x \dashv y := x$ and $x \vdash y := y$ for all $x, y \in G$. Select any element of G as 1 and define all x^{-1} as 1. Then $(G, \dashv, \vdash, ^{-1}, 1)$ becomes a digroup [19, Example 3.1] in which every element satisfies the axiom (D_4) and can be considered as a signature bar-unit of another digroup.

It turns out that the axioms of a digroup in Definition 4 are not independent. The system of independent axioms for the variety of digroups was found by Phillips [10, Theorem 2], where it was proved that the digroup Definition 4 can be simplified.

Theorem 3 ([10]). *A nonempty set G equipped with two binary operations $\dashv$ and $\vdash$, a unary operation $^{-1}$, and a nullary operation 1, is a digroup according to Definition 4 if and only if*

$$(D_0)\ (G,\dashv) \text{ and } (G,\vdash) \text{ are semigroups},$$
$$(D_2')\ x\vdash(x\dashv z)=(x\vdash x)\dashv z,$$
$$(D_4)\ 1\vdash x=x=x\dashv 1,$$
$$(D_5)\ x\vdash x^{-1}=1=x^{-1}\dashv x.$$

An arbitrary digroup we call *trivial* if its binary operations coincide and *non-trivial* otherwise.

If $(H,*)$ is a group, sometimes we refer to $(H,*)$ as a trivial digroup $(H,*,*)$. Obviously, for an arbitrary group $(H,*)$, we have $J((H,*,*)) = (H,*)$. In connection with this, it is natural to consider the following question: is there for an arbitrary group $\mathcal{H}$ a non-trivial digroup $\mathcal{G}$ such that $J(\mathcal{G})=\mathcal{H}$?

One of the main results of this paper is the following statement.

Theorem 4. *For an arbitrary group $\mathcal{H}$ there exists a non-trivial digroup such that the group part of this digroup coincides with $\mathcal{H}$.*

Proof. Let $\mathcal{H}=(H,*)$ be an arbitrary group with the unity 1, and let ξ be a fixed element of the center $Z(\mathcal{H})$. It is easy to see that $\xi\in Z((H,*,*))$, where $(H,*,*)$ is a trivial digroup and the center of a digroup is defined in similar way as for dimonoids (see Def. 2). Now we can define a unary operation $\dagger$ on $D(H)=H\cup(H\times H)$ by

$$x^{\dagger}=\begin{cases} x^{-1}, & x\in H,\\ x_2^{-1}*x_1^{-1}*\xi^{-1}, & x=(x_1,x_2)\in H\times H.\end{cases}$$

Denote the algebra $(D(H),\dashv,\vdash,\dagger,1)$ by $\mathcal{DG}_\xi(\mathcal{H})$. Here the operations $\dashv$ and $\vdash$ on $D(H)$ are defined in a similar way as in Proposition 1 (Sect. 2). By Theorem 1, $\mathcal{D}_\xi(\mathcal{H})=(D(H),\dashv,\vdash)$ is a dimonoid which contains $\mathcal{H}$. Consequently, axioms (D_0) and (D_2') from Definition 5 hold in $\mathcal{DG}_\xi(\mathcal{H})$.

Further for all $a\in H$ and $b=(b_1,b_2)\in H\times H$ we have

$$1\vdash a=1*a=a=a*1=a\dashv 1,$$

$$1 \vdash b = (1 * b_1, b_2) = b = (b_1, b_2 * 1) = b \dashv 1.$$

Thus, 1 is a bar-unit of the dimonoid $\mathcal{D}_\xi(\mathcal{H})$ and (D_4) holds. At the end, we check the last digroup axiom (D_5).

For every $x \in H$ there exists an inverse element $x^\dagger = x^{-1} \in H$ such that

$$x * x^\dagger = 1 = x^\dagger * x.$$

Moreover, for every pair $(x, y) \in H \times H$ there exists an inverse element $(x, y)^\dagger = y^{-1} * x^{-1} * \xi^{-1} \in H$ such that

$$(x, y) \vdash (x, y)^\dagger = (x, y) \vdash \left(y^{-1} * x^{-1} * \xi^{-1}\right) =$$
$$= x * y * \left(y^{-1} * x^{-1} * \xi^{-1}\right) * \xi = 1 =$$
$$= \left(y^{-1} * x^{-1} * \xi^{-1}\right) * x * y * \xi =$$
$$= \left(y^{-1} * x^{-1} * \xi^{-1}\right) \dashv (x, y) = (x, y)^\dagger \dashv (x, y).$$

According to Definition 5, $\mathcal{DG}_\xi(\mathcal{H})$ is a digroup. It is obvious that $J(\mathcal{DG}_\xi(\mathcal{H})) = \mathcal{H}$. □

Thus, Theorem 4 gives a new class of digroups which are defined by arbitrary groups.

Corollary 1. $E(\mathcal{DG}_\xi(\mathcal{H})) = \{1, (x, x^{-1} * \xi^{-1}) \mid x \in H\}$.

Corollary 2. *Let $\mathcal{H} = (H, *, ^{-1}, 1)$ be an arbitrary group and the nonnullary operations of $\mathcal{H}$ are extended to the binary operations $\dashv$ and $\vdash$, and the unary operation $\dagger$ on $H \cup (H \times H)$ as follows:*

$$a \dashv (b, c) = a * b * c,$$
$$(b, c) \dashv a = (b, c * a),$$
$$(a, b) \dashv (c, d) = (a, b * c * d),$$
$$a \vdash (b, c) = (a * b, c),$$
$$(b, c) \vdash a = b * c * a,$$
$$(a, b) \vdash (c, d) = (a * b * c, d),$$
$$(a, b)^\dagger = b^{-1} * a^{-1}.$$

Then the algebra $\mathcal{DG}_1(\mathcal{H}) = (H \cup (H \times H), \dashv, \vdash, \dagger, 1)$ is a digroup.

Example 4. Let $\mathcal{Z} = (\mathbb{Z}, +)$ be the additive group of all integers and let $\xi = 1 \in \mathbb{Z}$. By Theorem 4, $\mathcal{DG}_1(\mathcal{Z})$ is a digroup that contains $\mathcal{Z}$ as a subgroup. It is known that $\mathcal{DG}_1(\mathcal{Z})$ is generated by $(0, 0)$ and it is isomorphic to the free monogenic digroup (see [26]).

4. Free abelian generalized digroups

Let us consider the notion of a generalized digroup.

Definition 6. A dimonoid $(D, \dashv, \vdash)$ is called a *generalized digroup* (see, e.g., [12, 18]) if there exists a bar-unit $e \in D$, and for every $x \in D$ there exist elements $x_e^{\overset{\ell}{-1}}$ and $x_e^{\overset{r}{-1}}$ of D such that $x \vdash x_e^{\overset{r}{-1}} = e = x_e^{\overset{\ell}{-1}} \dashv x$.

If elements $x_e^{\overset{\ell}{-1}}$ and $x_e^{\overset{r}{-1}}$ coincide, then a generalized digroup is a digroup in the sense of Definition 3. If operations $\dashv$ and $\vdash$ of a generalized digroup coincide, then it becomes a group. As for digroups, the set of all bar-units of a generalized digroup $\mathcal{D} = (D, \dashv, \vdash)$ is called the *halo part* and it is denoted by $E(\mathcal{D})$.

Commutativity and abelianity in generalized digroups have different meanings. A generalized digroup $(D, \dashv, \vdash)$ is called *commutative* [18] if both semigroups $(D, \dashv)$ and $(D, \vdash)$ are commutative, and a generalized digroup $(D, \dashv, \vdash)$ is called *abelian* [12] if $x \dashv y = y \vdash x$ for all $x, y \in D$. It is obvious that for any generalized abelian digroup $(D, \dashv, \vdash)$ the semigroups $(D, \dashv)$ and $(D, \vdash)$ are anti-isomorphic.

In [18], it was proved that there do not exist commutative generalized digroups with different operations. From [12] it follows that any abelian generalized digroup is an abelian digroup, therefore further we will write "abelian digroup" instead of "abelian generalized digroup". Some models of free (generalized) digroups were considered in [11, 18, 26].

Now we give a model of the free monogenic generalized digroup [18].

As before, by $\mathbb{Z}$ we denote the set of integers. Define operations $\dashv$ and $\vdash$ on $\mathbb{Z} \times \mathbb{Z}$ by

$$(n, m) \dashv (p, s) = (n, m + p + s + 1),$$

$$(n, m) \vdash (p, s) = (n + m + p + 1, s)$$

for all $(n, m), (p, s) \in \mathbb{Z} \times \mathbb{Z}$. The algebra $(\mathbb{Z} \times \mathbb{Z}, \dashv, \vdash)$ is denoted by FD_1.

Theorem 5 ([18]). *FD_1 is a free monogenic (by $(0, 0)$) generalized digroup with the halo*

$$E(FD_1) = \{(n, m) \mid n + m + 1 = 0\}$$

and inverses with respect to the bar unit (n, m)

$$(p, s)_{(n,m)}^{\overset{\ell}{-1}} = (n, m - s - p - 1)$$

and

$$(p,s)^{\overset{r}{-1}}_{(n,m)} = (n-s-p-1, m),$$

where $(p,s) \in FD_1$.

Proposition 2. *The class of all generalized digroups does not form a variety.*

Proof. Consider the subset $\mathbb{N} \times \mathbb{N}$ of FD_1, where $\mathbb{N}$ is the set of all positive integers. Obviously, $\mathbb{N} \times \mathbb{N}$ forms a subdimonoid. At the same time, $E(\mathbb{N} \times \mathbb{N}) = \varnothing$. It means that the class of all generalized digroups is not closed under taking subalgebras and so, it is not a variety. □

The fact that a class of algebras, like (generalized) digroups defined above, is not a variety, does not prevent the presence of free objects in it. Now we show how to construct free abelian digroups of an arbitrary rank.

Let $(D, \dashv, \vdash)$ be a digroup, $\varepsilon \in D$ a bar-unit and A a subset of D. We say that A is a generating set of $(D, \dashv, \vdash)$ respect to ε if every element of D can be expressed as a product of elements of A and their inverses respect to ε.

Let X be an arbitrary nonempty set, and let $FA(X)$ be the free abelian group generated by X, where e is the empty word. Define operations $\dashv$ and $\vdash$ on $X \times FA(X)$ by

$$(x,a) \dashv (y,b) = (x, ayb), \quad (x,a) \vdash (y,b) = (y, xab)$$

for all $(x,a), (y,b) \in X \times FA(X)$. The obtained algebra is denoted by $FAD(X)$.

Theorem 6. $FAD(X)$ *is the free abelian digroup on* $X \times \{e\}$.

Proof. By [23], $FAD(X)$ is a dimonoid satisfying the identity $x \dashv y = y \vdash x$ for all $x, y \in FAD(X)$. We state that $E(FAD(X)) = \{(y,b) \mid yb = e\}$. Indeed,

$$(x,a) \dashv (y,b) = (x, ayb) = (x,a) = (x, yba) = (y,b) \vdash (x,a)$$

for all $(x,a) \in FAD(X)$ and $(y,b) \in E(FAD(X))$. The proof that $yb = e$ for each bar-unit (y,b) is obvious. For any $(x,a) \in FAD(X)$ the inverse element with respect to the bar-unit $(y,b) \in E(FAD(X))$ is

$$((x,a)^{\overset{\ell}{-1}}_{(y,b)} = (y, ba^{-1}x^{-1}) = (x,a)^{\overset{r}{-1}}_{(y,b)}.$$

Thus, we deduce that $FAD(X)$ is an abelian digroup.

Further we fix the bar-unit $(z, z^{-1}), z \in X$, and show that $X \times \{e\}$ is a generating set of $FAD(X)$ respect to (z, z^{-1}). It is clear that the set of all inverses for $X \times \{e\}$ respect to (z, z^{-1}) is the set $\{(z, z^{-1}x^{-1}) \,|\, x \in X\}$, and we should establish that every element of $FAD(X)$ has a representation in the form of the product of a finite number of elements from $X \times \{e\} \cup \{(z, z^{-1}x^{-1}) \,|\, x \in X\}$.

Observe that for every element w of an arbitrary abelian digroup $(D, \dashv, \vdash)$ and a positive integer n,

$$\underbrace{w \dashv \ldots \dashv w}_{n} = \underbrace{w \vdash \ldots \vdash w}_{n} = w^n.$$

Let $(x, a) \in FAD(X)$. Then there exist $g_i \in X$ and $n_i \in Z$, where $1 \le i \le p$, such that $(x, a) = (x, g_1^{n_1} \ldots g_p^{n_p})$. For every $1 \le i \le p$, let

$$\overline{g_i^{n_i}} = \begin{cases} (g_i, e)^{n_i}, \ n_i > 0, \\ ((g_i, e)^{-1})^{-n_i} = (z, z^{-1}g_i{}^{-1})^{-n_i}, \ n_i < 0. \end{cases}$$

It is not to hard to show that

$$(x, g_1^{n_1} \ldots g_p^{n_p}) = (x, e) \dashv \overline{g_1^{n_1}} \dashv \ldots \dashv \overline{g_p^{n_p}}.$$

If $n_i = 0$ for some $1 \le i \le p$, then we omit $\overline{g_i^{n_i}}$, regarding that it is equal to (z, z^{-1}). It is clear that such representation is unique up to an order of $\overline{g_i^{n_i}}$, $1 \le i \le p$. As a consequence, $FAD(X)$ is generated by $X \times \{e\}$ respect to (z, z^{-1}).

Let $(D, \dashv, \vdash)$ be an arbitrary abelian digroup and let $\alpha : X \to D$ be an arbitrary map. Further we will denote the inverse element (with respect to some fixed bar-unit) for $w \in D$ by w^{-1}. Suppose that $\beta : FAD(X) \to D$ is a map defined by the rule: $(x, e)\beta = x\alpha$ and

$$(x, y_1 y_2 \ldots y_m)\beta = x\alpha \dashv y_1\widetilde{\alpha} \dashv y_2\widetilde{\alpha} \dashv \ldots \dashv y_m\widetilde{\alpha}$$

for all $x \in X, y_1, y_2, \ldots, y_m \in X \cup X^{-1}$, where

$$y_i\widetilde{\alpha} = \begin{cases} y_i\alpha, \ y_i \in X, \\ (y_i^{-1}\alpha)^{-1}, \ y_i \in X^{-1} \end{cases} \qquad (1 \le i \le m).$$

The map β is a homomorphism of digroups. Indeed, for all $(x_1, x_2 \ldots x_k)$, $(y_1, y_2 \ldots y_m) \in FAD(X)$, where $x_1, y_1 \in X$, $x_i, y_j \in X \cup X^{-1}$, $2 \le i \le k$, $2 \le j \le m$, we obtain

$$((x_1, x_2 \ldots x_k) \dashv (y_1, y_2 \ldots y_m))\beta = (x_1, x_2 \ldots x_k y_1 y_2 \ldots y_m)\beta =$$

$$= x_1\alpha \dashv x_2\widetilde{\alpha} \dashv \ldots \dashv x_k\widetilde{\alpha} \dashv y_1\widetilde{\alpha} \dashv y_2\widetilde{\alpha} \dashv \ldots \dashv y_m\widetilde{\alpha} =$$
$$= (x_1\alpha \dashv x_2\widetilde{\alpha} \dashv \ldots \dashv x_k\widetilde{\alpha}) \dashv (y_1\alpha \dashv y_2\widetilde{\alpha} \dashv \ldots \dashv y_m\widetilde{\alpha}) =$$
$$= (x_1, x_2 \ldots x_k)\beta \dashv (y_1, y_2 \ldots y_m)\beta.$$

From Theorem 9 of [12], it follows that in an abelian digroup $(D, \dashv, \vdash)$, the semigroup $(D, \dashv)$ is right commutative; we used this fact above.

Finally, using that the map β is a homomorphism of the semigroup $(X \times FA(X), \dashv)$ into the semigroup $(D, \dashv)$ and abelianity of our digroups, we deduce that

$$((x_1, x_2 \ldots x_k) \vdash (y_1, y_2 \ldots y_m))\beta = ((y_1, y_2 \ldots y_m) \dashv (x_1, x_2 \ldots x_k))\beta =$$
$$= (y_1, y_2 \ldots y_m)\beta \dashv (x_1, x_2 \ldots x_k)\beta = (x_1, x_2 \ldots x_k)\beta \vdash (y_1, y_2 \ldots y_m)\beta.$$

□

If ρ is a congruence on a generalized digroup $(D, \dashv, \vdash)$ such that $(D, \dashv, \vdash)/\rho$ is an abelian digroup, we say that ρ is an abelian digroup congruence. If $\mu : D_1 \to D_2$ is a homomorphism of generalized digroups, the kernel of μ is denoted by Δ_μ, that is,

$$\Delta_\mu = \{(x, y) \in D_1 \times D_1 \,|\, x\mu = y\mu\}.$$

Now we present the least abelian digroup congruence on the free generalized digroup. For this, we recall the construction of the free generalized digroup [11].

Let $F(X)$ be the free group generated by X, where e is the empty word. Define operations $\dashv$ and $\vdash$ on $F(X) \times X \times F(X)$ by

$$(u, x, a) \dashv (v, y, b) = (u, x, avyb),$$
$$(u, x, a) \vdash (v, y, b) = (uxav, y, b)$$

for all $(u, x, a), (v, y, b) \in F(X) \times X \times F(X)$. The algebraic system $(F(X) \times X \times F(X), \dashv, \vdash)$ is denoted by $FD(X)$. By Proposition 4 from [11], $FD(X)$ is the free generalized digroup. Note that the free generalized digroup $FD(X)$ can be obtained from the more general digroup construction first described in [19].

Proposition 3. *The map*

$$\zeta : FD(X) \to FAD(X) : (u, x, a) \mapsto (u, x, a)\zeta = (x, ua)$$

is an epimorphism inducing the least abelian digroup congruence on the free generalized digroup $FD(X)$.

Proof. Take arbitrary elements $(u, x, a), (v, y, b) \in FD(X)$. We have

$$((u, x, a) \dashv (v, y, b))\zeta = (u, x, avyb)\zeta =$$

$$= (x, uavyb) = (x, ua) \dashv (y, vb) = (u, x, a)\zeta \dashv (v, y, b)\zeta,$$

$$((u, x, a) \vdash (v, y, b))\zeta = (uxav, y, b)\zeta =$$

$$= (y, uxavb) = (x, ua) \vdash (y, vb) = (u, x, a)\zeta \vdash (v, y, b)\zeta.$$

For any $(x, u) \in FAD(X)$ there exists $(e, x, u) \in FD(X)$ such that $(e, x, u)\zeta = (x, u)$, so ζ is surjective. Therefore, ζ is an epimorphism. Since by Theorem 6 $FAD(X)$ is the free abelian digroup, Δ_ζ is the least abelian digroup congruence on $FD(X)$. □

If ρ is a congruence on a generalized digroup $(D, \dashv, \vdash)$ such that the operations of $(D, \dashv, \vdash)/\rho$ coincide and it is a group, we say that ρ is a group congruence [18].

Now we present the least group congruence on the free abelian digroup.

Proposition 4. *The map*

$$\gamma : FAD(X) \to FA(X) : (x_1, x_2 \ldots x_k) \mapsto (x_1, x_2 \ldots x_k)\gamma = x_1x_2 \ldots x_k$$

is an epimorphism inducing the least group congruence on the free abelian digroup $FAD(X)$.

Proof. For arbitrary elements $(x_1, x_2 \ldots x_k), (y_1, y_2 \ldots y_m) \in FAD(X)$, where $x_1, y_1 \in X, x_i, y_j \in X \cup X^{-1}, 2 \leq i \leq k, 2 \leq j \leq m$, we get

$$((x_1, x_2 \ldots x_k) \dashv (y_1, y_2 \ldots y_m))\gamma = (x_1, x_2 \ldots x_k y_1 y_2 \ldots y_m)\gamma =$$

$$= x_1x_2 \ldots x_k y_1 y_2 \ldots y_m = (x_1x_2 \ldots x_k)(y_1y_2 \ldots y_m) =$$

$$= (x_1, x_2 \ldots x_k)\gamma\, (y_1, y_2 \ldots y_m)\gamma,$$

$$((x_1, x_2 \ldots x_k) \vdash (y_1, y_2 \ldots y_m))\gamma = (y_1, x_1x_2 \ldots x_k y_2 \ldots y_m)\gamma =$$

$$= y_1x_1x_2 \ldots x_k y_2 \ldots y_m = x_1x_2 \ldots x_k y_1 y_2 \ldots y_m =$$

$$= (x_1x_2 \ldots x_k)(y_1y_2 \ldots y_m) = (x_1, x_2 \ldots x_k)\gamma\, (y_1, y_2 \ldots y_m)\gamma.$$

The map γ is surjective since for any $xu, y^{-1}w \in FA(X)$, where $x, y \in X, u, v \in FA(X)$, there exist $(x, u), (y, y^{-1}vy^{-1}) \in FAD(X)$ such that $(x, u)\gamma = xu$ or $(y, y^{-1}vy^{-1})\gamma = y^{-1}v$. Thus, γ is an epimorphism. Since $FA(X)$ is a free abelian group, Δ_γ is the least group congruence on $FAD(X)$. □

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