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Decomposition of matrices from $SL_2(K[x, y])$

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Dedicated to Professor Yu. A. Drozd on the occasion of his 80th birthday

Abstract. Let $\mathbb{K}$ be an algebraically closed field of characteristic zero and $\mathbb{K}[x, y]$ the polynomial ring. The group $\mathrm{SL}_2(\mathbb{K}[x, y])$ of all matrices with determinant equal to 1 over $\mathbb{K}[x, y]$ can not be generated by elementary matrices. The known counterexample was pointed out by P. M. Cohn. Conversely, A. A. Suslin proved that the group $\mathrm{SL}_r(\mathbb{K}[x_1, \ldots, x_n])$ is generated by elementary matrices for $r \geq 3$ and arbitrary $n \geq 2$, the same is true for $n = 1$ and arbitrary r. It is proven that any matrix from $\mathrm{SL}_2(\mathbb{K}[x, y])$ with at least one entry of degree ≤ 2 is either a product of elementary matrices or a product of elementary matrices and of a matrix similar to the one pointed out by P. Cohn. For any matrix $\begin{pmatrix} f & g \\ -Q & P \end{pmatrix} \in \mathrm{SL}_2(\mathbb{K}[x, y])$, we obtain formulas for the homogeneous components P_i, Q_i for the unimodular row $(-Q, P)$ as combinations of homogeneous components of the polynomials f, g, respectively, with the same coefficients.

Introduction

Let $\mathbb{K}$ be a field and $A = \mathbb{K}[x_1, \ldots, x_n]$ the polynomial ring in n variables. The group $GL_r(A)$ of all invertible matrices and its subgroup $SL_r(A)$ of matrices with determinant of 1 was studied by many authors from different points of view (see, for example, [2, 3, 5, 6], the last paper contains

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an extensive literature review). One of the important questions in studyng $GL_r(A)$ (and $SL_{rn}(A)$) is the question about generators and relations of these groups. Classical papers of A. A. Suslin and P. M. Cohn answer this question. In [3], it was proved that the group $SL_r(A)$ is generated by elementary matrices (or, in other terminology, elementary transvections) if $r \geq 3$ for arbitrary $n \geq 2$; in case $n = 1, r \geq 2$ the proof is elementary. If $r = 2$ and $n \geq 2$, then the group $SL_r(A)$ cannot be generated by elementary matrices [2]. The counterexample from [2] is the matrix $\begin{pmatrix} x^2 & xy-1 \\ xy+1 & y^2 \end{pmatrix}$ from the group $SL_2(\mathbb{K}[x,y])$. A question arises: how typical is this counterexample? We prove (Theorem 1) that any matrix from $SL_2(\mathbb{K}[x,y])$ with at least one entry of degree ≤ 2 is either a product of elementary matrices or a product of a matrix similar to the one pointed out in [2] and elementary matrices.

We consider the group $SL_2(\mathbb{K}[x,y])$ over an algebraically closed field $\mathbb{K}$ of characteristic zero. Let us recall some definitions and notations. An elementary matrix from the group $SL_2(\mathbb{K}[x,y])$ is of the form $\begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix}$ or of the form $\begin{pmatrix} 1 & 0 \\ h & 1 \end{pmatrix}$, $h \in \mathbb{K}[x,y]$. A row $(f,g) \in (\mathbb{K}[x,y])^2$ is called unimodular if there exist polynomials $P, Q \in \mathbb{K}[x,y]$ such that $Pf + Qg = 1$ (about some properties of unimodular rows see, for example, [4]). The latter means that the matrix $\begin{pmatrix} f & g \\ -Q & P \end{pmatrix}$ has the determinant of 1. The unimodular row $(-Q, P)$ will be called associated with the row (f,g). Note that by multiplying a unimodular row (f,g) from the right by the matrix $\begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix}$, where $h \in \mathbb{K}[x,y]$, the result is the unimodular row $(f, g + fh)$. Multiplying unimodular rows (f,g) by an elementary matrix from the right defines a linear transformation of the free module of rank 2 over $\mathbb{K}[x,y]$. We call such a transformation an elementary transformation. The automorphism group $\mathrm{Aut}(\mathbb{K}[x,y])$ acts naturally on the group $SL_2(\mathbb{K}[x,y])$ by the rule: for any $\theta \in \mathrm{Aut}(\mathbb{K}[x,y])$ and $A = (a_{ij}) \in SL_2(\mathbb{K}[x,y])$ put $A^\theta = (a_{ij}^\theta)$ (note that for any elementary matrix B the matrix $B^\theta, \theta \in \mathrm{Aut}(\mathbb{K}[x,y])$ is also an elementary matrix). We will use unimodular rows, and then the main result will be reformulated in matrix language. In Lemmas 5, 6, and 7 we prove that any unimodular row (f,g) with $\deg f \leq 2$ is, up to action of an automorphism $\theta \in \mathrm{Aut}\,(\mathbb{K}[x,y])$, one of the forms: $(1,0)$, or $(x^2, \psi(y)x + \gamma)$ with arbitrary $\psi(y) \in \mathbb{K}[x,y]$, or $(xy - \gamma, x^k)$, $\gamma \in \mathbb{K}^*, k \in \mathbb{Z}$, $k \geq 2$. As a consequence, we obtain the main result (Theorem 1).

In the second part of the paper, we consider unimodular rows (f, g), $\deg f = \deg g = n, n \geq 1$. We obtain formulas for homogeneous components P_i, Q_i of the associated unimodular row $(-Q, P)$ as combinations of homogeneous components of polynomials f, g, respectively, with the same coefficients (Theorem 2). These formulas can be used for studying matrices from $SL_2(\mathbb{K}[x, y])$ with entries of any degree.

1. Some properties of unimodular rows over $\mathbf{K}[x, y]$

Here some technical results are collected about unimodular rows (of length 2) over the polynomial ring $\mathbb{K}[x, y]$.

Lemma 1. (1) *Let (f, g) be a unimodular row over the ring $\mathbb{K}[x, y]$ and P, Q be polynomials such that $Pf + Qg = 1$. Then the rows (P, Q), (P, g) and (f, Q) are also unimodular.*

(2) *If (f, g) is a unimodular row, then for any endomorphism θ of the ring $\mathbb{K}[x, y]$, the row $(\theta(f), \theta(g))$ is also unimodular.*

Lemma 2. *Let $f, g, P, Q \in \mathbb{K}[x, y]$ be nonconstant polynomials such that $Pf + Qg = 0$. Then there exist polynomials $h_1, h_2, \varphi, \psi \in \mathbb{K}[x, y]$ such that $f = \varphi h_1$, $g = \varphi h_2$, $\gcd(h_1, h_2) = 1$, $P = \psi h_2$, $Q = -\psi h_1$.*

Proof. Let $\varphi = \gcd(f, g)$, $\psi_0 = \gcd(P, Q)$. Then $f = \varphi h_1$, $g = \varphi h_2$ for coprime polynomials $h_1, h_2 \in \mathbb{K}[x, y]$. Analogously $P = \psi_0 P_0$, $Q = \psi_0 Q_0$ for some coprime P_0, Q_0. Then by the conditions of the lemma we have

$$0 = Pf + Qg = \varphi\psi_0 \left(P_0 h_1 + Q_0 h_2\right).$$

The latter equalities imply

$$P_0 h_1 + Q_0 h_2 = 0. \tag{1}$$

Since $\gcd(h_1, h_2) = 1$ we have $P_0 \mid h_2$ and $h_2 \mid P_0$. But then $P_0 = \alpha h_2$ for some $\alpha \in \mathbb{K}^*$. Analogously $Q_0 = \beta h_1$ for some $\beta \in \mathbb{K}^*$. It follows from 1 that $\beta = -\alpha$. Besides, $P = \psi_0 P_0 = \psi_0 \alpha h_2$ and $Q_0 = -\psi_0 \alpha h_1$. Denoting $\psi = \psi_0 \alpha$ we get $P = \psi h_2$, $Q = -\psi h_1$. □

Lemma 3. *Let (f, g) be a unimodular row of nonconstant polynomials over $\mathbb{K}[x, y]$ and let $f = f_0 + \cdots + f_n$, $g = g_0 + \cdots + g_l$ be decomposition of f and g, respectively, in sums of homogeneous components. Then the polynomials f_n and g_l are not coprime, i.e., $\deg\gcd(f_n, g_l) \geq 1$.*

Proof. By the conditions of the lemma there exist polynomials $P, Q \in \mathbb{K}[x, y]$ such that $Pf + Qg = 1$. Let

$$P = P_0 + \cdots + P_m,\ Q = Q_0 + \cdots + Q_k$$

be their decomposition into sums of homogeneous components. It follows from the equality $Pf + Qg = 1$ that $P_m f_n + Q_k g_l = 0$ (it is obvious that $m + n = k + l$). Assume on the contrary that $\gcd(f_n, g_l) = 1$. Then, by Lemma 2, we have

$$P_m = \psi g_l,\ Q_k = -\psi f_n \tag{2}$$

for a polynomial $\psi \in \mathbb{K}[x, y]$ with $\deg \psi = m - l = k - n$. Denote by Ω the set of all pairs (P, Q) of polynomials that satisfy the equality $Pf + Qg = 1$. Choose a pair $(P, Q) \in \Omega$ such that the sum $m + k = \deg P + \deg Q$ is minimal. Then the equalities (2) imply that

$$\deg(P - \psi g) + \deg(Q + \psi f) < \deg P + \deg Q.$$

Besides, the equality holds: $(P - \psi g)f + (Q + \psi f)g = 1$. The latter contradicts the choice of the pair (P, Q). This contradiction shows that $\deg \gcd(f_n, g_l) \geq 1$. □

Corollary 1. *Let $f, g \in \mathbb{K}[x, y]$ be a unimodular row. If $\deg f = 1$, then $g = hf + c$ for some $h \in \mathbb{K}[x, y]$, $c \in \mathbb{K}^*$.*

Proof. Let

$$f = f_0 + f_1,\ g = g_0 + g_1 + \cdots + g_l$$

be the decomposition of polynomials into sums of homogeneous components. Then by Lemma 3 we have $\deg \gcd(f_1, g_l) \geq 1$. The latter means that g_l is divisible by f_1, i.e. $g_l = h_1 f_1$ for some polynomial $h_1 \in \mathbb{K}[x, y]$. But then the row $(f, g - h_1 f)$ is unimodular and $\deg(g - h_1 f) < \deg g$. Continuing such considerations we obtain a unimodular row $(f, g - hf)$ for some $h \in \mathbb{K}[x, y]$ such that $\deg(g - hf) = 0$, i.e. $g - hf = c$. Obviously $c \neq 0$ and we get $g = hf + c$, $c \in \mathbb{K}^*$. □

Let us recall that any quadratic curve $f(x, y) = 0$, $\deg f = 2$ is reduced by linear transformations of variables to one of the known canonical forms. This can be reformulated as follows:

Lemma 4. *Let $f(x, y) \in \mathbb{K}[x, y]$, $\deg f = 2$. Then there exist an affine automorphism θ of the ring $\mathbb{K}[x, y]$ of the form $\theta(x) = \alpha_1 x + \beta_1 y + \gamma_1$, $\theta(y) = \alpha_2 x + \beta_2 y + \gamma_2$ such that $\theta(f)$ is a polynomial of the following type:*

(1) $f(x,y) = x^2 + \gamma$, $\gamma \in \mathbb{K}$;
(2) $f(x,y) = x^2 + y$;
(3) $f(x,y) = xy + \gamma$, $\gamma \in \mathbb{K}$.

Lemma 5. *Let (f,g) be a unimodular row such that $f = x^2+y$. Then this row is reduced to the row $(1,0)$ by elementary transformations, i.e. there exist elementary matrices $B_1,\dots,B_k$ such that $(f,g)B_1,\dots,B_k = (1,0)$.*

Proof. Let us write the polynomial g as a polynomial of x with coefficients depending on y,

$$g(x,y) = g_0(y) + g_1(y)x + \dots + g_k(y)x^k.$$

Denote

$$h(x,y) = g_2(y) + g_3(y)x + \dots + g_k(y)x^{k-2}.$$

Then

$$(f,g)\cdot\begin{pmatrix} 1 & -h \\ 0 & 1 \end{pmatrix} = (f, g_0(y) + g_1(y)x - yh(x,y)).$$

Note that the polynomial $g^{(1)} = g_0(y)+g_1(y)x-yh(x,y)$ is of degree $< k$ on x, i.e., $\deg_x g < \deg_x g^{(1)}$. Repeating this process for the unimodular row $(x^2+y, g^{(1)})$ we obtain as a result a unimodular row of the form $(x^2+y, g^{(s)})$ for some $s \geq 2$ with $\deg_x g^{(s)} \leq 1$. So we can assume without loss of generality that $g(x,y) = g_0(y) + g_1(y)x$. By the conditions of the lemma, there exist polynomials $P(x,y), Q(x,y) \in \mathbb{K}[x,y]$ such that

$$P(x,y)(x^2+y) + Q(x,y)(g_0(y) + g_1(y)x) = 1.$$

Putting here $y = -x^2$ we get the equality

$$Q(x,-x^2)(g_0(-x^2) + xg_1(-x^2)) = 1.$$

It follows from this equality that $g_0(-x^2)+xg_1(-x^2) = c$ for some $c \in \mathbb{K}^*$. Since $\deg_x g_0(-x^2)$ is even and $\deg_x xg_1(-x^2)$ is odd we get $g_1 = 0$ and $g_0(y) \in \mathbb{K}$. But then $g = g_0 \in \mathbb{K}^*$ and the unimodular row (x^2+y, g_0) obviously is reduced to the row $(1,0)$. □

Lemma 6. *Let (f,g) be a unimodular row, where $f = x^2+\gamma$, $\gamma \in \mathbb{K}$. Then this row can be reduced by elementary transformations to either the row $(1,0)$, or to the row $(x^2+\gamma, x\psi(y)+\delta)$, $\delta \in \mathbb{K}$, $\deg\psi(y) \geq 1$.*

Proof. Write down the polynomial $g(x,y)$ as a polynomial of x with coefficients in $\mathbb{K}[y]$

$$g = g_0(y) + g_1(y) + \dots + g_k(y)x^k.$$

Repeating the consideration from the proof of Lemma 5 one can assume without loss of generality that $g = g_0(y) + g_1(y)x$ for some polynomials $g_0(y)$ and $g_1(y)$. Since $(x^2 + \gamma, g_0(y) + g_1(y)x)$ is a unimodular row, there exist polynomials $P, Q \in \mathbb{K}[x, y]$ such that

$$P(x^2 + \gamma) + Q(g_0(y) + g_1(y)x) = 1.$$

Note that for any polynomial $A(x, y) \in \mathbb{K}[x, y]$, the polynomials

$$\overline{P}(x, y) = P(x, y) + A(x, y)g(x, y),\ \overline{Q}(x, y) = Q(x, y) - A(x, y)(x^2 + \gamma)$$

also satisfy the equality $(x^2+\gamma)\overline{P}+g(x, y)\overline{Q} = 1$. Therefore, without loss of generality, one can reduce the unimodular row (P, Q) by elementary transformations to the row $(P, Q_0(y) + Q_1(y)x)$ without changing the initial unimodular row $(x^2 + \gamma, g(x, y))$. We get the equality

$$P(x, y)(x^2 + \gamma) + (Q_0(y) + Q_1(y)x)(g_0(y) + g_1(y)x) = 1. \tag{3}$$

First, let $\gamma \neq 0$. Substituting in formulas (3) x for $\sqrt{-\gamma}$ and then x for $-\sqrt{-\gamma}$ we obtain two inclusions $g_1(y)\sqrt{-\gamma}+g_0(y) \in \mathbb{K}$ and $-g_1(y)\sqrt{-\gamma}+g_0(y) \in \mathbb{K}$. It follows from these inclusions that $g_0(y) \in \mathbb{K}$ and $g_1(y) \in \mathbb{K}$. But then from (3) we see that $Q_0(y), Q_1(y) \in \mathbb{K}$. The equality (3) shows also that $g_1 = 0$ and $Q_1 = 0$, i.e., $g(x, y) = c_1$ and $Q(x, y) = c_2$ for some $c_1, c_2 \in \mathbb{K}$. Therefore the unimodular row $(x^2 + \gamma, g)$ can be reduced (by elementary transformations) to the row $(1, 0)$.

Now let $\gamma = 0$, i.e., $f(x, y) = x^2$. Putting $x = 0$ in the equality (3) we get $Q_0(y)g_0(y) = 1$. Thus $Q_0, g_0 \in \mathbb{K}^*$. The latter means that $g = x\psi(y) + \delta$, where $\psi(y) = Q_1(y)$ and $\delta = Q_0$. Note that the unimodular row associated with $(x^2, x\psi(y)+\delta)$ is the row $(\frac{x\psi(y)-\delta}{\delta^2}, \delta^{-2}\psi^2(y))$ because the matrix $\begin{pmatrix} x^2 & x\psi(y) + \delta \\ \delta^{-2}(x\psi(y) - \delta) & \delta^{-2}\psi^2(y) \end{pmatrix}$ has the determinant 1. □

2. The main theorem

We need to consider the last case when the unimodular row is of the form $(xy + \gamma, g(x, y))$.

Lemma 7. *Let (f, g) be a unimodular row with $f(x, y) = xy + \gamma$, $\gamma \in \mathbb{K}$. Then this row can be reduced by elementary transformations to the unimodular row $(xy + \gamma, x^k)$ or to the row $(xy + \gamma, (-\gamma^{-1}y)^k)$ with integer $k \geq 2$, or to the row $(1, 0)$.*

Proof. By the conditions of the lemma we have an equality of the form

$$P(x,y)(xy+\gamma)+Q(x,y)g(x,y)=1 \tag{4}$$

for some polynomials $P, Q \in \mathbb{K}[x,y]$. Write down the polynomial $g(x,y)$ in the form $g(x,y) = \varphi(x) + \psi(y) + xyh(x,y)$ for some polynomials $\varphi(x), \psi(y), h(x,y) \in \mathbb{K}[x,y]$. Then we get the equality

$$(xy+\gamma, g)\begin{pmatrix} 1 & -h(x,y) \\ 0 & 1 \end{pmatrix} = (xy+\gamma, \varphi(x)+\psi(y)-\gamma h(x,y)).$$

If $h(x,y) \neq 0$ we can write $h(x,y) = \varphi_1(x) + \psi_1(y) + xyh_1(x,y)$ and repeat the previous considerations. As a result, we may assume without loss of generality that $g(x,y) = \varphi(x) + \psi(y)$. Analogously repeating considerations from the proof of Lemma 6 we may assume that $Q(x,y) = u(x) + v(y)$ for some polynomials $u(x), v(y) \in \mathbb{K}[x,y]$.

First, let $\gamma \neq 0$. Let us put $y = -\gamma/x$ in the equality (4). We get $(u(x) + v(-\gamma/x))(\varphi(x) + \psi(-\gamma/x)) = 1$. One can easily prove that an element $p(x, x^{-1})$ from ring $\mathbb{K}[x, x^{-1}]$ is invertible in this ring if and only if $p = \alpha x^k$ for some $k \in \mathbb{Z}$, $\alpha \in \mathbb{K}^*$. So, we have $g(x,y) = x^k$, $Q(x,y) = (-\gamma^{-1}y)^k$ or $g(x,y) = (-\gamma^{-1}y)^k$, $Q(x,y) = x^k$ for some $k \geq 0$. In any case, the polynomial $P(x,y)$ is of the form

$$P(x,y) = \frac{1-(-\gamma^{-1}xy)^k}{\gamma+xy} = \gamma^{-1}\left(1+\left(-\frac{xy}{\gamma}\right)+\cdots+\left(-\frac{xy}{\gamma}\right)^{k-1}\right).$$

As a result, we get two unimodular rows:

1) $(xy+\gamma, x^k)$ with the associated row $\left(-(-\gamma^{-1}y)^k, \frac{1-(-\gamma^{-1}xy)^k}{\gamma+xy}\right)$;

2) $(xy+\gamma, (-\gamma^{-1}y)^k)$ with the associated row $\left(-x^k, \frac{1-(-\gamma^{-1}xy)^k}{\gamma+xy}\right)$.

Note that one can assume that $k \geq 2$. Really, in other case the row $(xy+\gamma, x^k)$ is reduced to the row $(1,0)$ because of Corollary 1. Let now $\gamma = 0$. Let us replace x with 0 in the equality (4). Then we have $(u(0) + v(y))(\varphi(0) + \psi(y)) = 1$. This equality implies obviously $v(y), \psi(y) \in \mathbb{K}$. Analogously after substituting 0 instead of y in (4) we get $v(x), \psi(x) \in \mathbb{K}$. We see that in this case the polynomial $g(x,y)$ is constant and therefore the unimodular row can be reduced to the row $(1,0)$. The proof is complete. □

Theorem 1. *Let* $A = \begin{pmatrix} a_{11}(x,y) & a_{12}(x,y) \\ a_{21}(x,y) & a_{22}(x,y) \end{pmatrix} \in SL_2(\mathbb{K}[x,y])$. *If* $\deg a_{ij} = 2$ *for some* $i, j \in \{1,2\}$, *then there exists an automorphism* $\theta \in \mathrm{Aut}(\mathbb{K}[x,y])$ *such that* A^θ *is one of the types:*

1) $A^\theta = B_1 B_2 \dots B_k$*,* $k \geq 1$*,* B_i *are elementary matrices;*

2) $A^\theta = B_1 \dots B_s C B_{s+1} \dots B_k$*, where* $B_1 \dots B_s$*,* $B_{s+1} \dots B_k$ *are elementary matrices and* C *is one of the form:*

a) $\begin{pmatrix} x^2 & x\psi(y)+\delta \\ \frac{x\psi(y)-\delta}{\delta^2} & \frac{\psi(y)^2}{\delta^2} \end{pmatrix}$ *b)* $\begin{pmatrix} xy+\gamma & x^k \\ -(-\gamma^{-1}y)^k & \frac{1-(-\gamma^{-1}xy)^k}{\gamma+xy} \end{pmatrix}$

for some $\delta, \gamma \in \mathbb{K}^*$*,* $\psi(y) \in \mathbb{K}[x,y]$*,* $k \in \mathbb{Z}$*,* $k \geq 2$*.*

Proof. Multiplying the matrix A from the left or from the right by the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ we can assume without loss of generality that $i = j = 1$, i.e. $\deg a_{1\,1} \leq 2$. Applying a linear automorphism θ to the matrix A we can reduce (by Lemma 3) the element $a_{1\,1}(x,y)$ to one of the forms

1) $a_{1\,1}(x,y) = x^2 + y$;
2) $a_{1\,1}(x,y) = x^2 + \gamma$;
3) $a_{1\,1}(x,y) = xy + \gamma$.

First, let $a_{1\,1}(x,y) = x^2 + y$. Then applying Lemma 5 to the first row of the matrix A we get the matrix $\begin{pmatrix} 1 & 0 \\ b(x,y) & 1 \end{pmatrix}$ for a polynomial $b(x,y) \in \mathbb{K}[x,y]$, recall that multiplying from the left by elementary matrices makes elementary transformations in the first and second rows of A. The latter means that A is a product of elementary matrices. In the case $a_{1\,1}(x,y) = x^2 + \gamma$, $\gamma \in \mathbb{K}$ we get either a product of elementary matrices $A = B_1 \dots B_k$ or a product of the form $A = B_1 \dots B_{i-1} C B_{i+1} \dots B_k$, where B_i are elementary matrices and C is of the form

$$C = \begin{pmatrix} x^2 & x\psi(y)+\delta \\ \delta^{-2}(x\psi(y)-\delta) & \delta^{-2}\psi(y)^2 \end{pmatrix}.$$

By Lemma 7, the last case $a_{1\,1}(x,y) = xy + \gamma$, $\gamma \in \mathbb{K}$ yields the product $A = B_1 \dots B_{i-1} C B_{i+1} \dots B_k$ with C of the form

$$C_1 = \begin{pmatrix} xy+\gamma & x^k \\ -(-\gamma^{-1}y)^k & \frac{1-(-\gamma^{-1}xy)^k}{\gamma+xy} \end{pmatrix}$$

or of the form

$$C_2 = \begin{pmatrix} xy+\gamma & (-\gamma^{-1}y)^k \\ -x^k & \frac{1-(-\gamma^{-1}xy)^k}{\gamma+xy} \end{pmatrix}.$$

Note that the matrices C_1 and C_2 are conjugated by the automorphism $\theta : x \mapsto -\gamma^{-1}y,\ y \mapsto -\gamma x$. The proof is complete. □

3. Formulas for associated rows

If a unimodular row (f, g) is given, then there exists a unimodular row $(-Q, P)$ such that $Pf + Qg = 1$ (then the matrix $\begin{pmatrix} f & g \\ -Q & P \end{pmatrix}$ has determinant of 1). Such a row $(-Q, P)$ is unique up to a row $(-\lambda Q, \lambda P)$ for an arbitrary polynomial $\lambda \in \mathbb{K}[x, y]$. Really, if $P'f + Q'g = 1$ for a row (P', Q'), then $(P - P')f + (Q - Q')g = 0$. By Lemma 2, $P - P' = \lambda g$, $Q - Q' = \lambda f$ for some $\lambda \in \mathbb{K}[x, y]$ and therefore

$$(P', Q') = (P, Q) + (-\lambda g, \lambda f).$$

Let us point out how one can write homogeneous components of polynomials P, Q using homogeneous components of g and f respectively. We restrict ourselves only to polynomials f, g of the same degree. Let $\deg f = \deg g = n$. Then obviously $\deg P = \deg Q = m$ for some m. Write down polynomials f, g, P, Q as sums of their homogeneous components

$$f = f_0 + \cdots + f_n, \; g = g_0 + \cdots + g_n,$$
$$P = P_0 + \cdots + P_m, \; Q = Q_0 + \cdots + Q_m.$$

Denote $\varphi = \gcd(f_n, g_n)$. We assume that all the polynomials f, g, P, Q are nonconstant ones. Then by the Lemma 3, $\deg \varphi \geq 1$. It turns out that $\varphi^{i+1} P_{m-i}$ and $\varphi^{i+1} Q_{m-i}$ can be written as linear combinations of $g_i's$ and $f_i's$, respectively, with the same polynomial coefficients.

Theorem 2. *There exist homogeneous polynomials $\alpha_0, \ldots, \alpha_m$ such that for $0 \leq i \leq m$*

$$\varphi^{i+1} P_{m-i} = \sum_{j=0}^{\min(i,n)} \varphi^j \alpha_{i-j} g_{n-j},$$
$$-\varphi^{i+1} Q_{m-i} = \sum_{j=0}^{\min(i,n)} \varphi^j \alpha_{i-j} f_{n-j}. \qquad (\star)$$

Proof. Induction on i. The case $i = 0$ is a consequence of Lemma 2. Really, we have $P_m f_n + Q_m g_n = 0$. Let

$$\varphi = \gcd(f_n, g_n), \; h_1 = f_n/\varphi, \; h_2 = g_n/\varphi.$$

By Lemma 2 $P_m = \psi h_2$, $Q_m = -\psi h_1$ for some $\psi \in \mathbb{K}[x, y]$. Then

$$\varphi P_m = \psi \varphi h_2 = \psi g_n, \; -\varphi Q_m = \psi \varphi h_1 = \psi f_n.$$

Putting $\alpha_0 = \psi$ we get the case $i = 0$. Let the formulas $(\star)$ be true for $i' < i$, let us prove it for i. Since $Pf + Qg = 1$ we have equalities for homogeneous components in the left side of the later equality: $(Pf + Qg)_{m+n-i} = 0$ for $0 \le i \le m$. But the left side of the latter equality can be written in the form

$$\sum_{k=0}^{\min(i,n)} (P_{m-i+k} f_{n-k} + Q_{m-i+k} g_{n-k}) = 0.$$

After multiplying this equality by φ^{i+1} we can rewrite it for $0 \le i \le m$ in the form

$$\sum_{k=0}^{\min(i,n)} \varphi^k \left(\varphi^{i-k+1} P_{m-i+k} f_{n-k} + \varphi^{i-k+1} Q_{m-i+k} g_{n-k}\right) = 0.$$

Replacing P_{m-i+k} and Q_{m-i+k}, $k \ge 1$ by their expressions due to the induction hypothesis we obtain the equality (we denote $min(i,n)$ by $i \wedge n$ for brevity in the next part of the proof):

$$0 = \varphi^{i+1} (P_{m-i} f_n + Q_{m-i} g_n) + \\ + \sum_{k=1}^{i\wedge n} \varphi^k \left(f_{n-k} \sum_{j=0}^{i\wedge n} \varphi^j \alpha_{i-k-j} g_{n-j} - g_{n-k} \sum_{j=0}^{i\wedge n} \varphi^j \alpha_{i-k-j} f_{n-j} \right).$$

The last equality can be rewritten in the form

$$\varphi^{i+1} (P_{m-i} f_n + Q_{m-i} g_n) + g_n \sum_{k=1}^{i\wedge n} \varphi^k \alpha_{i-k} f_{n-k} - f_n \sum_{k=1}^{i\wedge n} \varphi^k \alpha_{i-k} g_{n-k} + \\ + \sum_{\substack{1 \le j,k \le n \\ j+k \le i}} \varphi^{j+k} \alpha_{i-k-j} f_{n-k} g_{n-j} - \sum_{\substack{1 \le j,k \le n \\ j+k \le i}} \varphi^{j+k} \alpha_{i-k-j} f_{n-j} g_{n-k} = 0.$$

Note that the last two sums in this equality give as result 0 and we can write the last equality as

$$\left(\varphi^{i+1} P_{m-i} - \sum_{k=1}^{i\wedge n} \varphi^k \alpha_{i-k} g_{n-k}\right) f_n + \\ + \left(\varphi^{i+1} Q_{m-i} - \sum_{k=1}^{i\wedge n} \varphi^k \alpha_{i-k} f_{n-k}\right) g_n = 0.$$

It follows from Lemma 2 that there exists a polynomial α_i such that

$$\varphi^{i+1} P_{m-i} - \sum_{k=1}^{i\wedge n} \varphi^k \alpha_{i-k} g_{n-k} = \alpha_i g_n,$$

$$\varphi^{i+1}Q_{m-i} + \sum_{k=1}^{i\wedge n} \varphi^k \alpha_{i-k} f_{n-k} = -\alpha_i f_n.$$

These equalities can be rewritten (in the initial notation) in the form

$$\varphi^{i+1}P_{m-i} = \sum_{k=0}^{\min(i,n)} \varphi^k \alpha_{i-k} g_{n-k}, \qquad -\varphi^{i+1}Q_{m-i} = \sum_{k=0}^{\min(i,n)} \varphi^k \alpha_{i-k} f_{n-k}.$$

The proof is complete. □

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