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The semisimple dimension of a certain type of module

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Abstract. In this paper we define and study a new kind of dimension called, semisimple dimension, that measures how far a module is from being semisimple. Like other kinds of dimensions, this is an ordinal valued invariant. It is shown that a Noetherian module with semisimple dimension is an Artinian module. Moreover, we involve the theory of graphs within of such dimension achieving some applications for the edge ideal of a graph.

Introduction

Throughout this paper, R is a commutative ring with non-zero identity.

In the literature, many kinds of dimensions have been defined and investigated for modules and they have important roles in the study of ring and module theory. For example the Krull dimension has been first defined in 1928 by W. Krull for a commutative Noetherian ring, motivated by E. Noether's studies about the relationship between the chain of prime ideals and dimension of algebraic varieties. After that this theory has been investigated and developed for non commutative rings and modules by many authors, such as W. Krull, G. Krause, A. V. Jategaonkar, and many other people. Also, uniform module and uniform dimension of a

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module have been introduced and studied by Goldie (1958 – 1960). It is also referred to as Goldie dimension. Recently, [3] introduced uniserial dimension of a module. Furthermore, [4] introduced the dual version couniserial dimension. Each of these dimensions provides useful technical tools for the study of ring and module structure.

In this paper we define a notion of dimension of modules, called semisimple dimension. Similar to many other kinds of dimension, the value of this dimension is an ordinal. In fact, semisimple dimension of a module M shows in some sense that how far is M from being semisimple. In order for to define semisimple dimension for modules over a ring R, we define, by transfinite induction, for every ordinal $\alpha \geq 1$ the class $\mathfrak{X}_\alpha$ of R-modules. In the first step, let $\mathfrak{X}_1$ be the class of all non-zero semisimple modules. Consider an ordinal $\alpha > 1$; and assume that $\mathfrak{X}_\beta$ has been defined for ordinals $\beta < \alpha$. Now suppose that $\mathfrak{X}_\alpha$ be the class of all R-modules M for which each of non-zero submodules N of M, which is not a direct summand of M, we have

$$N \in \bigcup_{\beta<\alpha} \mathfrak{X}_\beta.$$

If there exists some $\mathfrak{X}_\alpha$ containing the R-module M, then we say that the semisimple dimension of M is defined or M has semisimple dimension and the least such α is called the semisimple dimension of M, denoted by $\mathrm{s.s.\,dim}(M)$. For $M = 0$, we define $\mathrm{s.s.\,dim}(M) = 0$. If for a non-zero module M there exists no $\mathfrak{X}_\alpha$ containing M, then we say that $\mathrm{s.s.\,dim}(M)$ is not defined, or that M has no semisimple dimension. It is clear by definition that every submodule of a module with semisimple dimension must have semisimple dimension.

In the Section 2, we put some definitions and prerequisites for a better understanding of the theory and results. We introduce preliminaries of the theory of graphs which involving the edge ideal of a graph G; associated to the graph G is a monomial ideal

$$I\left(G\right) = \left(v_i v_j \mid v_i v_j \text{ is an edge of } G\right),$$

with $v_i v_j = v_j v_i$ and with $i \neq j$, in the polynomial ring $R = K[v_1, v_2, \ldots, v_s]$ over a field K, called the **edge ideal** of G. The preliminaries of the theory of graphs were introduced in this Section 2 together with the concepts suitable for the work.

In the Section 3, we prove some properties of the semisimple dimension, properties that involve the edge ideal of a graph G, which is a graph simple and finite with no isolated vertices.

Throughout of the paper, we mean by a graph G, a finite simple graph with the vertex set $V(G)$ and with no isolated vertices. Here we use properties of commutative algebra and homological algebra for the development of the results (see [2] and [5]).

1. Prerequisites of the graph theory

Let us present in this section the concepts of the graphs theory that we will use in the course of this work.

1.1. Edge ideal of a graph

This section is in accordance with [1] and [6].

Let $R = K[v_1, \ldots, v_s]$ be a polynomial ring over a field K, and let $Z = \{z_1, \ldots, z_q\}$ be a finite set of monomials in R. The **monomial subring** spanned by Z is the K-subalgebra,

$$K[Z] = K[z_1, \ldots, z_q] \subset R.$$

In general, it is very difficult to certify whether $K[Z]$ has a given algebraic property - e.g., Cohen-Macaulay, normal - or to obtain a measure of its numerical invariants - e.g., Hilbert function. This arises because the number q of monomials is usually large.

Thus, consider any graph G, simple and finite without isolated vertices, with vertex set $V(G) = \{v_1, \ldots, v_s\}$.

Let Z be the set of all monomials $v_i v_j = v_j v_i$, with $i \neq j$, in $R = K[v_1, \ldots, v_s]$, such that $\{v_i v_j\}$ is an edge of G, i.e., the graph finite and simple G, with no isolated vertices, is such that the squarefree monomials of degree two are defining the edges of the graph G.

Definition 1. A *walk* of length s in G is an alternating sequence of vertices and edges $w = \{v_1, z_1, v_2, \ldots, v_{s-1}, z_h, v_s\}$, where $z_i = \{v_{i-1} v_i\}$ is the edge joining v_{i-1} and v_i.

Definition 2. A walk is **closed** if $v_1 = v_s$. A walk may also be denoted by $\{v_1, \ldots, v_s\}$, the edges being evident by context. A **cycle** of length s is a closed walk, in which the points $v_1, \ldots, v_s$ are distinct.

A **path** is a walk with all the points distinct. A **tree** is a connected graph without cycles and a graph is **bipartite** if all its cycles are even. A vertex of degree one will be called an **end point**.

Definition 3. A subgraph $G^{'} \subseteq G$ is called **induced** if $v_i v_j = v_j v_i$, with $i \neq j$, is an edge of $G^{'}$ whenever v_i and v_j are vertices of $G^{'}$ and $v_i v_j$ is an edge of G.

The **complement** of a graph G, for which we write G^c, is the graph on the same vertex set in which $v_i v_j = v_j v_i$, with $j \neq i$, is an edge of G^c if and only if it is not an edge of G. Finally, let C_k denote the cycle on k vertices; a **chord** is an edge which is not in the edge set of C_k. A cycle is called **minimal** if it has no a chord.

If G is a graph without isolated vertices, simple and finite, then let R denote the polynomial ring on the vertices of G over some fixed field K.

Definition 4 ([1])**.** According to the previous context, the **edge ideal** of a finite simple graph G, with no isolated vertices, is defined by

$$I(G) = (v_i v_j \mid v_i v_j \text{ is an edge of } G),$$

with $v_i v_j = v_j v_i$, and with $i \neq j$.

2. The results about semisimple dimension of the edge ideal of a graph

In this section, we presented some results about the semisimple dimension which involve the theory of graphs together with the edge ideal of a graph G, which is simple and finite and with no isolated vertices.

Here, we take K a fixed field and we consider $K[v_1, v_2, \dots, v_s]$ the ring polynomial over the field K. Since K is a field, we have that K is a Noetherian ring and then $K[v_1, \dots, v_s]$ is also a Noetherian ring (Theorem of the Hilbert Basis).

Remark 1. By the previous context, $R = K[v_1, v_2, \dots, v_s]$ is a Noetherian ring. Thus, the edge ideal $I(G)$ is an R-module, and thus we can to get characterizations for this module under certain hypothesis.

As we defined in the introduction, semisimple dimension of an R-module is an ordinal valued invariant. For basic concepts about ordinals the reader is refered to [7]. We start this section with a remark on the definition of semisimple dimension.

Remark 2. It can be easily seen from the definition that, if an R-module M has semisimple dimension and $N \leq M$, then N has also semisimple dimension and $\text{s.s.}\dim(N) \leq \text{s.s.}\dim(M)$. Moreover, if $\text{s.s.}\dim(M) =$

s.s. dim(N), where N is a submodule of M, then N is a direct summand of M. Also, since every set of ordinal numbers has a supremum, it is an immediate consequence of the definition that M has semisimple dimension if and only if for every submodule N of M which N is not a direct summand of M, s.s. dim(N) is defined. In the latter case, if

$$\alpha = \sup \{\text{s.s.}\dim(N) \mid N \leq M, N \text{ is not a direct summand of } M\},$$

then s.s. $\dim(M) \leq \alpha + 1$.

The following proposition helps us to determine that the edge ideal of a graph G has semisimple dimension.

Proposition 1. *Let $R = K[v_1, \ldots, v_s]$ be the ring polynomial, $I(G)$ the edge ideal in R of a finite simple graph G, with no isolated vertices. The following assertions are equivalent.*

(1) *$I(G)$ has semisimple dimension.*

(2) *For every descending chain of submodules $G_1 \geq G_2 \geq \ldots$, there exists $n \geq 1$ such that G_j is a direct summand of G_i for all $n \leq i \leq j$.*

Proof. (1) ⇒ (2). Let $G_1 \geq G_2 \geq \ldots$ be a chain of submodules of $I(G)$. Put

$$\gamma = \inf \{\text{s.s.}\dim(G_n) \mid n \geq 1\}.$$

So $\gamma = \text{s.s.}\dim(G_n)$ for some $n \geq 1$. By Remark 2, G_j is a direct summand of G_i, for all $n \leq i \leq j$, because γ is the infimum.
(2) ⇒ (1). Let on contrary, $I(G)$ does not have semisimple dimension. Then there exists a submodule G_1 of $I(G)$ which is not a direct summand of $I(G)$ and G_1 does not have semisimple dimension. So there exists a submodule G_2 of G_1 such that G_2 is not a direct summand of G_1 and G_2 does not have semisimple dimension. Continuing this process, we obtain a descending chain of submodules $G_1 \geq G_2 \geq \ldots$, such that for every $1 \leq i \leq j$, G_j is not a direct summand of G_i and this is a contradiction. □

The following corollary is an immediate consequence of Proposition 1.

Corollary 1. *Let $R = K[v_1, \ldots, v_s]$ be the ring polynomial, $I(G)$ the edge ideal in R of a finite simple graph G, with no isolated vertices. Suppose that $I(G)$ is an Artinian R-module. Then $I(G)$ has semisimple dimension.*

In the following proposition, we characterize Artinian modules in terms of having semisimple dimension and finite uniform dimension.

Proposition 2. *Let $R = K[v_1, \dots, v_s]$ be the ring polynomial, $I(G)$ the edge ideal in R of a finite simple graph G, with no isolated vertices. Thus, the R-module $I(G)$ has semisimple dimension and finite uniform dimension if and only if it is an Artinian module.*

Proof. ($\Rightarrow$): Suppose that $I(G)$ is the R-module with finite uniform dimension $\mathrm{u.dim}(I(G)) = m$, and consider a descending chain $G_1 \geq G_2 \geq \dots$ of submodules of $I(G)$. By Proposition 1, there exists $n \geq 1$ such that G_j is a direct summand of G_i for all $n \leq i \leq j$. Thus there exist submodules $K_1, \dots, K_{m-1}$ such that

$$K_1 \oplus K_2 \oplus \dots \oplus K_{m-1} \oplus G_{n+m} = G_n.$$

But since $\mathrm{u\,dim}(G_n) \leq m$ we have $G_t = G_{n+m}$ for all $t \geq n+m$ and so $I(G)$ is Artinian.
($\Leftarrow$): The assertion holds by Corollary 1 and [8, Theorems 31.1 and 21.3]. □

For a right R-module M, we say that M is of finite length if it has a composition series. A right R-module M is of finite length if and only if M is both right Noetherian and right Artinian.

The length of any two composition series of M_R are the same and it is said to be the length of M_R and is denoted by $\mathrm{length}(M)$.

A module of finite length obviously has semisimple dimension.

The following proposition shows a relation between semisimple dimension of the edge ideal $I(G)$ with finite length and $\mathrm{length}(I(G))$.

Proposition 3. *Let $R = K[v_1, \dots, v_s]$ be the ring polynomial, $I(G)$ the edge ideal in R of a finite simple graph G, with no isolated vertices. Suppose that $I(G)$ has semisimple dimension. If $I(G)$ is a Noetherian R-module, then we have the following:*

(1) *$I(G)$ is an Artinian R-module.*

(2) $\mathrm{s.s.dim}(I(G)) \leq \mathrm{length}(I(G))$.

(3) *If $I(G) \neq 0$ then* $\mathrm{length}(I(G)) - \mathrm{u.dim}(I(G)) + 1 \leq \mathrm{s.s.dim}(I(G))$.

(4) *If $I(G)$ is uniform, then* $\mathrm{s.s.dim}(I(G)) = \mathrm{length}(I(G))$.

(5) *For an exact sequence*

$$0 \to G_1 \to I(G) \to I(G)/G_1 \to 0$$

with $I(G) \neq 0$*,* $\text{s.s.}\dim(G_1)+\text{s.s.}\dim(I(G)/G_1) \leq \text{s.s.}\dim(I(G)) + \text{u.}\dim(I(G)) - 1$*. In particular, if* G_1 *and* $I(G)/G_1$ *are uniform, then* $\text{s.s.}\dim(G_1) + \text{s.s.}\dim(I(G)/G_1) = \text{s.s.}\dim(I(G))$*.*

Proof. (1) By Proposition 2, $I(G)$ is Artinian.
(2) We prove this part by induction on $\text{length}(M) = n$. The case $n = 1$ is trivial. Now let $n \geq 1$ and K be a proper submodule of $I(G)$. Then, by assumption, $\text{s.s.}\dim(K) \leq \text{length}(K) < \text{length}(M)$. Now using Remark 2, $\text{s.s.}\dim(M) \leq \text{length}(M)$.
(3) Set $\text{length}(M) = n$ and $\text{u.}\dim(M) = m$. Since $I(G)$ is Artinian, we have $\text{soc}(I(G)) \leq_e I(G)$. Hence $\text{length}(I(G)/\text{soc}(I(G)))+\text{u.}\dim(I(G)) = \text{length}(I(G))$. Now, we consider the chain

$$\text{soc}(I(G)) = G_1 \leq G_2 \leq \ldots \leq G_{n-m-1} \leq G_{n-m} = I(G)$$

such that G_i/G_{i-1} is simple for every $2 \leq i \leq n-m$. Thus we have $G_2 \in \mathfrak{X}_2, \ldots, G_{n-m} \in \mathfrak{X}_{n-m}$. This shows that $\text{s.s.}\dim(I(G)) \geq n-m+1$, and so the assertion holds.
(4) Follows immediately from (2) and (3).
(5) It is well-known that $\text{length}(G_1) + \text{length}(I(G)/G_1) = \text{length}(I(G))$. On the other hand, by (2) and (3), we have $\text{s.s.}\dim(I(G)) \leq \text{length}(I(G)) \leq \text{s.s.}\dim(I(G)) + \text{u.}\dim(I(G)) - 1$.

Thus we have

$$\text{s.s.}\dim(G_1) + \text{s.s.}\dim(I(G)/G_1) \leq \text{length}(G_1) + \text{length}(I(G)/G_1)$$

and so, since

$$\text{length}(G_1) + \text{length}(I(G)/G_1) = \text{length}(I(G)),$$

we have that

$$\begin{aligned}\text{s.s.}\dim(G_1) + \text{s.s.}\dim(I(G)/G_1) &\leq \text{length}(I(G)) \\ &\leq \text{s.s.}\dim(I(G)) + \text{u.}\dim(I(G)) - 1,\end{aligned}$$

as desired. The last statement it follows immediately from (4). □

Corollary 2. *Let* $R = K[v_1, \ldots, v_s]$ *be the ring polynomial,* $I(G)$ *the edge ideal in* R *of a finite simple graph* G*, with no isolated vertices. Suppose that* $I(G)$ *is an* R*-module non-zero. If* $I(G)$ *is of finite length and* $I(G) = G_1 \oplus \ldots \oplus G_n$*, then*

$$\text{s.s.}\dim(G_1) + \ldots + \text{s.s.}\dim(G_n) \leq \text{s.s.}\dim(I(G)) + \text{u.}\dim(I(G)) - 1.$$

In particular, in case $I(G)$ *is a uniform or a semisimple module, then the equality holds.*

Proof. The proof follows immediately from part (5) of Proposition 3. □

Proposition 4. *Let $R = K[v_1, \dots, v_s]$ be the ring polynomial, $I(G)$ the edge ideal in R of a finite simple graph G, with no isolated vertices. Suppose that $I(G)$ is an R-module finitely generated. If $I(G)$ has semisimple dimension and $I(G) \cong I(G) \oplus I(G_1)$ for some R-module $I(G_1)$, where G_1 is a graph finite and simple with no isolated vertices, then $I(G_1)$ has finite length.*

Proof. First we show that $I(G_1)$ is Noetherian. If N_1 is a submodule of $I(G_1)$ that is not finitely generated, then $I(G)$ contains a submodule K_1 isomorphic to $I(G) \oplus N_1$ which is not finitely generated. Since $K_1 = I(G) \cong N_1$, there exists a submodule K_2 of K_1 isomorphic to $I(G)$. Continuing this process in a similar way, we obtain a chain

$$I(G) \geq K_1 \geq K_2 \geq \dots$$

such that $K_i \cong I(G) \oplus N_1$ is not finitely generated for i odd, and $K_i \cong I(G)$ for i even. But $I(G)$ is finitely generated with semisimple dimension, which gives us a contradiction. Thus $I(G_1)$ is Noetherian. Now, by Proposition 3, $I(G_1)$ is also Artinian, and so $I(G_1)$ has finite length. This completes the proof. □

We know that every Artinian module has non-zero socle. In the following theorem we show that this is the case for every edge ideal with semisimple dimension.

Theorem 1. *Let $R = K[v_1, \dots, v_s]$ be the ring polynomial, $I(G)$ the edge ideal in R of a finite simple graph G, with no isolated vertices. Suppose that $I(G)$ is an R-module non-zero. If $I(G)$ has semisimple dimension the following assertions hold:*

(1) *If* $\text{s.s.}\dim(I(G)) = \gamma$*, then for any ordinal β with $0 \leq \beta \leq \gamma$, there exists a submodule L of $I(G)$ such that* $\text{s.s.}\dim(L) = \beta$*.*

(2) *$I(G)$ has non-zero socle.*

Proof. (1) We prove this part using transfinite induction on $\text{s.s.}\dim(I(G)) = \gamma$. The assertion is trivial for the case $\gamma = 1$. For $\gamma \geq 1$ let $0 \leq \beta < \gamma$. Then by Remark 2, there exists a submodule K of $I(G)$ which is not a direct summand of $I(G)$ and $\text{s.s.}\dim(K) \geq \beta$. As $\beta \leq \text{s.s.}\dim(K) < \gamma$, using induction hypothesis, there exists a submodule L of K such that $\text{s.s.}\dim(L) = \beta$.
(2) It is clear by part (1). □

3. Conflict of interest

The author of the article declares that there is no conflict of interest.

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