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RESEARCH ARTICLE

The inverse semigroup of all fence-preserving injections and its maximal subsemigroups

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Abstract. In this paper, we study the inverse semigroup IF_n of all partial injections α on an n-element set such that both α and α^{-1} are fence-preserving (preserve the zig-zag order). The main result of this paper is the characterization of the maximal subsemigroups of IF_n: There are five types of maximal subsemigroups, whenever n is odd; if n is even, then the maximal semigroups are of the form $IF_n \setminus \{\alpha\}$, where α belongs to the least generating set of IF_n. Moreover, we describe the i-conjugate elements in IF_n.

Introduction

Let S be a semigroup. A subsemigroup S' of S is called a maximal subsemigroup of S if $S' \neq S$ (i.e., S is a proper subsemigroup of S) and S' is not contained in any proper other subsemigroup of S. Maximal subsemigroups were characterized for a variety of semigroups. The paper [6] by N. Graham, R. Graham, and J. Rhodes is basically for the study of maximal subsemigroups of finite semigroups. A semigroup S is called inverse

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if for all $x \in S$, there is a unique $y \in S$ with $xyx = x$ and $yxy = y$. Such an element y is called the inverse of x and is denoted by x^{-1}. Let n be a positive integer and let $\bar{n} = \{1, \ldots, n\}$ be the set of the first n positive integers. For a set $A \subseteq \bar{n}$, a mapping $\alpha : A \rightarrow \bar{n}$ is called partial transformation on$\bar{n}$. The set of all partial transformations on $\bar{n}$ is denoted by PT_n. The set A is called domain of α (in symbols: $A = \mathrm{dom}(\alpha)$) and $\mathrm{im}(\alpha) = \{\mathrm{x}\alpha : \mathrm{x} \in \mathrm{dom}(\alpha)\}$ is the image (range) of α. The cardinality of $\mathrm{im}(\alpha)$ is called the rank of α (in symbols: $\mathrm{rank}(\alpha) = |\mathrm{im}(\alpha)|$). If $A = \bar{n}$, then α is called a full transformation. If α is injective, then α is called a partial injection. The set I_n of all partial injections on $\bar{n}$ forms a monoid with the identity mapping $id_{\bar{n}}$ on $\bar{n}$ as identity element, called the symmetric inverse semigroup on $\bar{n}$. For any $A \subseteq \bar{n}$, the transformation $id_{\bar{n}}$ restricted to A, i.e., the transformation α with $\mathrm{dom}(\alpha) = \mathrm{A}$ and $x\alpha = x$ for all $x \in \mathrm{dom}(\alpha)$ is called a partial identity.

Several subsemigroups of I_n have already been intensively studied. In particular, the maximal subsemigroups were determined. In [5], the authors characterized the maximal subsemigroups of the monoid IO_n consisting of all order-preserving (isotone) partial injections on $\bar{n}$. In [3], the authors characterized the maximal subsemigroups of the ideals of IO_n. The maximal subsemigroups of the ideals of the monoid IM_n consisting of all monotone, i.e., order-preserving or order-reversing (isotone or antitone) partial injections on $\bar{n}$ were determined in [3], as well. Recall that $\alpha \in I_n$ is called order-preserving (order-reversing) if $x < y \Rightarrow x\alpha < y\alpha$ ($x < y \Rightarrow x\alpha > z\alpha$) for all $x, y \in \mathrm{dom}(\alpha)$. D-B. Li, W-T. Zhang, and Y-F. Luo determined the maximal subsemigroups of the monoid of all orientation-preserving extensive partial injections on $\bar{n}$ [9]. The maximal subsemigroups of the ideals of the inverse semigroup of all order-, fence-, and parity-preserving partial injections on $\bar{n}$ were determined by J. Koppitz and A. Sareeto in [11].

Let $\prec$ be the binary relation on $\bar{n}$ defined by

$$i \prec i+1, \text{ whenever } i+1 \in \bar{n} \text{ and } i \prec i-1, \text{ whenever } i-1 \in \bar{n}$$

for all even integers $i \in \bar{n}$. This relation $\prec$ is a partial order on $\bar{n}$, which is called zig-zag order (or fence, regarded as poset $(\bar{n},\prec)$). Every element in $\bar{n}$ is either maximal or minimal with respect to $\prec$.

A partial transformation α is called fence-preserving if for all $x, y \in \mathrm{dom}(\alpha)$, the following implication holds:

$$x \prec y \Rightarrow x\alpha \prec y\alpha.$$

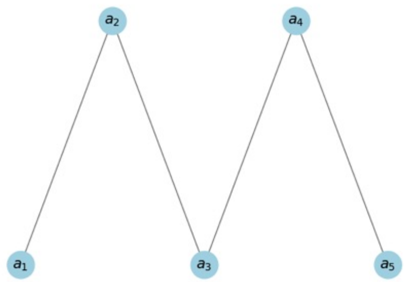

Figure 1: zig-zag order.

Then set of all fence-preserving full transformations on $\bar{n}$ forms a monoid and it was first studied by J. Currie and T. Visentin in 1991 [2] as well as by A. Rutkowski in 1992 [10]. Extending the results of [2], A. Rutkowski gives the number of fence-preserving full transformations on $\bar{n}$ for an even as well as for an odd positive integer n. For a long period, fence-preserving transformations were not considered. In 2015, a paper by K. Jendrdana and R. Srithus apears, which studies coregular semigroups of fence-preserving full transformations [7]. One year later, R. Tanyawong, R. Srithus, and R. Chinram determined the regular semigroups of fence-preserving full transformations [12]. In 2017, I. Dimitrova and J. Koppitz [4] studied the monoid IF_n of all $\alpha \in I_n$ such that both α and α^{-1} are fence-preserving. Note that IF_n is an inverse semigroup. In contrast, the monoid of all fence-preserving partial injections is not an inverse semigroup. For example, if $n = 4$, then the inverse element of the fence-preserving partial injection $\begin{pmatrix} 1 & 4 \\ 1 & 2 \end{pmatrix}$ is $\begin{pmatrix} 1 & 2 \\ 1 & 4 \end{pmatrix}$, where $\begin{pmatrix} 1 & 2 \\ 1 & 4 \end{pmatrix}$ does not be fence-preserving since $2 \prec 1$ but $4 \not\prec 1$. We can characterize the elements in IF_n as in the following proposition.

Proposition 1. *Let*

$$\alpha = \begin{pmatrix} d_1 & d_2 & \dots & d_p \\ m_1 & m_2 & \dots & m_p \end{pmatrix} \in I_n.$$

Then $\alpha \in IF_n$ if and only if for all $i, j \in \{1, 2, \dots, p\}$, the following both conditions hold:

(1) $|d_i - d_j| = 1$ if and only if $|m_i - m_j| = 1$;
(2) if $|d_i - d_j| = 1$, then d_i and m_i have the same parity.

Proof. Let $i, j \in \{1, 2, \ldots, p\}$.
(1) Suppose that $|d_i - d_j| = 1$. Then $|m_i - m_j| = 1$ since α is fence-preserving. Suppose that $|m_i - m_j| = 1$ and $|d_i - d_j| \neq 1$, which implies that α^{-1} is not fence-preserving. So, $\alpha^{-1} \notin IF_n$, a contradiction. Thus, $|d_i - d_j| = 1$.
(2) Assume that $|d_i - d_j| = 1$. Then $d_i \prec d_j$ or $d_j \prec d_i$. Since α is fence-preserving, we have $m_i \prec m_j$ and $m_j \prec m_i$, respectively. So, d_i and m_i have the same parity.

Conversely, suppose that both conditions (1) and (2) hold. Let $x, y \in \text{dom}(\alpha)$ with $x \prec y$, which means $|x - y| = 1$. Let $x = d_i$ and $y = d_j$ for some $i, j \in \{1, 2, \ldots, p\}$. By (1) and $|d_i - d_j| = 1$, we get $|m_i - m_j| = 1$, i.e., $x\alpha \prec y\alpha$ or $y\alpha \prec x\alpha$. From (2) and $|d_i - d_j| = 1$, we get that x and $x\alpha$ have the same parity. Hence, $x\alpha \prec y\alpha$. Let $x, y \in \text{dom}(\alpha^{-1}) = \text{im}(\alpha)$ with $x \prec y$, which means $|x - y| = 1$. Let $x = m_i$ and $y = m_j$ for some $i, j \in \{1, 2, \ldots, p\}$. Then $m_i = d_i\alpha$ and $m_j = d_j\alpha$. By (1) and $|m_i - m_j| = 1$, we obtain $|x\alpha^{-1} - y\alpha^{-1}| = |d_i - d_j| = 1$. Thus, $d_i \prec d_j$ or $d_j \prec d_i$. Since $|d_i - d_j| = 1$ and as (2), we get that $d_i = m_i\alpha$ and m_i have the same parity. Therefore, $m_i\alpha^{-1} \prec m_j\alpha^{-1}$. Hence, $x\alpha^{-1} \prec y\alpha^{-1}$. □

We call $Y \subseteq \bar{n}$ an insterval of $\bar{n}$ if Y is a consecutive set, i.e., $Y = \{i, i+1, \ldots, i+r\}$ for some $i \in \{1, 2, \ldots, n\}$ and $r \in \{0, 1, \ldots, n-i\}$, such that $i-1, i+r+1 \notin Y$. From Proposition 1, we obtain immediately a characterization of the restrictions of an $\alpha \in IF_n$ to the intervals in $\text{dom}(\alpha)$.

Remark 1. Let $\alpha \in IF_n$, let $k \in \{1, \ldots, n\}$ and let $p \in \{0, \ldots, n-k\}$ such that $A = \{k, k+1, \ldots, k+p\}$ be an interval in $\text{dom}(\alpha)$. Then

$$\alpha|_A = \begin{pmatrix} k & k+1 & \ldots & k+p \\ l & l+1 & \ldots & l+p \end{pmatrix} \text{ or } \alpha|_A = \begin{pmatrix} k & k+1 & \ldots & k+p \\ l & l-1 & \ldots & l-p \end{pmatrix}$$

for some $l \in \{1, \ldots, n\}$, where k and l have the same parity, whenever $p > 0$. With other words, the restriction of α to an interval in $\text{dom}(\alpha)$ is an order-preserving or order-reversing partial transformations on $\bar{n}$, with a consecutive image set. So, Remark 1 gives a vivid description of the elements in IF_n.

In [4], the authors show that IF_n is generated by the set $\{\alpha : \alpha \in IF_n, \text{rank}(\alpha) \geq \text{n}-2\}$. First, we consider an even positive integer n. Let

$$G_2 = \left\{ \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\}.$$

For $n \geq 4$, we define

$$\sigma_1 = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ n & - & 1 & \cdots & n-2 \end{pmatrix};$$
$$\sigma_2 = \sigma_1^{-1} = \begin{pmatrix} 1 & \cdots & n-2 & n-1 & n \\ 3 & \cdots & n & - & 1 \end{pmatrix};$$
$$\gamma_i = \begin{pmatrix} 1 & \cdots & i-1 & i & i+1 & \cdots & n \\ i-1 & \cdots & 1 & - & i+1 & \cdots & n \end{pmatrix} \text{ for } i \in \{4, 6, \ldots, n\};$$
$$\delta_i = \begin{pmatrix} 1 & \cdots & i-1 & i & i+1 & \cdots & n \\ 1 & \cdots & i-1 & - & n & \cdots & i+1 \end{pmatrix} \text{ for } i \in \{1, 3, \ldots, n-3\};$$

and

$$G_n = \{id_{\bar{n}}, \sigma_1, \sigma_2\} \cup \{\gamma_i : i \in \{4, 6, \ldots n\}\} \cup \{\delta_i : i \in \{1, 3, \ldots, n-3\}\}.$$

Theorem 1 ([4]). *If n is an even positive integer, then $G\nearrow_n$ is the least generating set of IF_n.*

Since G_n is the least generating set of IF_n, one can easy determine the maximal subsemigroups IF_n:

Proposition 2. *Let n be an even positive integer and let T be a subsemigroup of IF_n. Then T is a maximal subsemigroup of IF_n if and only if $T = IF_n \setminus \{\alpha\}$ for some $\alpha \in G_n$.*

For example, $T_1 = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}$,
$T_2 = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\}$, and $T_3 = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\}$
are the maximal subsemigroups of IF_2.

We consider now the case that n is odd. The authors of [8] give a generating set of IF_n of minimal size. We define particular elements in IF_n as follows:

$$\gamma_n = \begin{pmatrix} 1 & 2 & \ldots & n-1 & n \\ n & n-1 & \ldots & 2 & 1 \end{pmatrix};$$
$$\alpha_i = \begin{pmatrix} 1 & 2 & \ldots & i-1 & i & i+1 & i+2 & \ldots & n \\ 1 & 2 & \ldots & i-1 & - & n & n-1 & \ldots & i+1 \end{pmatrix} \text{ for } i \in \{2, 4, \ldots, n-1\};$$
$$\alpha_i = \begin{pmatrix} 1 & 2 & \ldots & i-1 & i & i+1 & i+2 & \ldots & n \\ 1 & 2 & \ldots & i-1 & - & i+1 & i+2 & \ldots & n \end{pmatrix} \text{ for } i \in \{1, 3, \ldots, n\};$$
$$\beta_2^{odd} = \begin{pmatrix} 1 & 2 & 3 & 4 & \ldots & n \\ 2 & - & - & 4 & \ldots & n \end{pmatrix};$$
$$\beta_2^{even} = (\beta_2^{odd})^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & \ldots & n \\ - & 1 & - & 4 & \ldots & n \end{pmatrix}.$$

In the case, $n \geq 5$ for $i \in \{4, 6, \ldots, n-1\}$, we define
$\beta_i^{odd} = \begin{pmatrix} 1 & 2 & 3 & \ldots & i & i+1 & i+2 & \ldots & n \\ i & - & 1 & \ldots & i-2 & - & i+2 & \ldots & n \end{pmatrix}$ and
$\beta_i^{even} = (\beta_i^{odd})^{-1} \begin{pmatrix} 1 & \ldots & i-2 & i-1 & i+2 & \ldots & n \\ 3 & \ldots & i & - & i+2 & \ldots & n \end{pmatrix}$.

In case $n \geq 7$, we define
$\alpha_{i,j} = \begin{pmatrix} 1 & 2 & \ldots & i-1 & i & i+1 & \ldots & j-1 & j & j+1 & \ldots & n \\ - & 2 & \ldots & i-1 & - & j-1 & \ldots & i+1 & - & j+1 & \ldots & n \end{pmatrix}$
for $i, j \in \{2, 3, \ldots, n-1\}$ with $4 \leq j - i < n - 1$ and $i < n - j + 1$;
$\alpha_{1,j} = \begin{pmatrix} 1 & 2 & \ldots & j-1 & j & j+1 & \ldots & n \\ - & j-1 & \ldots & 2 & - & j+1 & \ldots & n \end{pmatrix}$ for $j \in \{5, 7, \ldots, n-2\}$;
$\alpha_{i,n} = \begin{pmatrix} 1 & \ldots & j-1 & j & j+1 & \ldots & n-1 & n \\ 1 & \ldots & j-1 & - & n-1 & \ldots & j+1 & - \end{pmatrix}$ for $i \in \{3, 5, \ldots, n-4\}$
and
$\alpha_{1,n} = \begin{pmatrix} 1 & 2 & \ldots & n-1 & n \\ - & n-1 & \ldots & 2 & - \end{pmatrix}$.

Further, let $G_n = \{\gamma_n\} \cup \{\alpha_i \ : \ i \in \{1, 3, \ldots, \left(\frac{n+1}{2}\right)_o\}\}$
$\cup \{\alpha_i \ : \ i \in \{2, 4, \ldots, n-3\}\}$
$\cup \{\beta_i^{odd}, \beta_i^{even} \ : \ i \in \{2, 4, \ldots, \left(\frac{n+1}{2}\right)_e\}\}$
$\cup \{\alpha_{i,j} \ : \ i, j \in \{1, 3, \ldots, n\}, 4 \leq j - i < n - 1, i \leq n - j + 1\}$,
where $\left(\frac{n+1}{2}\right)_o$ $(\left(\frac{n+1}{2}\right)_e)$ denotes the odd (even) number in the set $\{\frac{n+1}{2}, \frac{n+1}{2} - 1\}$.

Theorem 2 ([8])**.** *G_n is a generating set of IF_n of minimal size for all odd integers $n \geq 5$ and $G_3 = \{\gamma_3, \alpha_1, \alpha_2, \beta_2^{odd}, \beta_2^{even}\}$ is a generating set of IF_3 of minimal size.*

In particular, we can observe that IF_n has no least generating set, whenever $n \geq 3$ is odd. For example, we can replace α_1 in G_n by $\gamma_n \alpha_1 \gamma_n \notin G_n$ and obtain a new generating set of IF_n of minimal size. This suggests that the question for the maximal subsemigroups of IF_n ($n \geq 3$ is odd) does not have a straightforward answer. The main target of this paper will be the characterization of the maximal subsemigroups of IF_n. Moreover, the paper will also fill two further gaps in the study of the inverse semigroup IF_n.

Back to any positive integer n. We will also consider the conjugacy in the inverse semigroup IF_n. In [1], J. Araújo et al. characterize the i-conjugate elements in the symmetric inverse semigroup I_n. We are going to find a charcterization of the i-conjugate elements in the subsemigroup IF_n of the symmetric inverse semigroup. This is the subject of the next section.

1. Conjugacy in IF_n

Two elements a and b of an inverse semigroup S with identity element are called i-conjugate, denoted $a \sim_i b$, if there exists $g \in S$ such that $g^{-1}ag = b$ and $gbg^{-1} = a$. By [1], $\alpha, \beta \in I_n$ are i-conjugate if and only if there is a full transformation $\sigma \in I_n$ (σ is a permutation) such that $\sigma^{-1}\alpha\sigma = \beta$. The aim of this section is to give a description of the conjugacy in the inverse semigroup IF_n, for all positive integers n. Let us fix a positive integer n for that section.

Lemma 1. *Let $\alpha, \beta \in IF_n$ with*

$$\alpha = \begin{pmatrix} a_1 & a_2 & \dots & a_p \\ a_{p+1} & a_{p+2} & \dots & a_{2p} \end{pmatrix} \text{ and } \beta = \begin{pmatrix} b_1 & b_2 & \dots & b_q \\ b_{q+1} & b_{q+2} & \dots & b_{2q} \end{pmatrix}.$$

If $\alpha \sim_i \beta$, then $p = q$.

Proof. Suppose that $\alpha \sim_i \beta$. Since $\alpha \sim_i \beta$, we have $\alpha = \gamma\beta\gamma^{-1}$ and $\beta = \gamma^{-1}\alpha\gamma$ for some $\gamma \in IF_n$. Let $|\mathrm{dom}(\alpha)| = p = |\mathrm{im}(\alpha)|$ and $|\mathrm{dom}(\beta)| = q = |\mathrm{im}(\beta)|$. Clearly, $\beta = \gamma^{-1}\alpha\gamma$ implies $q = |\mathrm{im}(\beta)| \leq |\mathrm{im}(\alpha)| = p$ and $\alpha = \gamma\beta\gamma^{-1}$ implies $q = |\mathrm{im}\ (\alpha)| \leq |\mathrm{im}\ (\beta)| = p$. Then $p = q$. □

For any positive integer m, let S_m be the set of all full transformations in I_m (i.e., all permutations on $\{1, 2, \dots, m\}$). Let $p \in \bar{n}, j \in \{1, 2, \dots, 2p\}$, and $\sigma \in S_{2p}$. Then we put

$$j^\sigma = \begin{cases} j\sigma & \text{if } j \in \{1, 2, \dots, p\}, \\ (1-p)\sigma + p & \text{if } j \in \{p+1, p+2, \dots, 2p\}. \end{cases}$$

Theorem 3. *Let $\alpha, \beta \in IF_n$ with*

$$\alpha = \begin{pmatrix} a_1 & a_2 & \dots & a_p \\ a_{p+1} & a_{p+2} & \dots & a_{2p} \end{pmatrix} \text{ and } \beta = \begin{pmatrix} b_1 & b_2 & \dots & b_q \\ b_{q+1} & b_{q+2} & \dots & b_{2q} \end{pmatrix}.$$

Then $\alpha \sim_i \beta$ if and only if $p = q$ and there is $\sigma \in S_{2p}$ with $\sigma|_{\{1,2,\dots,p\}} \in S_p$ such that

(a) $a_i = a_j$ *if and only if* $b_{i^\sigma} = b_{j\sigma}$ *for all* $1 \leq i < j \leq 2p$*;*
(b) either a_i *and* b_{i^σ} *have the same parity and*
$|a_i - a_j| = |b_{i^\sigma} - b_{j^\sigma}| = 1$ *or* $|a_i - a_j|, |b_{i^\sigma} - b_{j^\sigma}| > 1$
for all $1 \leq i < j \leq 2p$*.*

Proof. Suppose that $\alpha \sim_i \beta$. By Lemma 1, we obtain $p = q$. We show now (a) and (b). From $\alpha \sim_i \beta$, it follows $\alpha = \gamma\beta\gamma^{-1}$ for some $\gamma \in IF_n$. Define $\sigma : \{1, 2, \ldots, 2p\} \to \{1, 2, \ldots, 2p\}$ with $\sigma|_{\{1,2,\ldots,p\}} : \{1, 2, \ldots, p\} \to \{1, 2, \ldots, p\}$ by

$$a_j\gamma = \begin{cases} b_{j\sigma} & \text{for } j \in \{1, 2, \ldots, p\}, \\ b_{(j-p)\sigma+p} & \text{for } j \in \{p+1, p+2, \ldots, 2p\}. \end{cases}$$

It is easy to verify that σ is well-defined, $\sigma \in S_{2p}$, and $\sigma|_{\{1,2,\ldots,p\}} \in S_p$. Suppose that $1 \leq i < j \leq 2p$.
(a) By the definition of σ and since $\alpha = \gamma\beta\gamma^{-1}$, we get $a_k\gamma = b_{k^\sigma}$ and $a_k = b_{k^\sigma}\gamma^{-1}$ for $k \in \{i, j\}$. Suppose that $a_i = a_j$. Then $b_{i^\sigma} = a_i\gamma = a_j\gamma = b_{j^\sigma}$. Suppose that $b_{i^\sigma} = b_{j^\sigma}$. Then $a_i = b_{i^\sigma}\gamma^{-1} = b_{j^\sigma}\gamma^{-1} = a_j$.
(b) Assume that $|a_i - a_j| \leq 1$ or $|b_{i^\sigma} - b_{j^\sigma}| \leq 1$. We will prove that a_i and b_{i^σ} have the same parity. If $|a_i - a_j| \leq 1$, then $|a_i - a_j| = 0$ or $|a_i - a_j| = 1$. Since $i < j$, we get $a_i \neq a_j$ which means $|a_i - a_j| = 1$. Thus, $a_i \prec a_j$ or $a_j \prec a_i$. Without loss of generality, let $a_i \prec a_j$. By the fence-preserving property of γ, we have $a_i\gamma \prec a_j\gamma$. Since $a_i \prec a_j$ and $a_i\gamma \prec a_j\gamma$, we can conclude that a_i and $a_i\gamma = b_{i^\sigma}$ have the same parity. If $|b_{i^\sigma} - b_{j^\sigma}| \leq 1$, then we can similarly prove that a_i and b_{i^σ} have the same parity, substituting a_i with b_{i^σ} and a_j with b_{j^σ}. Next, we will prove that $|a_i - a_j| = |b_{i^\sigma} - b_{j^\sigma}| = 1$. If $|a_i - a_j| \leq 1$, then, as already shown, $|a_i - a_j| = 1$ and either $a_i \prec a_j$ or $a_j \prec a_i$. Without loss of generality, let $a_i \prec a_j$. Then $b_{i^\sigma} = a_i\gamma \prec a_j\gamma = b_{j^\sigma}$ by the fence-preserving property of γ. Therefore, $|b_{i^\sigma} - b_{j^\sigma}| = 1$. If $|b_{i^\sigma} - b_{j^\sigma}| \leq 1$, then we can similarly prove $|a_i - a_j| = |b_{i^\sigma} - b_{j^\sigma}| = 1$, substituting a_i with b_{i^σ} and a_j with b_{j^σ}.

Suppose that $p = q$ and there is $\sigma \in S_{2p}$ with $\sigma|_{\{1,2,\ldots,p\}} \in S_p$ such that (a) and (b) hold. We define a partial transformation γ on $\bar{n}$ by $a_i\gamma = b_{i^\sigma}$ for all $i \in \{1, 2, \ldots, 2p\}$. The mapping γ is well defined and injective by (a). Let $i \in \{1, 2, \ldots, p\}$. From $a_i\gamma = b_{i^\sigma}$, we get $a_i = b_{i^\sigma}\gamma^{-1}$. Then we consider

$$b_{i^\sigma}(\gamma^{-1}\alpha\gamma) = (b_{i^\sigma}\gamma^{-1})\alpha\gamma = a_i\alpha\gamma = (a_i\alpha)\gamma = a_{i+p}\gamma = b_{(i+p)^\sigma} = b_{i\sigma+p}$$
and $b_{i^\sigma}\beta = b_{i\sigma}\beta = b_{i\sigma+p}$.

We obtain that $b_{i^\sigma}(\gamma^{-1}\alpha\gamma) = b_{i^\sigma}\beta$. Next, we consider

$a_i(\gamma\beta\gamma^{-1}) = (a_i\gamma)\beta\gamma^{-1} = b_{i^\sigma}\beta\gamma^{-1} = (b_{i^\sigma}\beta)\gamma^{-1} = b_{i\sigma+p}\gamma^{-1} = b_{(i+p)^\sigma}\gamma^{-1}$
$= a_{i+p}$ and $a_i\alpha = a_{i+p}$.

Thus, we obtain that $a_i(\gamma\beta\gamma^{-1}) = a_i\alpha$. We have now shown that $\beta = \gamma^{-1}\alpha\gamma$ and $\alpha = \gamma\beta\gamma^{-1}$. It remains to prove that $\gamma \in IF_n$. Let $i < j \in \{1, 2, \ldots, p\}$.
(1) Suppose that $|a_i - a_j| = 1$. From (b) and $|a_i - a_j| = 1$, we have $|a_i\gamma - a_j\gamma| = |b_{i^\sigma} - b_{j^\sigma}| = |a_i - a_j| = 1$. Let $|a_i\gamma - a_j\gamma| = 1$. Then $|b_{i^\sigma} - b_{j^\sigma}| = 1$. From (b) and $|b_{i^\sigma} - b_{j^\sigma}| = 1$, we have $|a_i - a_j| = |b_{i^\sigma} - b_{j^\sigma}| = 1$.
(2) Suppose $|a_i - a_j| = 1$. By (b) and $|a_i - a_j| = 1$, we get that a_i and $a_i\gamma = b_{i^\sigma}$ have the same parity.
By Proposition 1, we obtain that $\gamma \in IF_n$. □

2. Maximal Subsemigroup of IF_n

In this section, let n be an odd integer greater than or equal to 3. Note that, the semigroups consisting of the identity mapping on $\{1\}$ and the empty transformation, respectively, are both maximal subsemigroups of IF_1. We will characterize the maximal subsemigroups of IF_n. First, we observe the following general fact:

Lemma 2. *Let T be a maximal subsemigroup of any subsemigroup S of PT_n. Then there are $r \in \{0, 1, \ldots, n\}$ and $J \subseteq \{\alpha \in S \ : \ rank(\alpha) = r\}$ such that $T = S \setminus J$.*

Proof. Let $J = S \setminus T$. Suppose there exist $\alpha, \beta \in J$ such that rank$(\alpha) <$ rank(β). Since $\alpha \notin T$ and T is a maximal subsemigroup of S, we have $\langle T, \alpha\rangle = S$. Therefore, $\beta = \gamma_1 \ldots \gamma_k$, for some $\gamma_1, \ldots, \gamma_k \in T \cup \{\alpha\}$. As $\beta \notin T$, $\gamma_i = \alpha$, for some $i = 1, \ldots, k$. Hence, rank$(\beta) =$ rank$(\gamma_1 \ldots \gamma_\mathrm{k}) \leq$ rank$(\alpha) <$ rank(β), which is a contradiction. Thus rank$(\alpha) =$ rank(β) for all $\alpha, \beta \in J$. □

Let us now define the following sets:

$$A_i = \{\alpha_i, \gamma_n\alpha_i, \alpha_i\gamma_n, \gamma_n\alpha_i\gamma_n\} \text{ for } i \in \{1, 3, \ldots, \left(\frac{n+1}{2}\right)_o\};$$
$$B_i = \{\beta_i^{odd}, \gamma_n\beta_i^{odd}, \beta_i^{odd}\gamma_n, \gamma_n\beta_i^{odd}\gamma_n, \alpha_2\beta_i^{odd}, \gamma_n\alpha_2\beta_i^{odd}, \alpha_2\beta_i^{odd}\gamma_n,$$
$$\gamma_n\alpha_2\beta_i^{odd}\gamma_n\} \text{ for } i \in \{2, 4, \ldots, \left(\frac{n+1}{2}\right)_e\};$$
$$C_i = \{\beta_i^{even}, \gamma_n\beta_i^{even}, \beta_i^{even}\gamma_n, \gamma_n\beta_i^{even}\gamma_n, \beta_i^{even}\alpha_2, \gamma_n\beta_i^{even}\alpha_2,$$

$$\beta_i^{even}\alpha_2\gamma_n, \gamma_n\beta_i^{even}\alpha_2\gamma_n\} \text{ for } i \in \{2,4,\ldots,\left(\frac{n+1}{2}\right)_e\};$$
$$D_i = \{\alpha_i, \gamma_n\alpha_i, \alpha_i\gamma_n, \gamma_n\alpha_i\gamma_n, \alpha_{n-i+1}, \gamma_n\alpha_{n-i+1}, \alpha_{n-i+1}\gamma_n, \gamma_n\alpha_{n-i+1}\gamma_n\} \text{ for } i \in \{2,4,\ldots,\left(\frac{n+1}{2}\right)_e\};$$
$$E_{i,j} = \{\alpha_{i,j}, \gamma_n\alpha_{i,j}, \alpha_{i,j}\gamma_n, \gamma_n\alpha_{i,j}\gamma_n\} \text{ for } i,j \in \{1,3,\ldots,n\} \text{ with } 4 \le j-i < n-1 \text{ and } i \le n-j+1\}, \text{ whenever } n \ge 7.$$

Note that for $n = 3$, we have $\beta_2\gamma_n = \beta_2, \alpha_2\beta_2 = \beta_2\alpha_2 = \beta_2$, and $\alpha_2 = \gamma_n\alpha_2\gamma_n$. Hence,

$$B_2 = \{\beta_2^{odd}, \gamma_3\beta_2^{odd}\},$$
$$C_2 = \{\beta_2^{even}, \gamma_3\beta_2^{even}\}, \text{ and}$$
$$D_2 = \{\alpha_2, \gamma_3\alpha_2\}.$$

Now, we will consider five types of subsemigroups of IF_n, for which, we will verify that these are the maximal ones.

Lemma 3. *$IF_n \setminus A_i$ is a maximal subsemigroup of IF_n for all $i \in \{1,3,\ldots,\left(\frac{n+1}{2}\right)_o\}$.*

Proof. Let $i \in \{1,3,\ldots,\left(\frac{n+1}{2}\right)_o\}$. First, we show that $IF_n \setminus A_i$ is a subsemigroup of IF_n. Assume that there are $\alpha, \beta \in IF_n \setminus A_i$ such that

$$\alpha\beta = \alpha_i.$$

Without loss of generality, we can assume that $\beta \neq id_{\bar{n}}$. So, $\mathrm{im}(\alpha_i) \subseteq \mathrm{im}(\beta)$ and by Remark 1, it is easy to verify that there are three possible cases for β:

(1) $\beta = \alpha_i\gamma_n \in A_i$,
(2) $\beta = \alpha_i \in A_i$, and
(3) $\beta = \gamma_n$.

In case (3), we get $\alpha = \alpha_i\gamma_n \in A_i$. These observations show that $\beta \in A_i$ or $\alpha \in A_i$, a contradiction. Then $\alpha_i \notin \langle IF_n \setminus A_i\rangle$.

It follows that $\alpha_i, \gamma_n\alpha_i, \alpha_i\gamma_n, \gamma_n\alpha_i\gamma_n \notin \langle IF_n \setminus A_i\rangle$ since $\gamma_n \notin A_i$ and $\gamma_n^2 = id_{\bar{n}}$. Therefore, $IF_n \setminus A_i$ is a subsemigroup of IF_n. It remains to show that $IF_n \setminus A_i$ is a maximal subsemigroup of IF_n. For this, let $x \in A_i$. There are $r, s \in \{1,2\}$ such that $x = \gamma_n^r\alpha_i\gamma_n^s$. Then

$$A_i = \{\gamma_n^{r+1}\alpha\gamma_n^s,\ \gamma_n^r\alpha_i\gamma_n^{s+1},\ \gamma_n^{r+1}\alpha\gamma_n^{s+1},\ \gamma_n^r\alpha_i\gamma_n^s\} \subseteq \langle IF_n \setminus A_i, x, \gamma_n\rangle.$$

Since $\gamma_n \notin A_i$, this shows that $\langle IF_n \setminus A_i, x\rangle = IF_n$, i.e., $IF_n \setminus A_i$ is a maximal subsemigroup of IF_n. □

Lemma 4. *Both $IF_n \setminus B_i$ and $IF_n \setminus C_i$ are maximal subsemigroups of IF_n for all $i \in \{2, 4, \dots, \left(\frac{n+1}{2}\right)_e\}$.*

Proof. Let $i \in \{2, 4, \dots, \left(\frac{n+1}{2}\right)_e\}$. We prove that $IF_n \setminus B_i$ is a maximal subsemigroup of IF_n. Note that $\beta_i^{even} = (\beta_i^{odd})^{-1}$. So, we can prove that $IF_n \setminus C_i$ is a maximal subsemigroup of IF_n using the same method. First, we show that $IF_n \setminus B_i$ is a subsemigroup of IF_n.

Suppose that $i = 2$. Assume that there are $\alpha, \beta \in IF_n \setminus B_2$ such that

$$\alpha\beta = \beta_2^{odd}.$$

Thus, $\mathrm{im}(\beta_2^{\mathrm{odd}}) \subseteq \mathrm{im}(\beta)$ and by Remark 1, it is easy to verify that there are six possible cases for β:

(1) $\beta = \beta_2^{odd} \in B_2$;

(2) $\beta = \begin{pmatrix} n-1 & n & 1 & 2 & \dots & n-2 \\ - & 2 & - & 4 & \dots & n \end{pmatrix} = \gamma_n \alpha_2 \beta_2^{odd} \in B_2$;

(3) $\beta = \begin{pmatrix} n-1 & n & n-2 & n-3 & \dots & 1 \\ - & 2 & - & 4 & \dots & n \end{pmatrix} = \gamma_n \beta_2^{odd} \in B_2$;

(4) $\beta = \begin{pmatrix} n & 1 & 2 & n-1 & \dots & 3 \\ - & 2 & - & 4 & \dots & n \end{pmatrix} = \alpha_2 \beta_2^{odd} \in B_2$;

(5) β is a partial identity with $\mathrm{im}(\beta_2^{\mathrm{odd}}) \subseteq \mathrm{im}(\beta)$, and

(6) $\gamma_n\beta$ is a partial identity with $\mathrm{im}(\beta_2^{\mathrm{odd}}) \subseteq \mathrm{im}(\beta)$.

In cases (5) and (6), we can calculate that $\alpha = \beta_2^{odd} \in B_2$ and $\alpha = \beta_2^{odd}\gamma_n \in B_2$, respectively. These observations show that $\alpha \in B_2$ or $\beta \in B_2$, a contradiction. So, $\beta_2^{odd} \notin \langle IF_n \setminus B_2\rangle$. This implies $\beta_2^{odd}, \gamma_n\beta_2^{odd}$, $\beta_2^{odd}\gamma_n, \gamma_n\beta_2^{odd}\gamma_n, \alpha_2\beta_2^{odd}, \gamma_n\alpha_2\beta_2^{odd}, \alpha_2\beta_2^{odd}\gamma_n, \gamma_n\alpha_2\beta_2^{odd}\gamma_n \notin \langle IF_n \setminus B_2\rangle$ since $\gamma_n, \alpha_2 \notin B_2, \alpha_2^2\beta_2^{odd} = \beta_2^{odd}$ and $\gamma_n^2 = id_{\bar{n}}$. Hence, $IF_n \setminus B_2$ is a subsemigroup of IF_n.

Suppose $i \in \{4, 6, \dots, \left(\frac{n+1}{2}\right)_e\}$. We show that $IF_n \setminus B_i$ is a subsemigroup of IF_n. Suppose that there are $\alpha, \beta \in IF_n \setminus B_i$ with

$$\alpha\beta = \beta_i^{odd}.$$

We get $B_i \cap \langle IF_n \setminus B_i \rangle = \emptyset$ by the same arguments as in the case $i = 2$.

It remains to show that $IF_n \setminus B_i$ is a maximal subsemigroup of IF_n. For this, let $x \in B_i$. Then there are $r, s \in \{1, 2\}$ such that $x = \gamma_n^r \alpha_2 \beta_i^{odd} \gamma_n^s$ or $x = \gamma_n^r \beta_i^{odd} \alpha_2 \gamma_n^s$. Suppose $x = \gamma_n^r \alpha_2 \beta_i^{odd} \gamma_n^s$. Because of $\alpha_2 \gamma_n^r \gamma_n^r \alpha_2 \beta_i^{odd} = \alpha_2 \alpha_2 \beta_i^{odd} = \beta_i^{odd}$, we can conclude that $B_i = \{\beta_i^{odd}, \gamma_n \beta_i^{odd}, \beta_i^{odd} \gamma_n, \gamma_n \beta_i^{odd} \gamma_n, \alpha_2 \beta_i^{odd}, \gamma_n \alpha_2 \beta_i^{odd}, \alpha_2 \beta_i^{odd} \gamma_n, \gamma_n \alpha_2 \beta_i^{odd} \gamma_n\} \subseteq \langle IF_n \setminus B_i, x, \gamma_n \rangle$. Since $\gamma_n \notin B_i$, this shows that $\langle IF_n \setminus B_i, x \rangle = IF_n$. Suppose that $x = \gamma_n^r \beta_i^{odd} \gamma_n^s$. We have $\gamma_n \gamma_n \beta_i^{odd} = \beta_i^{odd}$. Hence, we get that $B_i \subseteq \langle IF_n \setminus B_i, \gamma_n, \alpha_2, x \rangle (i = 2$ and $B_2 \subseteq \langle IF_n \setminus B_2, \gamma_n \rangle$, whenever $n = 3)$. Since $\gamma_n, \alpha_2 \notin B_i$, this shows that $\langle IF_n \setminus B_i, x \rangle = IF_n$. Altogether, we can conclude that $IF_n \setminus B_i$ is a maximal subsemigroup of IF_n. $\square$

Lemma 5. *$IF_n \setminus D_i$ is a maximal subsemigroup of IF_n for all $i \in \{2, 4, \ldots, \left(\frac{n+1}{2}\right)_e\}$.*

Proof. Let $i \in \{2, 4, \ldots, \left(\frac{n+1}{2}\right)_e\}$. First, we show that $IF_n \setminus D_i$ is a subsemigroup of IF_n. Suppose that there are $\alpha, \beta \in IF_n \setminus D_i$ with

$$\alpha\beta = \alpha_{n-i+1}.$$

Without loss of generality, we can assume that $\beta \neq id_{\bar{n}}$. Since $\mathrm{im}(\alpha_{\mathrm{n-i+1}}) \subseteq im(\beta)$, by Remark 1, it is easy to verify that there are nine possible cases for β:

(1) $\beta = \alpha_{n-i+1} \in D_i$;

(2) $\beta = \begin{pmatrix} n-i & n-i-1 & \ldots & 1 & n-i+1 & n-i+2 & \ldots & n \\ 1 & 2 & \ldots & n-i & - & n-i+2 & \ldots & n \end{pmatrix}$
$= \gamma_n \alpha_i \gamma_n \in D_i$;

(3) $\beta = \begin{pmatrix} n & n-1 & \ldots & i+1 & i & 1 & 2 & \ldots & i-1 \\ 1 & 2 & \ldots & n-i & - & n-i+2 & n-i+3 & \ldots & n \end{pmatrix}$
$= \gamma_n \alpha_{n-i+1} \in D_i$;

(4) $\beta = \begin{pmatrix} i+1 & i+2 & \ldots & n & i & i-1 & i-2 & \ldots & 1 \\ 1 & 2 & \ldots & n-i & - & n-i+2 & n-i+3 & \ldots & n \end{pmatrix}$
$= \alpha_i \gamma_n \in D_i$;

(5) $\beta = \begin{pmatrix} 1 & 2 & \ldots & n-i & n-i+1 & n-i+2 & n-i+3 & \ldots & n \\ 1 & 2 & \ldots & n-i & - & n-i+2 & n-i+3 & \ldots & n \end{pmatrix}$;

(6) $\beta = \begin{pmatrix} n-i & \ldots & 1 & n-i+1 & n & \ldots & n-i+2 \\ 1 & \ldots & n-i & - & n-i+2 & \ldots & n \end{pmatrix}$;

(7) $\beta = \begin{pmatrix} i+1 & \ldots & n & i & 1 & 2 & \ldots & i-1 \\ 1 & \ldots & n-i & - & n-i+2 & n-i+3 & \ldots & n \end{pmatrix}$;

(8) $\beta = \begin{pmatrix} n & n-1 & \ldots & i+1 & i & i-1 & i-2 & \ldots & 1 \\ 1 & 2 & \ldots & n-i & - & n-i+2 & n-i+3 & \ldots & n \end{pmatrix}$, and

(9) $\beta = \gamma_n$.

In the cases (5), (6), (7), (8), and (9), we can easily calculate that $\alpha = \alpha_{n-i+1} \in D_i$, $\alpha = \gamma_n\alpha_i\gamma_n \in D_i$, $\alpha = \gamma_n\alpha_i \in D_i$, $\alpha = \alpha_{n-i+1}\gamma_n \in D_i$, and $\alpha = \alpha_{n-i+1}\gamma_n \in D_i$, respectively. These observations show that $\alpha \in D_i$ or $\beta \in D_i$, a contradiction. So, $\alpha_{n-i+1} \notin \langle IF_n \setminus D_i \rangle$. Similarly, we can verify that $\alpha_i \notin \langle IF_n \setminus D_i \rangle$. Finally, $\gamma_n^2 = id_{\bar{n}}$ and $\alpha_i, \alpha_{n-i+1} \notin \langle IF_n \setminus D_i \rangle$ imply that $\alpha_i, \gamma_n\alpha_i, \alpha_i\gamma_n, \gamma_n\alpha_i\gamma_n, \alpha_{n-i+1}, \gamma_n\alpha_{n-i+1}, \alpha_{n-i+1}\gamma_n$, $\gamma_n\alpha_{n-i+1}\gamma_n \notin \langle IF_n \setminus D_i \rangle$. Hence, $IF_n \setminus D_i$ is a subsemigroup of IF_n.

It remains to show that $IF_n \setminus D_i$ is a maximal subsemigroup of IF_n. For this, let $x \in D_i$. Then there are $r, s \in \{1, 2\}$ such that $x = \gamma_n^r\alpha_i\gamma_n^s$ or $x = \gamma_n^r\alpha_{n-i+1}\gamma_n^s$. If $n = 3$, then $i = 2$ and $\alpha_i = \alpha_{n-i+1} = \alpha_2$. Since $\alpha_2\gamma_3 = \gamma_3\alpha_2$ and $\gamma_3 \notin D_2$, we obtain that $x = \alpha_2$ or $x = \gamma_3\alpha_2$. Since $\gamma_3 \notin D_2$, we can conclude that $\langle IF_3 \setminus D_2, x\rangle = IF_3$. So, we have to consider the case $n \geq 5$. Suppose $x = \gamma_n^r\alpha_i\gamma_n^s$. Because of $(\gamma_n\alpha_i\gamma_n)^2\alpha_{n-i+1} = \alpha_{n-i+1}$ and $\gamma_n^2 = id_{\bar{n}}$, we can conclude

$$\begin{aligned} D_i &= \{\alpha_i, \gamma_n\alpha_i, \alpha_i\gamma_n, \gamma_n\alpha_i\gamma_n, \alpha_{n-i+1}, \gamma_n\alpha_{n-i+1}, \alpha_{n-i+1}\gamma_n, \gamma_n\alpha_{n-i+1}\gamma_n\} \\ &\subseteq \langle IF_n \setminus D_i, x, \gamma_n, (\gamma_n\alpha_i\gamma_n)\alpha_{n-i+1}\rangle. \end{aligned}$$

It is easy to verify that $\gamma_n\alpha_i\gamma_n\alpha_{n-i+1} \notin D_i$. Since $\gamma_n \notin D_i$, this shows that $\langle IF_n \setminus D_i, x\rangle = IF_n$. Suppose that $x = \gamma_n^r\alpha_{n-i+1}\gamma_n^s$. Dually, we get $D_i \subseteq \langle IF_n \setminus D_i, x, \gamma_n, \gamma_n\alpha_{n-i+1}\gamma_n\rangle$ and $\langle IF_n \setminus D_i, x\rangle = IF_n$. Altogether, we can conclude that $IF_n \setminus D_i$ is a maximal subsemigroup of IF_n. □

Lemma 6. *$IF_n \setminus E_{i,j}$ is a maximal subsemigroup of IF_n for all $i, j \in \{1, 3, \ldots, n\}$ with $4 \leq j - i < n - 1$ and $i \leq n - j + 1$, whenever $n \geq 7$.*

Proof. Let $i, j \in \{1, 3, \ldots, n\}$ with $4 \leq j - i < n - 1$ and $i \leq n - j + 1$. First, we show that $IF_n \setminus E_{i,j}$ is a subsemigroup of IF_n. Note that $i = 1$ and $j = n$ is not possible.

Suppose that $i = 1$. Then $j \in \{5, 7, \ldots, n-2\}$. Suppose that there are $\alpha, \beta \in IF_n \setminus E_{1,j}$ with

$$\alpha\beta = \alpha_{1,j}.$$

Since $\text{im}(\alpha_{1,j}) \subseteq \text{im}(\beta)$, by Remark 1, it is easy to verify that there are four possible cases for β:

(1) $\beta = \alpha_{1,j} \in E_{1,j}$;

(2) $\beta = \begin{pmatrix} n & n-j+2 & \dots & n-1 & n-j+1 & n-j & \dots & 1 \\ - & 2 & \dots & j-1 & - & j+1 & \dots & n \end{pmatrix}$
$= \gamma_n \alpha_{1,j} \in E_{1,j}$;

(3) β is partial identity with $\text{im}(\alpha_{1,j}) \subseteq \text{im}(\beta)$, and

(4) $\gamma_n \beta$ is partial indentity with $\text{im}(\alpha_{1,j}) \subseteq \text{im}(\beta)$.

It is easy to see that the cases (3) and (4) imply $\alpha = \alpha_{1,j} \in E_{1,j}$ and $\alpha = \alpha_{1,j}\gamma_n \in E_{1,j}$, respectively. These observations show that $\alpha \in E_{1,j}$ or $\beta \in E_{1,j}$, which contradicts with $\alpha, \beta \in IF_n \setminus E_{1,j}$. So, $\alpha_{1,j} \notin \langle IF_n \setminus E_{1,j} \rangle$ which implies that $\alpha_{1,j}, \gamma_n\alpha_{1,j}, \alpha_{1,j}\gamma_n, \gamma_n\alpha_{1,j}\gamma_n \notin \langle IF_n \setminus E_{1,j} \rangle$. Therefore, $IF_n \setminus E_{1,j}$ is a subsemigroup of IF_n.

If $j = n$, then $i \in \{3, 5, \dots, n-1\}$ and we can show by the same arguments that $IF_n \setminus E_{i,n}$ is a subsemigroup of IF_n.

We have still to consider the case that $i \neq 1$ and $j \neq 1$. Assume that there are $\alpha, \beta \in IF_n \setminus E_{i,j}$ with

$$\alpha\beta = \alpha_{i,j}.$$

Since $\text{im}(\alpha_{i,j}) \subseteq \text{im}(\beta)$, by Remark 1, it is easy to verify that there are four possible cases for β:

(1) $\beta = \alpha_{i,j} \in E_{i,j}$;

(2) $\beta = \begin{pmatrix} n & \dots & n-i+2 & i_n & n-j+2 & \dots & n-i & j_n & n-j & \dots & 1 \\ 1 & \dots & i-1 & - & i+1 & \dots & j-1 & - & j+1 & \dots & n \end{pmatrix}$
$= \gamma_n \alpha_{i,j} \in E_{i,j}$ with $k_n = n-k+1$ for $k \in \{i,j\}$;

(3) β is a partial identity with $\text{im}(\alpha_{i,j}) \subseteq \text{im}(\beta)$, and

(4) $\gamma_n\beta$ is a partial identity with $\text{im}(\alpha_{i,j}) \subseteq \text{im}(\beta)$.

Considering the cases (3) and (4), it is easy to see that $\alpha = \alpha_{i,j} \in E_{i,j}$ and $\alpha = \alpha_{i,j}\gamma_n \in E_{i,j}$, respectively. These observations show that $\alpha \in E_{i,j}$ or $\beta \in E_{i,j}$, a contradiction. So, $\alpha_{i,j} \notin \langle IF_n \setminus E_{i,j} \rangle$ which implies that $\alpha_{i,j}, \gamma_n\alpha_{i,j}, \alpha_{i,j}\gamma_n, \gamma_n\alpha_{i,j}\gamma_n \notin \langle IF_n \setminus E_{i,j} \rangle$. Hence, $IF_n \setminus E_{i,j}$ is a subsemigroup of IF_n.

It remains to show that $IF_n \setminus E_{i,j}$ is a maximal subsemigroup of IF_n. For this, let $x \in E_{i,j}$. Then there are $r, s \in \{1, 2\}$ such that $x = \gamma_n^r \alpha_{i,j} \gamma_n^s$ and $E_{i,j} = \{\alpha_{i,j}, \gamma_n\alpha_{i,j}, \alpha_{i,j}\gamma_n, \gamma_n\alpha_{i,j}\gamma_n\} \subseteq \langle IF_n \setminus E_{i,j}, x, \gamma_n \rangle$. Since $\gamma_n \notin$

$E_{i,j}$, this show that $\langle IF_n \setminus E_{i,j}, x\rangle = IF_n$, i.e., $IF_n \setminus E_{i,j}$ is maximal subsemigroup of IF_n. □

Since γ_n and $id_{\bar{n}} = \gamma_n^2$ are only permutations in IF_n, we can conclude:

Lemma 7. *$IF_n \setminus \{\gamma_n\}$ is a maximal subsemigroup of IF_n.*

It remains to show that the maximal subsemigroup of IF_n, presented in the previous lemmas, are exactly the maximal ones.

Theorem 4. *Let T be a maximal subsemigroup of IF_n. Then T has one of the following forms:*

(1) $T = IF_n \setminus \{\gamma_n\}$;

(2) $T = IF_n \setminus A_i$ *for some* $i \in \{1, 3, \ldots, \left(\frac{n+1}{2}\right)_o\}$;

(3) $T = IF_n \setminus B_i$ *or* $T = IF_n \setminus C_i$ *for some* $i \in \{2, 4, \ldots, \left(\frac{n+1}{2}\right)_e\}$;

(4) $T = IF_n \setminus D_i$ *for some* $i \in \{2, 4, \ldots, \left(\frac{n+1}{2}\right)_e\}$;

(5) $T = IF_n \setminus E_{i,j}$ *for some* $i, j \in \{1, 3, \ldots, n-1\}$ *with* $4 \le j - i < n - 1$ *and* $i \le n - j + 1$.

Proof. By Lemmas 3-7, all the given sets are maximal subsemigroups of IF_n.

Conversely, let T be a maximal subsemigroup of IF_n. Then by Lemma 2, there is a set $J \subseteq \{\alpha \in IF_n : rank(\alpha) = r\}$ for some $r \in \{0, 1, \ldots, n\}$ such that $T = IF_n \setminus J$. First, we observe that $J \cap G_n \neq \emptyset$, since G_n is a generating set of IF_n by Theorem 2. This implies that $r \in \{n, n-1, n-2\}$.

Suppose that $r = n$. Then $\gamma_n \in J$ since $IF_n \setminus J$ is a (maximal) subsemigroup of IF_n by Lemma 7, we conclude that $J = \{\gamma_n\}$, i.e., $T = IF_n \setminus \{\gamma_n\}$.

Suppose that $r = n - 1$. Then $J \cap G_n \neq \emptyset$ implies $\alpha_i \in J_n$ for some $i \in \{1, 3, \ldots, \left(\frac{n+1}{2}\right)_o\}$ or $i \in \{2, 4, \ldots, n-3\}$. If $i \in \{1, 3, \ldots, \left(\frac{n+1}{2}\right)_o\}$, then we have $\gamma_n a_i, a_i\gamma_n, \gamma_n a_i \gamma_n \in J$, i.e., $A_i \subseteq J$. Since $IF_n \setminus A_i$ is a maximal subsemigroup of IF_n (by Lemma 3) and $A_i \subseteq J$, we conclude that $J = A_i$, and hence, $T = IF_n \setminus A_i$. If $i \in \{2, 4, \ldots, n-3\}$, then $\gamma_n\alpha_i, \alpha_i\gamma_n, \gamma_n\alpha_i\gamma_n \in J$. Let $b_1, b_2, \ldots, b_p \in IF_n \setminus \{id_{\bar{n}}, \alpha_i\gamma_n\alpha_i, \alpha_i\gamma_n, \gamma_n\alpha_i\gamma_n\}$ such that $\alpha_i = b_1 b_2 \ldots b_p$. Then $\mathrm{dom}(b_i) = \bar{n} \setminus \{i\}$ or $\mathrm{dom}(\gamma_n b_i) = \bar{n} \setminus \{i\}$ and $\mathrm{im}(b_i) = \bar{n} \setminus \{i\}$ or $\mathrm{im}(b_i\gamma_n) = \bar{n} \setminus \{i\}$ for all $i \in \{1, 2, \ldots, p\}$.

Moreover, there is $k \in \{1, 2, \ldots, p\}$ such that b_k is not a partial identity. Hence, $b_k \in \{\alpha_{n-i+1}, \alpha_{n-i+1}\gamma_n, \gamma_n\alpha_{n-i+1}, \gamma_n\alpha_{n-i+1}\gamma_n\}$. This implies $\{\alpha_{n-i+1}, \alpha_{n-i+1}\gamma_n, \gamma_n\alpha_{n-i+1}, \gamma_n\alpha_{n-i+1}\gamma_n\} \cap J \neq \emptyset$, and thus, $\{\alpha_{n-i+1}, \alpha_{n-i+1}\gamma_n, \gamma_n\alpha_{n-i+1}, \gamma_n\alpha_{n-i+1}\gamma_n\} \subseteq J$ since $\gamma_n \notin J$. Let

$$j = \begin{cases} i, & \text{if } i \leq \left(\dfrac{n+1}{2}\right)_e; \\ n-i, & \text{if } i > \left(\dfrac{n+1}{2}\right)_e. \end{cases}$$

Note that if $i > \left(\dfrac{n+1}{2}\right)_e$, then we can consider two cases:

If $\left(\dfrac{n+1}{2}\right)_e = \dfrac{n+1}{2}$, then we get $n-i < n-\left(\dfrac{n+1}{2}\right)_e = n-\dfrac{n+1}{2} = \dfrac{n+1}{2}-1 < \left(\dfrac{n+1}{2}\right)_e$.

If $\left(\dfrac{n+1}{2}\right)_e = \dfrac{n+1}{2}-1$, then we get $n-i \leq n-(\left(\dfrac{n+1}{2}\right)_e)-1 = n-\dfrac{n+1}{2}+1-1 = \dfrac{n+1}{2}-1 = \left(\dfrac{n+1}{2}\right)_e$.

This implies that $\alpha_{n-i+1} \in D_j = \{\alpha_j, \gamma_n\alpha_j, \alpha_j\gamma_n, \gamma_n\alpha_j\gamma_n, \alpha_{n-j+1}, \gamma_n\alpha_{n-j+1}, \alpha_{n-j+1}\gamma_n, \gamma_n\alpha_{n-j+1}\gamma_n\}$ with $j \in \{2, 4, \ldots, \left(\dfrac{n+1}{2}\right)_e\}$. Hence, $D_j \subseteq J$. Since $IF_n \setminus D_j$ is a maximal subsemigroup of IF_n (by Lemma 5), we can conclude that $J = D_j$, i.e., $T = IF_n \setminus D_j$.

Suppose that $r = n-2$. Then there is $x \in G_n \cap J$. Let $i, j \in \{1, 3, \ldots, n\}$ with $4 \leq j-i < n-1$ and $i \leq n-j+1$ such that $x = \alpha_{i,j}$. Then $\gamma_n\alpha_{i,j}, \alpha_{i,j}\gamma_n, \gamma_n\alpha_{i,j}\gamma_n \in J$. Since $IF_n \setminus E_{i,j}$ is a maximal subsemigroup of IF_n by Lemma 6, we conclude that $J = E_{i,j}$, i.e., $T = IF_n \setminus E_{i,j}$.

Suppose that $x = \beta_i^{odd}$ or $x = \beta_i^{even}$ for some $i \in \{2, 4, \ldots, \left(\dfrac{n+1}{2}\right)_e\}$. Then $\gamma_n\beta_i^{odd}, \beta_i^{odd}\gamma_n, \gamma_n\beta_i^{odd}\gamma_n \in J$ and $\gamma_n\beta_i^{even}, \beta_i^{even}\gamma_n, \gamma_n\beta_i^{even}\gamma_n \in J$. Assume that $\alpha_2\beta_i^{odd} \notin J$, i.e., $\alpha_2\beta_i^{odd} \in T$. Since $\operatorname{rank}(\alpha_2) = n-1$, we can conclude that $\alpha_2 \in T$. So, $\beta_i^{odd} = \alpha_2\alpha_2\beta_i^{odd} \in T = IF_n \setminus J$. This contradicts $\beta_i^{odd} \in J$. So, we have

$$\alpha_2\beta_i^{odd}, \gamma_n\alpha_2\beta_i^{odd}, \alpha_2\beta_i^{odd}\gamma_n, \gamma_n\alpha_2\beta_i^{odd}\gamma_n \in J.$$

We obtain by the same arguments that $\beta_i^{even}\alpha_2 \in J$. Hence,

$$\beta_i^{even}\alpha_2, \gamma_n\beta_i^{even}\alpha_2, \beta_i^{even}\alpha_2\gamma_n, \gamma_n\beta_i^{even}\alpha_2\gamma_n \in J.$$

Since $IF_n \setminus B_i$ and $IF_n \setminus C_i$, respectively, is a maximal subsemigroup of IF_n by Lemma 4, we can conclude that $J = B_i$ and $J = C_i$, respectively, i.e., $T = IF_n \setminus B_i$ and $T = IF_n \setminus C_i$, respectively. □

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