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RESEARCH ARTICLE

Formal functional calculus for copolynomials over a commutative ring

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Abstract. We study the copolynomials, i.e. K-linear mappings from the ring of polynomials $K[x_1, ..., x_n]$ into the commutative ring K. With the help of the Cauchy-Stieltjes transform of a copolynomial we introduce and study a multiplication of copolynomials. We build a counterpart of formal functional calculus for the case of a finite number of copolynomials. We obtain an analogue of the spectral mapping theorem and analogues of the Taylor formula and the Riesz-Dunford formula.

1. Introduction

Due to the classical Riesz-Dunford holomorphic functional calculus [6] for a holomorphic function f in a neighborhood of the spectrum of a continuous linear operator A in a Banach space one can determine the operator $f(A)$. In [10], assuming that A is a weakly locally nilpotent operator in a Fréchet space [11], we extended this functional calculus to the case of all formal power series. In [9] we introduced the multiplication operation for K-linear functionals in the ring of polynomials $K[x]$ over the commutative ring K such that this operation is consistent with the differentiation (see also Section 3 of the present paper, where it is considered the case of several variables). The resulting ring $K[x]'$ has been called

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the ring of copolynomials (see also [13]). We consider the ring $K[x]'$ as an algebraic analogue of a space of distributions (see [9, 12, 13]). We note that several non-equivalent constructions of a multiplication are considered in classical theories of distributions. For example, in the Colombeau theory [5] the square of the δ-function is well-defined, but in some other theories it is not defined [1]. Presence of multiplication and natural topology in $K[x]'$ make it possible to consider power series in the ring $K[x]'$. It appeared that many properties of copolynomials associated with formal power series are similar to corresponding properties of weakly locally nilpotent operators. In the present paper we build a counterpart of a formal functional calculus for the case of a finite number of copolynomials (see theorems 4.8 and 4.9). Furthermore, we obtain a counterpart of the spectral mapping theorem (Corollary 4.7) and analogues of the Taylor formula (Theorem 4.11) and the Riesz-Dunford formula (Theorem 4.12). In Section 4 we consider examples of the evaluation $f(\delta)$, where δ is a copolynomial δ-function (see examples 4.1–4.3), for some interesting formal power series.

Linear functionals in the space of polynomials were extensively studied from different points of views in algebra, combinatorics, and the theory of orthogonal polynomials (cf., for example, [8, 17, 7, 19]). In a classical case ($K = \mathbb{R}$ or $K = \mathbb{C}$), series with respect to derivatives of the δ-function are intensively studied because of their applications to differential and functional-differential equations and the theory of orthogonal polynomials [7].

2. Preliminaries

Let K be an arbitrary commutative integral domain with identity and let $K[x_1, ..., x_n]$ be a ring of polynomials with coefficients in K.

Definition 2.1. By a copolynomial over the ring K we mean a K-linear functional defined on the ring $K[x_1, ..., x_n]$, i.e. a homomorphism from the module $K[x_1, ..., x_n]$ into the ring K.

We denote the module of copolynomials over K by $K[x_1, .., x_n]'$. Thus $T \in K[x_1, ..., x_n]'$ if and only if $T : K[x_1, ..., x_n] \to K$ and T has the property of K-linearity: $T(ap+bq) = aT(p)+bT(q)$ for all $p, q \in K[x_1, ..., x_n]$ and $a, b \in K$. If $T \in K[x_1, .., x_n]'$ and $p \in K[x_1, ..., x_n]$, the for the value of T on p we use the notation (T, p). We also write the copolynomial $T \in K[x_1, ..., x_n]'$ in the form $T(x)$, where $x = (x_1, ..., x_n)$ is regarded as the argument of polynomials $p(x) \in K[x_1, ..., x_n]$ subjected to the action of the K-linear mapping T. In this case, the result of action of T upon p can be represented in the form $(T(x), p(x))$.

Let $\mathbb{N}_0$ be the set of nonnegative integers. For a multi-index $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{N}_0^n$ we put

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_n^{\alpha_n}}, \quad |\alpha| = \sum_{j=1}^{n} \alpha_j,$$

$$x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}, \quad \alpha! = \alpha_1! \alpha_2! \cdots \alpha_n!.$$

For multi-indexes $\alpha, \beta \in \mathbb{N}_0^n$ the relation $\alpha \leq \beta$ means that $\alpha_j \leq \beta_j$ for all $j = 1, ..., n$. If $\alpha \leq \beta$, then we will use the notation $\binom{\beta}{\alpha} = \prod\limits_{j=1}^{n} \binom{\beta_j}{\alpha_j}$.

Let $p(x) = \sum\limits_{|\alpha| \leq m} a_\alpha x^\alpha \in K[x_1, ..., x_n]$. If $h = (h_1, ..., h_n)$, then the polynomial $p(x + h) \in K[x_1, ..., x_n][h_1, ..., h_n]$ can be represented in the form

$$p(x + h) = \sum_{|\alpha| \leq m} p_\alpha(x) h^\alpha,$$

where $p_\alpha(x) \in K[x_1, ..., x_n]$. Since in the case of a field with zero characteristic $p_\alpha(x) = \frac{D^\alpha p(x)}{\alpha!}$, we also assume that, by definition, $\frac{D^\alpha p(x)}{\alpha!} = p_\alpha(x), \quad |\alpha| \leq m$ is true for any commutative ring K. For $m < |\alpha|$ we assume that $\frac{D^\alpha p(x)}{\alpha!} = 0$.

Now we introduce the notion of shift for a copolynomial [13]. For $T \in K[x_1, ..., x_n]'$ and fixed $h = (h_1, ..., h_n) \in K^n$ we define the copolynomial $T(x + h)$ by

$$(T(x + h), p) = (T, p(x - h)), \quad p \in K[x_1, ..., x_n].$$

Definition 2.2. The partial derivative $\frac{\partial T}{\partial x_j}$ of a copolynomial $T \in K[x_1, ..., x_n]'$ with respect to the variable x_j $(j = 1, ..., n)$ is defined as in the classical case by the formula

$$\left(\frac{\partial T}{\partial x_j}, p\right) = -\left(T, \frac{\partial p}{\partial x_j}\right), \quad p \in K[x_1, ..., x_n]. \tag{2.1}$$

By using this formula, we arrive at the following expression for the derivative $D^\alpha T$:

$$(D^\alpha T, p) = (-1)^{|\alpha|}(T, D^\alpha p), \quad p \in K[x_1, ..., x_n].$$

Therefore

$$(D^\alpha T, p) = 0, \text{ where } p \in K[x_1, ..., x_n] \text{ and } |\alpha| > \text{deg}p.$$

By virtue of the equality

$$\left(\frac{D^\alpha T}{\alpha!}, p\right) = (-1)^{|\alpha|}\left(T, \frac{D^\alpha p}{\alpha!}\right), \quad p \in K[x_1, ..., x_n],$$

the copolynomials $\frac{D^\alpha T}{\alpha!}$ are well defined for any $T \in K[x_1, ..., x_n]'$ and $\alpha \in \mathbb{N}_0^n$.

Example 2.3. The copolynomial δ-function is given by the formula

$$(\delta, p) = p(0), \quad p \in K[x_1, ..., x_n].$$

Therefore

$$(D^\alpha \delta, p) = (-1)^{|\alpha|}(\delta, D^\alpha p) = (-1)^{|\alpha|} D^\alpha p(0), \quad \alpha \in \mathbb{N}_0^n. \tag{2.2}$$

Example 2.4. Let $K = \mathbb{R}$ and let $f : \mathbb{R}^n \to \mathbb{R}$ be a Lebesgue-integrable function such that

$$\int\limits_{\mathbb{R}^n} |x^\alpha f(x)| dx < +\infty, \quad \alpha \in \mathbb{N}_0^n.$$

Then f generates the regular copolynomial T_f:

$$(T_f, p) = \int\limits_{\mathbb{R}^n} p(x) f(x) dx, \quad p \in \mathbb{R}[x_1, ..., x_n].$$

Note that, in this case, unlike the classical theory, all copolynomials are regular [7, Theorem 7.3.4].

We now consider the problem of convergence in the space $K[x_1, ..., x_n]'$. In the ring K, we consider the discrete topology. Further, in the module of copolynomials $K[x_1, ..., x_n]'$, we consider the topology of pointwise convergence. It is easy to show that the last topology is generated by the following metric:

$$d(T_1, T_2) = \sum_{|\alpha|=0}^{\infty} \frac{d_0((T_1, x^\alpha), (T_2, x^\alpha))}{2^{|\alpha|}},$$

where d_0 is the discrete metric on K. The metric d provides $K[x_1, ..., x_n]'$ in a complete metric space. The convergence of a sequence $\{T_k\}_{k=1}^{\infty}$ to T in $K[x_1, ..., x_n]'$ means that for every polynomial $p \in K[x_1, ..., x_n]$ there exists a number $k_0 \in \mathbb{N}$ such that

$$(T_k, p) = (T, p), \quad k = k_0, k_0 + 1, k_0 + 2, \dots .$$

The series $\sum\limits_{k=0}^{\infty} T_k$ converges in $K[x_1, ..., x_n]'$ if a sequence of its partial sums $\sum\limits_{k=0}^{N} T_k$ converges in $K[x_1, ..., x_n]'$.

The following lemma shows the possibility of the decomposition of an arbitrary copolynomial in series about the system $\frac{D^\alpha \delta}{\alpha!}$, $\alpha \in \mathbb{N}_0^n$ (see [8, Proposition 2.3] in the case $n = 1$ and $K = \mathbb{C}$).

Lemma 2.5. *Let $T \in K[x_1, ..., x_n]'$. Then*

$$T = \sum_{|\alpha|=0}^{\infty} (-1)^{|\alpha|} (T, x^\alpha) \frac{D^\alpha \delta}{\alpha!}. \tag{2.3}$$

We now consider the following linear differential operator of infinite order on $K[x_1, ..., x_n]'$:

$$\mathcal{F} = \sum_{|\alpha|=0}^{\infty} a_\alpha D^\alpha,$$

where $a_\alpha \in K$. This operator acts upon a copolynomial $T \in K[x_1, ..., x_n]'$ by the following rule: if $p \in K[x_1, ..., x_n]$ and $m = \deg p$, then

$$(\mathcal{F}T, p) = \left(\sum_{|\alpha|=0}^{\infty} a_\alpha D^\alpha T, p \right) =$$

$$= \sum_{|\alpha| \le m} (-1)^{|\alpha|} a_\alpha (T, D^\alpha p) = \sum_{|\alpha| \le m} a_\alpha (D^\alpha T, p).$$

Thus, the differential operator $\mathcal{F} : K[x_1, ..., x_n]' \to K[x_1, ..., x_n]'$ is well defined and for any polynomial p of degree at most m the equality

$$(\mathcal{F}T, p) = \sum_{|\alpha| \le m} a_\alpha (D^\alpha T, p). \tag{2.4}$$

is true.

Remark 2.6. Let $n = 1$ and $h \in K$. It was shown in [16] that shift $T(x+h)$ of the copolynomial $T \in K[x]'$ generates the following infinite order differential operator $\tau_h = \sum_{k=0}^{\infty} h^k \frac{1}{k!} \frac{d^k}{dx^k}$, i.e.

$$T(x+h) = \tau_h(T) = \sum_{k=0}^{\infty} h^k \frac{1}{k!} \frac{d^k T}{dx^k}, \quad T \in K[x]'. \tag{2.5}$$

3. Multiplication of copolynomials

3.1. The Cauchy-Stieltjes transform

Let $z = (z_1, ..., z_n)$ and let $K\left[\left[z_1, ..., z_n, \frac{1}{z_1}, ..., \frac{1}{z_n}\right]\right]$ be the module of formal Laurent series with coefficients in K. For the multi-index $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{Z}^n$ we put $z^\alpha = z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_n^{\alpha_n}$. For $g(z) = \sum_{\alpha \in \mathbb{Z}^n} g_\alpha z^\alpha \in K\left[\left[z_1, ..., z_n, \frac{1}{z_1}, ..., \frac{1}{z_n}\right]\right]$ we naturally define the formal residue:

$$Res(g(z)) = g_{(-1,...,-1)}.$$

Definition 3.1. Let $T \in K[x_1, ..., x_n]'$ and $s = (s_1, ..., s_n)$. Consider the following formal Laurent series from the ring $\frac{1}{s_1 s_2 \cdots s_n} K[[\frac{1}{s_1}, \frac{1}{s_2}, ..., \frac{1}{s_n}]]$:

$$C(T)(s) = \sum_{|\alpha|=0}^{\infty} \frac{(T, x^\alpha)}{s^{\alpha+\iota}},$$

where $\iota = (1, ..., 1) \in \mathbb{N}_0^n$. The Laurent series $C(T)(s)$ will be called the *Cauchy-Stieltjes transform* of a copolynomial T.

We may write informally as follows:

$$C(T)(s) = \left(T, \frac{1}{(s_1 - x_1)(s_2 - x_2) \cdots (s_n - x_n)}\right).$$

Obviously, that the mapping

$$C : K[x_1, ..., x_n]' \to \frac{1}{s_1 s_2 \cdots s_n} K\left[\left[\frac{1}{s_1}, \frac{1}{s_2}, ..., \frac{1}{s_n}\right]\right]$$

is an isomorphism of K-modules.

Proposition 3.2 (the inversion formula)**.** *Let* $T \in K[x_1, ..., x_n]'$ *and* $p \in K[x_1, ..., x_n]$*. Then*

$$(T, p) = Res(C(T)(s)p(s)).$$

Proof. It is sufficient to consider the case $p(x) = x^\beta$ for some multi-index $\beta \in \mathbb{N}_0^n$. We have

$$C(T)(s)s^\beta = \sum_{|\alpha|=0}^{\infty} \frac{(T, x^\alpha)s^\beta}{s^{\alpha+\iota}}.$$

Therefore, $Res(C(T)(s)s^\beta) = (T, x^\beta)$. □

Example 3.3 (the "integral" Cauchy formula)**.** We put $(\delta_z, p) = p(z_1, ..., z_n)$ for $z = (z_1, ..., z_n) \in K^n$. Then $C(\delta_z)(s) = \sum\limits_{|\alpha|=0}^{\infty} \frac{z^\alpha}{s^{\alpha+\iota}}$. In particular,

$$C(\delta)(s) = \frac{1}{s_1 s_2 \cdots s_n}. \tag{3.1}$$

We may write informally $p(z_1, ..., z_n) = Res(\frac{p(s_1,...,s_n)}{(s_1-z_1)(s_2-z_2)\cdots(s_n-z_n)})$, if to identify the rational function $\frac{1}{(s_1-z_1)(s_2-z_2)\cdots(s_n-z_n)}$ with the Laurent series $\sum\limits_{|\alpha|=0}^{\infty} \frac{z^\alpha}{s^{\alpha+\iota}}$.

The following assertion yields commutativity of the Cauchy-Stieltjes transform and the differentiating operation.

Proposition 3.4. *For each* $T \in K[x_1, ..., x_n]'$ *the following equality*

$$C\left(\frac{\partial T}{\partial x_j}\right) = \frac{\partial}{\partial s_j} C(T), \quad j = 1, ..., n \tag{3.2}$$

holds.

Proof. Without loss of generality we assume $j = 1$. We have

$$C\left(\frac{\partial T}{\partial x_1}\right)(s) = \sum_{|\alpha|=0}^{\infty} \frac{(\frac{\partial T}{\partial x_1}, x^\alpha)}{s^{\alpha+\iota}} = -\sum_{\alpha:\alpha_1\geq 1} \frac{\alpha_1(T, x_1^{\alpha_1-1}x_2^{\alpha_2}\cdots x_n^{\alpha_n})}{s_1^{\alpha_1+1}s_2^{\alpha_2+1}\cdots s_n^{\alpha_n+1}} =$$

$$= -\sum_{|\alpha|=0}^{\infty} \frac{(\alpha_1+1)(T, x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_n^{\alpha_n})}{s_1^{\alpha_1+2}s_2^{\alpha_2+1}\cdots s_n^{\alpha_n+1}},$$

$$\frac{\partial}{\partial s_1}C(T)(s) = \frac{\partial}{\partial s_1}\sum_{|\alpha|=0}^{\infty}\frac{(T, x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_n^{\alpha_n})}{s_1^{\alpha_1+1}s_2^{\alpha_2+1}\cdots s_n^{\alpha_n+1}} =$$

$$= -\sum_{|\alpha|=0}^{\infty}\frac{(\alpha_1+1)(T, x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_n^{\alpha_n})}{s_1^{\alpha_1+2}s_2^{\alpha_2+1}\cdots s_n^{\alpha_n+1}}.$$

□

3.2. Multiplication of copolynomials and its properties

The Cauchy-Stieltjes transform and Proposition 3.4 allow to introduce the multiplication operation on the module of copolynomials such that this operation is consistent with the differentiation.

Definition 3.5. Let $T_1, T_2 \in K[x_1, ..., x_n]'$, i.e. T_1, T_2 are copolynomials. Define their *product* by the following equality:

$$C(T_1T_2) = C(T_1)C(T_2), \tag{3.3}$$

i.e.

$$T_1T_2 = C^{-1}\left(C(T_1)C(T_2)\right),$$

where $C : K[x_1, ..., x_n]' \to \frac{1}{s_1s_2\cdots s_n}K[[\frac{1}{s_1}, \frac{1}{s_2}, ..., \frac{1}{s_n}]]$ is a Cauchy-Stieltjes transform.

In the following lemma the action of the product of copolynomials on monomials is expressed through the action of multipliers on monomials.

Lemma 3.6. *Let $T_1, T_2 \in K[x_1, ..., x_n]'$ and $\alpha \in \mathbb{N}_0^n$. Then*

$$(T_1T_2, x^\alpha) = \begin{cases} \sum\limits_{\beta\le\alpha-\iota}(T_1, x^\beta)(T_2, x^{\alpha-\iota-\beta}), & \alpha \ge \iota, \\ 0, \quad \text{in another case.} \end{cases} \tag{3.4}$$

Proof. By Formula (3.3) we have

$$C(T_1T_2)(s) = C(T_1)(s)C(T_2)(s) = \sum_{|\beta|=0}^{\infty}\sum_{|\gamma|=0}^{\infty}\frac{(T_1, x^\beta)(T_2, x^\gamma)}{s^{\beta+\gamma+2\iota}} =$$

$$= \sum_{\alpha\ge\iota}\sum_{\beta\le\alpha-\iota}(T_1, x^\beta)(T_2, x^{\alpha-\iota-\beta})\frac{1}{s^{\alpha+\iota}}.$$

Applying the inversion formula to both sides of this equality (See Proposition 3.2), we obtain (3.4). □

Remark 3.7. Definition 3.5 means that the module of copolynomials $K[x_1, ..., x_n]'$ with the introduced product is an associative commutative ring, which isomorphic to the ring of formal Laurent series $\frac{1}{s_1 s_2 \cdots s_n} K[[\frac{1}{s_1}, \frac{1}{s_2}, ..., \frac{1}{s_n}]]$ with a natural product operation. In particular, the ring of copolynomials is an integral domain and this is a ring without identity.

Example 3.8. Let $n = 1$. We find the square of δ-function:

$$C(\delta^2)(s) = (C(\delta))^2(s) = \frac{1}{s^2} = \left(\frac{-1}{s}\right)' = (-C(\delta))' = C(-\delta'),$$

i.e.

$$\delta^2 = -\delta'.$$

Moreover, by (2.2) and (3.1) we have

$$C\left(\frac{\delta^{(k)}}{k!}\right)(s) = \sum_{j=0}^{\infty} \left(\frac{\delta^{(k)}}{k!}, x^j\right) \frac{1}{s^{j+1}} = \sum_{j=0}^{\infty} \left(\delta, \frac{1}{k!} \frac{d^k x^j}{dx^k}\right) \frac{(-1)^k}{s^{j+1}} =$$

$$= \frac{(-1)^k}{s^{k+1}} = (-1)^k (C(\delta))^{k+1},$$

so that

$$\frac{(-1)^k \delta^{(k)}}{k!} = \delta^{k+1}, \quad k = 0, 1, 2, \dots . \tag{3.5}$$

The generalization of equalities (3.5) on n variables is the following formulas:

$$\frac{\partial^{nk} \delta}{\partial x_1^k \dots \partial x_n^k} = (-1)^{nk} (k!)^n \delta^{k+1}, \quad k = 0, 1, 2, \dots . \tag{3.6}$$

Definition 3.5 implies that for any multi-index $\beta \in \mathbb{N}_0^n$ and for any copolynomials $T_1, ..., T_m \in K[x_1, ..., x_n]'$ the formula

$$\left(\prod_{k=1}^{m} T_k, x^\beta\right) = 0, \quad m > |\beta| + 1 \tag{3.7}$$

holds. Let $T_1, ..., T_m \in K[x_1, ..., x_n]'$ and $\gamma = (\gamma_1, ..., \gamma_m) \in \mathbb{N}_0^m$, $\gamma \neq 0$. We define T^γ by the equality

$$T^\gamma = \prod_{j=1, \gamma_j \neq 0}^{m} T_j^{\gamma_j}. \tag{3.8}$$

In particular, $T^{\iota} = \prod_{j=1}^{m} T_j$. If $\gamma = 0$ and $S \in K[x_1, ..., x_n]'$, then we define $T^{\gamma}S = S$. Equality (3.7) implies that for any multi-indexes $\beta \in \mathbb{N}_0^n$ and $\gamma \in \mathbb{N}_0^m$ the formula

$$(T^{\gamma}, x^{\beta}) = 0, \quad |\gamma| > |\beta| + 1 \tag{3.9}$$

is satisfied.

4. Main results

Let $K_0[[z_1, ..., z_m]] = \{g \in K[[z_1, ..., z_m]] : g(0) = g_0 = 0\}$ and $g(z) = \sum_{|\gamma|=1}^{\infty} g_{\gamma} z^{\gamma} \in K_0[[z_1, ..., z_m]]$. The ring $K_0[[z_1, ..., z_m]]$ is an algebra over the ring K. We will consider the Krull topology on the algebra $K_0[[z_1, ..., z_m]]$ [15, Chapter 1, §3, Section 4], [3, Chapter IV, §5, Section 10]. The equality (3.9) implies that the series $g(T_1, ..., T_m) = \sum_{|\gamma|=1}^{\infty} g_{\gamma} T^{\gamma}$ converges in the topology $K[x_1, ..., x_n]'$.

Example 4.1. Let $n = 1$ and $a \in K$. Consider the formal power series $g(z) = \sum_{k=0}^{\infty} a^k k! z^{k+1} \in K_0[[z]]$. Then taking into account (3.5)

$$g(\delta) = \sum_{k=0}^{\infty} a^k k! \delta^{k+1} = \sum_{k=0}^{\infty} (-1)^k a^k \delta^{(k)}.$$

If $K = \mathbb{R}$ and $a > 0$, then

$$g(\delta) = \frac{1}{a}\, \theta(x) e^{-x/a}, \tag{4.1}$$

where $\theta(x)$ is a Heaviside function and the equality (4.1) means that

$$(g(\delta), p) = \frac{1}{a} \int_{-\infty}^{\infty} \theta(x) e^{-x/a} p(x) dx = \frac{1}{a} \int_{0}^{\infty} e^{-x/a} p(x) dx, \quad p \in \mathbb{R}[x]$$

(see examples 4 and 5 in [14] and Example 3.1 in [12]).

Example 4.2. Let $n = 1$ and $a \in K$. Consider the formal power series $g(z) = \sum_{k=0}^{\infty} a^k z^{k+1} \in K_0[[z]]$. Then (3.5) implies

$$g(\delta) = \sum_{k=0}^{\infty} a^k \delta^{k+1} = \sum_{k=0}^{\infty} (-1)^k a^k \frac{\delta^{(k)}}{k!} = \delta(x - a).$$

Example 4.3. Assume that $a \in K$ and $\mathcal{F} = \sum\limits_{k=0}^{\infty} a^k \frac{\partial^{nk}}{\partial x_1^k \ldots \partial x_n^k}$ is an infinite order differential operator. Then (3.6) implies

$$\mathcal{F}(\delta) = \sum_{k=0}^{\infty} a^k (-1)^{kn} (k!)^n \delta^{k+1}.$$

Remind that we consider the construction $g(T_1, ..., T_m)$ for the series $g(z) \in K_0[[z_1, ..., z_m]]$ only. If $g(z) = \sum\limits_{|\gamma|=0}^{\infty} g_\gamma z^\gamma \in K[[z_1, ..., z_m]]$, where $g_0 \neq 0$ and $S \in K[x_1, ..., x_n]'$, then the product $g(T_1, ..., T_m)S$ means the sum of following copolynomials:

$$g_0 S + \sum_{|\gamma|=1}^{\infty} g_\gamma T^\gamma S. \tag{4.2}$$

Consider the ring $AK[x_1, ..., x_n]'$, which is obtained by adjoining identity to the ring $K[x_1, ..., x_n]'$ (see, for example, [3, Chapter VIII, Applications], where these arguments were conducted for a field). This ring consists of elements (c, T), where $T \in K[x_1, ..., x_n]'$ and $c \in K$. Moreover, $(c_1, T_1) + (c_2, T_2) = (c_1 + c_2, T_1 + T_2)$, $(c_1, T_1) \cdot (c_2, T_2) = (c_1 c_2, T_1 T_2 + c_2 T_1 + c_1 T_2)$ for all $T_1, T_2 \in K[x_1, ..., x_n]'$ and $c_1, c_2 \in K$. The identity of this ring is the element $(1, 0)$. In what follows the element $(c, T) \in AK[x_1, ..., x_n]'$ we will write as $c + T$ or $T + c$. We denote by $\sigma(T)$ the *spectrum* of the copolynomial $T \in K[x_1, ..., x_n]'$, i.e. the set of $\lambda \in K$, for which $T - \lambda$ is a non-invertible element of the ring $AK[x_1, ..., x_n]'$. We denote by K^* the set of invertible elements of K.

The following theorem establishes the criterion of the invertibility of $T - \lambda$ in the ring $AK[x_1, ..., x_n]'$.

Theorem 4.4. *Let $T \in K[x_1, ..., x_n]'$ and $\lambda \in K$. The element $T - \lambda$ is invertible in the ring $AK[x_1, ..., x_n]'$ if and only if $\lambda \in K^*$. Therefore, $\sigma(T) = K \setminus K^*$.*

Proof. Indeed, $T - \lambda$ is an invertible element if and only if there exist $S \in K[x_1, ..., x_n]'$ and $\mu \in K$ such that $(T - \lambda)(S - \mu) = 1$. This means that $TS - \lambda S - \mu T + \lambda\mu = 1$, and it is equivalent to equalities

$$\lambda\mu = 1, \quad TS - \lambda S - \mu T = 0. \tag{4.3}$$

The first equation of the system (4.3) implies that λ is an invertible element of the ring K and $\mu = \lambda^{-1}$. The second equation of the system

(4.3) is rewritten as

$$TS = \lambda S + \lambda^{-1} T.$$

By this taking into account (3.4) for any multi-index $\alpha \in \mathbb{N}_0^n$ we have

$$\lambda(S, x^\alpha) + \lambda^{-1}(T, x^\alpha) = \begin{cases} \sum\limits_{\beta \le \alpha - \iota} (T, x^\beta)(S, x^{\alpha-\iota-\beta}), & \alpha \ge \iota, \\ 0, & \text{in the other case.} \end{cases}$$

Then

$$(S, x^\alpha) = -\lambda^{-2}(T, x^\alpha) + \begin{cases} \sum\limits_{\beta \le \alpha - \iota} (T, x^\beta)(S, x^{\alpha-\iota-\beta}), & \alpha \ge \iota, \\ 0, & \text{in the other case,} \end{cases}$$

i.e. the copolynomial S is uniquely restored by the copolynomial T. □

Theorem 4.4 implies the following assertion about a description of spectra of T and $g(T)$, where $g(z) \in K_0[[z]]$ and K is a field.

Corollary 4.5. *Assume that K is a field, $T \in K[x_1, ..., x_n]'$ and $g(z) \in K_0[[z]]$. Then $\sigma(T) = \{0\}$ and $\sigma(g(T)) = \{g(0)\} = \{0\}$, i.e. $\sigma(g(T)) = g(\sigma(T))$. Moreover, $AK[x_1, ..., x_n]'$ is a local ring with unique maximal ideal $K[x_1, ..., x_n]'$.*

Proof. If K is a field then $K^* = K \setminus \{0\}$. Then by Theorem 4.4 $\sigma(T) = \{0\}$ and $\sigma(g(T)) = \{g(0)\} = \{0\}$, i.e.

$$\sigma(g(T)) = g(\sigma(T)). \tag{4.4}$$

The locality of the ring $AK[x_1, ..., x_n]'$ follows from the invertibility of all elements $T - \lambda$, where $\lambda \neq 0$ (see [2, Chapter 1, Proposition 1.6]). □

Remark 4.6. In the case where K is not a field the equality (4.4) is absurd, because the substitution $g(\lambda)$ for $\lambda \neq 0$ is not defined, if $g(z)$ is not a polynomial. Moreover, if even $g \in K[z]$, $g(0) = 0$, then in general $g(K \setminus K^*) \neq K \setminus K^*$ and in this case the equality (4.4) is not correct.

For copolynomials $T_1, ..., T_m \in K[x_1, ..., x_n]'$ we denote by $\sigma(T_1, ..., T_m)$ their *joint spectrum*, i.e. the set of elements $(\lambda_1, ..., \lambda_m) \in K^m$ such that the ideal generating by $T_1 - \lambda_1, ..., T_m - \lambda_m$ is not contain with $AK[x_1, ..., x_n]'$ [4, Chapter 1, Section 3.5]. For $m = 1$ the notion of the joint spectrum coincides with the notion of the spectrum, which was considered above. Corollary 4.5 implies the following assertion which generalizes the assertion of this corollary.

Corollary 4.7 (Spectral mapping theorem)**.** *Assume that K is a field, $T_1, ..., T_m \in K[x_1, ..., x_n]'$ and $g(z) \in K_0[[z_1, ..., z_m]]$. Then*

$$\sigma(T_1, ..., T_m) = \{0\}$$

and $\sigma(g(T_1, ..., T_m)) = \{g(0)\} = \{0\}$, i.e.

$$\sigma(g(T_1, ..., T_m)) = g(\sigma(T_1, ..., T_m)).$$

Proof. Indeed if $\lambda_j \neq 0$ for some $j = 1, ..., m$, then by Corollary 4.5 the element $T_j - \lambda_j$ is invertible in the ring $AK[x_1, ..., x_n]'$. Then ideal generating by this element coincides with the ring $AK[x_1, ..., x_n]'$. □

Now consider the mapping $\theta_{T_1,...,T_m} : K_0[[z_1, ..., z_m]] \to K[x_1, ..., x_n]'$, $\theta_{T_1,...,T_m}(g) = g(T_1, ..., T_m)$, $g \in K_0[[z_1, ..., z_m]]$. In what follows for $T_1, ..., T_m \in K[x_1, ..., x_n]'$ and $\gamma \in \mathbb{N}_0^m$ the expression T^γ is defined by (3.8).

Theorem 4.8. *The mapping $\theta_{T_1,...,T_m}$ is a unique continuous homomorphism from the algebra $K_0[[z_1, ..., z_m]]$ to the algebra $K[x_1, ..., x_n]'$, which maps z_j to T_j $(j = 1, ..., m)$.*

Proof. Obviously that $\theta_{T_1,...,T_m}$ is a K-linear mapping. We show that

$$\theta_{T_1,...,T_m}(fg) = \theta_{T_1,...,T_m}(f)\theta_{T_1,...,T_m}(g)$$

for all $f, g \in K_0[[z_1, ..., z_m]]$, i.e.

$$(f \cdot g)(T_1, ..., T_m) = f(T_1, ..., T_m)g(T_1, ..., T_m). \tag{4.5}$$

Let $h = fg$. Then $h \in K_0[[z_1, ..., z_m]]$,

$$h(z) = \sum_{|\alpha|=0}^{\infty} h_\alpha z^\alpha, \quad \text{where} \quad h_\alpha = \sum_{\beta \leq \alpha} f_\beta g_{\alpha-\beta}.$$

Therefore,

$$h(T_1, ..., T_m) = \sum_{|\alpha|=0}^{\infty} h_\alpha T^\alpha = \sum_{|\alpha|=0}^{\infty} \sum_{\beta \leq \alpha} f_\beta g_{\alpha-\beta} T^\alpha = \sum_{|\beta|=0}^{\infty} \sum_{\alpha \geq \beta} f_\beta g_{\alpha-\beta} T^\alpha =$$

$$= \sum_{|\beta|=0}^{\infty} \sum_{|\gamma|=0}^{\infty} f_\beta g_\gamma T^{\beta+\gamma} = \sum_{|\beta|=0}^{\infty} f_\beta T^\beta \sum_{|\gamma|=0}^{\infty} g_\gamma T^\gamma =$$

$$= f(T_1, ..., T_m)g(T_1, ..., T_m),$$

where the all expressions of the kind cT^0 means as zero by definition, if $c = 0$.

Now we prove the continuous of the homomorphism $\theta_{T_1,...,T_m}$. Let $\{g_k(z)\}_{k=0}^{\infty}$ be a sequence of formal power series from $K_0[[z_1, ..., z_m]]$, which converges to zero as $k \to \infty$. By (3.9) for any $\beta \in \mathbb{N}_0^n$ we have

$$(\theta_{T_1,...,T_n}(g_k), x^\beta) = \sum_{|\gamma|=1}^{|\beta|+1} g_{k\gamma}(T^\gamma, x^\beta), \quad k \in \mathbb{N}. \tag{4.6}$$

If $g_k(z) = \sum\limits_{|\gamma|=0}^{\infty} g_{k\gamma} z^\gamma$, then there exists a number $k_0 = k_0(\beta) \in \mathbb{N}$ such that $g_{k\gamma} = 0$ for all $k \geq k_0$ and $1 \leq |\gamma| \leq |\beta| + 1$. Now by (4.6) $\lim\limits_{k\to\infty} \theta_{T_1,...,T_n}(g_k) = 0$ in the topology of $K[x_1, ..., x_n]'$.

Now let $T_1, ..., T_m \in K[x_1, ..., x_n]'$ and let

$$\theta : K_0[[z_1, .., z_m]] \to K[x_1, ..., x_n]'$$

be a continuous homomorphism of algebras such that $\theta(z_j) = T_j$ $(j = 1, ..., m)$. We show that

$$\theta(g) = g(T_1, ..., T_m), \quad g \in K_0[[z_1, ..., z_m]]. \tag{4.7}$$

For all $g(z) = \sum\limits_{|\gamma|=1}^{\infty} g_\gamma z^\gamma \in K_0[[z_1, ..., z_m]]$ the sequence $f_k(z) = \sum\limits_{|\gamma|=1}^{k} g_\gamma z^\gamma$ converges in the topology of $K_0[[z_1, ..., z_m]]$ to g as $k \to \infty$. Since θ is a continuous homomorphism and the series $\sum\limits_{|\gamma|=1}^{\infty} g_\gamma T^\gamma$ converges in the topology of $K[x_1, ..., x_n]'$, we have

$$\theta(g) = \lim_{k\to\infty} \theta(f_k) = \lim_{k\to\infty} \sum_{|\gamma|=1}^{k} g_\gamma \theta(z^\gamma) = \sum_{|\gamma|=1}^{\infty} g_\gamma T^\gamma = g(T_1, ..., T_m).$$

This implies (4.7). The proof is complete. □

The following theorem establishes a connection between the homomorphism $\theta_{T_1,...,T_m}$ and the composition of formal power series.

Theorem 4.9. *Assume that* $T_1, ...T_m \in K[x_1, ..., x_n]'$, $f(z) = \sum_{|\gamma|=1}^{\infty} f_\gamma z^\gamma \in K_0[[z_1, ..., z_m]]$ *and* $g(s) = \sum_{k=1}^{\infty} g_k s^k \in K_0[[s]]$. *Then* $g \circ f \in K_0[[z_1, ..., z_m]]$ *and*

$$g(f(T_1, ...T_m)) = (g \circ f)(T_1, ...T_m), \tag{4.8}$$

i.e. $\theta_{T_1,...,T_m}(g \circ f) = \theta_{f(T_1,...,T_m)}(g)$.

Proof. Since $f(0) = 0$ and $g(0) = 0$, the following series is well defined:

$$g(f(z)) = \sum_{k=1}^{\infty} g_k (f(z))^k$$

and $g \circ f \in K_0[[z_1, ..., z_m]]$. If $g(s) = s^N$ $(N \in \mathbb{N})$, then $(g \circ f)(z) = (f(z))^N$, and the equality (4.8) directly follows from Theorem 4.8 (see Formula (4.5)). Therefore the equality (4.8) holds also for any polynomial $g(s) \in K[s]$, where $g(0) = 0$.

Now let $g(s) = \sum_{k=1}^{\infty} g_k s^k$ be an arbitrary element of $K_0[[s]]$. Put $g_N(s) = \sum_{k=1}^{N} g_k s^k$. Since $g(s) = \lim_{N\to\infty} g_N(s)$ in the topology $K_0[[s]]$, we have $(g \circ f)(z) = \lim_{N\to\infty} (g_N \circ f)(z)$ (see [18, Chapter 1, §1, Section 4; Chapter 4, §1, Section 1]). The continuity of the homomorphism $\theta_{T_1,...,T_m}$ implies

$$(g \circ f)(T_1, ..., T_m) = \theta_{T_1,...,T_m}(g \circ f) = \lim_{N\to\infty} \theta_{T_1,...,T_m}(g_N \circ f) =$$

$$= \lim_{N\to\infty} g_N(f(T_1, ..., T_m)) = g(f(T_1, ..., T_m)).$$

The proof is complete. □

Now we obtain a counterpart of the Taylor expansion for the constructed formal functional calculus. The formal power series

$$\frac{D^\beta g}{\beta!} = \sum_{\gamma \geq \beta} \binom{\gamma}{\beta} g_\gamma z^{\gamma-\beta} \in K[[z_1, ..., z_m]] \tag{4.9}$$

is well-defined for the multi-index $\beta \in \mathbb{N}_0^m$ and the formal power series $g(z) = \sum_{|\gamma|=0}^{\infty} g_\gamma z^\gamma \in K[[z_1, ..., z_m]]$. If $T_1, ..., T_m, S_1, ..., S_m \in K[x_1, ..., x_n]'$ and $\gamma = 0 \in \mathbb{N}_0^m$, then by definition $T^\gamma S = ST^\gamma = S$.

The following lemma is one of counterparts for the binomial formula [3, Chapter 1, Section 5.1].

Lemma 4.10. *Let* $T_1, ..., T_m, S_1, ..., S_m \in K[x_1, ..., x_n]'$. *Then*

$$\prod_{j=1,\beta_j\neq 0}^{m} (T_j + S_j)^{\beta_j} = \sum_{\gamma\leq\beta} \binom{\beta}{\gamma} T^{\beta-\gamma} S^{\gamma} \tag{4.10}$$

for any nonzero multi-index $\beta \in \mathbb{N}_0^m$.

Theorem 4.11. *Let* $T_1, ..., T_m, S_1, ..., S_m \in K[x_1, ..., x_n]'$ *and let* $f(z) = \sum_{|\gamma|=1}^{\infty} f_\gamma z^\gamma \in K_0[[z_1, ..., z_m]]$. *Then the series* $\sum_{|\gamma|=0}^{\infty} \frac{D^\gamma f(T_1,..,T_m)}{\gamma!} S^\gamma$ *converges in the topology of* $K[x_1, ..., x_n]'$ *to the copolynomial* $f(T_1 + S_1, ..., T_m + S_m)$, *i.e. the following Taylor expansion holds:*

$$f(T_1 + S_1, ..., T_m + S_m) = \sum_{|\gamma|=0}^{\infty} \frac{(D^\gamma f)(T_1, ..., T_m)}{\gamma!} S^\gamma. \tag{4.11}$$

Proof. The convergence of the series in right-hand part of (4.11) follows from (3.7), (3.9). By virtue of (4.9), of the Newton binomial formula (4.10) and of Definition 4.2, we have

$$\sum_{|\gamma|=0}^{\infty} \frac{(D^\gamma f)(T_1, ..., T_m)}{\gamma!} S^\gamma = \sum_{|\gamma|=0}^{\infty} \sum_{\beta\geq\gamma} \binom{\beta}{\gamma} f_\beta T^{\beta-\gamma} S^\gamma =$$

$$= \sum_{|\beta|=0}^{\infty} f_\beta \sum_{\gamma\leq\beta} \binom{\beta}{\gamma} S^\gamma T^{\beta-\gamma} = \sum_{|\beta|=0}^{\infty} f_\beta \sum_{\gamma\leq\beta} \binom{\beta}{\gamma} S^\gamma T^{\beta-\gamma} =$$

$$= f(T_1 + S_1, ..., T_m + S_m).$$

□

Now we turn to the counterpart of the Riesz-Dunford formula for the formal functional calculus. Consider formal Laurent series

$$R_{T_1,...,T_m}(z) = \sum_{|\gamma|=1}^{\infty} \frac{T^\gamma}{z^{\gamma+\iota}} \in \frac{1}{z_1 z_2 \cdots z_m} K[x_1, ..., x_n]' \left[\left[\frac{1}{z_1}, ..., \frac{1}{z_m}\right]\right]$$

for copolynomials $T_1, ..., T_m \in K[x_1, ..., x_n]'$. Then we can define the product $f(z)R_{T_1,...,T_m}(z)$, where $f(z) = \sum_{|\gamma|=1}^{\infty} f_\gamma z^\gamma \in K_0[[z_1, ..., z_m]]$:

$$f(z)R_{T_1,...,T_m}(z) \stackrel{def}{=} \sum_{\beta\in\mathbb{Z}^m} \left(\sum_{\gamma\in\mathbb{N}_0^m\setminus\{0\}:\gamma\geq\beta-\iota}^{\infty} f_{\gamma+\iota-\beta} T^\gamma \right) \frac{1}{z^\beta},$$

which is an element of the ring $K[x_1, ..., x_n]' \left[\left[z_1, ..., z_m, \frac{1}{z_1}, ..., \frac{1}{z_m}\right]\right]$, because coefficients at each powers z are convergent series in $K[x_1, ..., x_n]'$. Calculating the formal residue of the series $f(z)R_{T_1,...,T_m}(z)$, we obtain the following counterpart of the Riesz-Dunford formula.

Theorem 4.12. *Let* $T_1, ..., T_m \in K[x_1, ..., x_n]'$ *and* $f(z) = \sum_{|\gamma|=1}^{\infty} f_\gamma z^\gamma \in K_0[[z_1, ..., z_m]]$. *Then*

$$f(T_1, ..., T_m) = Res(f(z)R_{T_1,...,T_m}(z)).$$

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