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Rings of differential operators on singular generalized multi-cusp algebras

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Dedicated to Professor Yu. A. Drozd
on the occasion of his 80th birthday

Abstract. The aim of the paper is to study the ring of differential operators $\mathcal{D}(A(m))$ on the generalized multi-cusp algebra $A(m)$ where $m \in \mathbb{N}^n$ (of Krull dimension n). The algebra $A(m)$ is singular apart from the single case when $m = (1, \dots, 1)$. In this case, the algebra $A(m)$ is a polynomial algebra in n variables. So, the n'th Weyl algebra $A_n = \mathcal{D}(A(1, \dots, 1))$ is a member of the family of algebras $\mathcal{D}(A(m))$. We prove that the algebra $\mathcal{D}(A(m))$ is a central, simple, $\mathbb{Z}^n$-graded, finitely generated Noetherian domain of Gelfand-Kirillov dimension $2n$. Explicit finite sets of generators and defining relations is given for the algebra $\mathcal{D}(A(m))$. We prove that the Krull dimension and the global dimension of the algebra $\mathcal{D}(A(m))$ is n. An analogue of the Inequality of Bernstein is proven. In the case when $n = 1$, simple $\mathcal{D}(A(m))$-modules are classified.

1. Introduction

The following notation will remain fixed throughout the paper (if it is not stated otherwise): K is a field of characteristic zero (not necessarily algebraically closed), module means a left module, $P_n = K[x_1, \dots, x_n]$

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is a polynomial algebra over K, $\partial_1 := \frac{\partial}{\partial x_1}, \dots, \partial_n := \frac{\partial}{\partial x_n} \in \mathrm{Der}_K(P_n)$, $A_n = K\langle x_1, \dots, x_n, \partial_1, \dots, \partial_n\rangle \subseteq \mathrm{End}_K(P_n)$ is the n'th *Weyl algebra* ($A_n = \mathcal{D}(P_n)$ is the algebra of differential operators on the polynomial algebra P_n), $\mathbb{N} = \{0, 1, \dots\}$ is the set of natural numbers and $\mathbb{N}_{\geq s} = \{i \in \mathbb{N} \mid i \geq s\}$. In the case $n = 1$, we usually drop the subscript '1'. So, $P = K[x]$ is a polynomial algebra in a variable x, $A_1 = K\langle x, \partial \mid \partial x - x\partial = 1\rangle$ is the *Weyl algebra*, i.e., $A_1 = \mathcal{D}(P)$ is the ring of differential operators on the polynomial algebra P.

The algebra of regular functions on the cusp $y^2 = x^3$ is isomorphic to the subalgebra $A(2) = K + \sum_{i\geq 2} Kx^i$ of the polynomial algebra $P = K[x]$. For each $m \geq 1$, $A = A(m) = K + \sum_{i\geq m} Kx^i$ is a subalgebra of P which is caleld the **generalized cusp algebra**. Clearly, $A(1) = K[x]$ is a polynomial algebra and $A(2)$ is the cusp algebra.

Definition. Let $m = (m_1, \dots, m_n) \in \mathbb{N}^n$, the subalgebra of the polynomial algebra $P_n = K[x_1, \dots, x_n]$,

$$A(m) = \bigotimes_{i=1}^{n} A(m_i), \quad \text{where} \quad A(m_i) = K + \sum_{j\geq m_i} Kx_i^j \subseteq K[x_i],$$

is called the **generalized multi-cusp algebra** of rank n (GMCA, for short).

Clearly, if $m = (1, \dots, 1)$ then $\mathcal{D}(A(m)) = A_n$ is the n'th Weyl algebra. If $m = (2, \dots, 2)$ then $A(m) \simeq A(2)^{\otimes n}$ is the algebra of regular functions on the direct product of n copies of the cusp.

The aim of the paper is to study algebraic properties of the algebra $\mathcal{D}(A(m))$ of differential operators of the generalized multi-cusp algebra $A(m)$ of rank n. We are mostly interested in the case when $m = (m_1, \dots, m_n) \in \mathbb{N}^n_{\geq 2}$ since for an arbitrary m the algebra $A(m)$ is isomorphic to the tensor product $P_s \otimes A(m')$ where $m' \in \mathbb{N}^{n-s}_{\geq 2}$ and $\mathcal{D}(A(m)) \simeq A_s \otimes \mathcal{D}(A(m'))$.

Generators and defining relations for the algebra $\mathcal{D}(A(m))$. Theorem 1.1 gives an explicit finite sets of generators and defining relations of the algebra $\mathcal{D}(A(m))$.

Theorem 1.1. *Let* $m = (m_1, \dots, m_n) \in \mathbb{N}^n$. *Then*

1. $\mathcal{D}(A(m)) \simeq \bigoplus_{i=1}^{n} \mathcal{D}(A(m_i))$.

2. *For each* $i = 1, \dots, n$, *let* $\mathcal{G}_i$ *and* $\mathcal{R}_i$ *be the set of generators and defining relations of the algebra* $\mathcal{D}(A(m_i))$ *as in Theorem 2.2.(4). Then the algebra* $\mathcal{D}(A(m))$ *is generated by the finite set of elements* $\mathcal{G} = \bigcup_{i=1}^{n} \mathcal{G}_i$ *that satisfy the defining relations* $\mathcal{R}_1, \dots, \mathcal{R}_n$ *and* $g_i g_j = g_j g_i$ *for all* $g_i \in \mathcal{G}_i$ *and* $g_j \in \mathcal{G}_j$ *for all* $i \neq j$.

A K-algebra R is called *central* if its centre $Z(R)$ is equal to the field K. Theorem 1.2 is about general properties of the algebra $\mathcal{D}(A(m))$.

Theorem 1.2. *Let* $m = (m_1, \dots, m_n) \in \mathbb{N}^n$. *Then the algebra* $\mathcal{D}(A(m))$ *is a central, simple,* $\mathbb{Z}^n$*-graded, finitely generated Noetherian domain of Gelfand-Kirillov dimension* $2n$.

An analogue of the Inequality of Bernstein for the algebras $\mathcal{D}(A(m))$**.** The starting point of the $\mathcal{D}$-module theory is the *Inequality of Bernstein*: *For all nonzero finitely generated* A_n*-modules* M, $\mathrm{GK}\,(M) \geq n$.

Theorem 1.3. *Let* $m = (m_1, \dots, m_n) \in \mathbb{N}^n$. *For all nonzero finitely generated* $\mathcal{D}(A(m))$*-modules* M, $\mathrm{GK}\,(M) \geq n$.

The Krull and global dimensions of the algebra $\mathcal{D}(A(m))$**.** The Krull dimension of the Weyl algebra A_n is n, [21].

Theorem 1.4. *Let* $m = (m_1, \dots, m_n) \in \mathbb{N}^n$. *The Krull dimension of the algebra* $\mathcal{D}(A(m))$ *is* n.

The global dimension of the Weyl algebra A_n is n, [21].

Theorem 1.5. *Let* $m = (m_1, \dots, m_n) \in \mathbb{N}^n$. *The global dimension of the algebra* $\mathcal{D}(A(m))$ *is* n.

Classification of simple $\mathcal{D}(A)$**-modules where** $A = A(m) = K + \sum_{i \geq m} Kx^i$**.**

The set $\widehat{\mathcal{D}(A)}$ of isomorphism classes of simple $\mathcal{D}(A)$-modules is a disjoint union of two subsets: the set of D-torsion and the set of D-torsion free simple $\mathcal{D}(A)$-modules where $D = K[h]$ and $h = \partial x$. The sets of simple D-torsion and D-torsion free $\mathcal{D}(A)$-modules are classified in Theorem 3.8 and Theorem 3.12, respectively.

2. Generators and defining relations of the algebra $\mathcal{D}(A)$

The aim of this section is to find generators and defining relatyions of the algebra $\mathcal{D}(A)$ of differential operators on the algebra $A = A(m)$ (Theorem 2.1). It is proven that the algebra $\mathcal{D}(A)$ is a central simple Noetherian domain of Gelfand-Kirillov dimension 2 (Theorem 2.2.(1)). The Krull dimension of the algebra $\mathcal{D}(A)$ is 1 (Theorem 2.10). Furthermore, for all nonzero left ideals I of the algebra $\mathcal{D}(A)$, the $\mathcal{D}(A)$-module $\mathcal{D}(A)/I$ has finite length (Theorem 2.9). We introduce two generalized Weyl algebras $\mathbb{A}$ and $\mathbb{B}$ such that $\mathbb{A} \subset \mathcal{D}(A) \subset \mathbb{B} = T^{-1}\mathbb{A} \simeq T^{-1}\mathcal{D}(A)$. The properties of the algebra $\mathcal{D}(A)$ is a mixture of properties of the algebras $\mathbb{A}$ and $\mathbb{B}$.

Generalized Weyl algebras $D(\sigma, a)$ of rank 1, [2–9]. Let D be a ring, σ be a ring automorphism of D, a is a *central* element of D. The **generalized Weyl algebra** of rank 1 (GWA, for short) $D(\sigma, a) = D[X, Y; \sigma, a]$ is a ring generated by the ring D and two elements X and Y that are subject to the defining relations:

$$\begin{gathered} Xd = \sigma(d)X \text{ and } Yd = \sigma^{-1}(d)Y \text{ for all } d \in D, \\ YX = a \text{ and } XY = \sigma(a). \end{gathered} \tag{1}$$

The ring D is called the *base ring* of the GWA. The automorphism σ and the element a are called the *defining automorphism* and the *defining element* of the GWA, respectively.

The algebra $A = \bigoplus_{n\in\mathbb{Z}} A_n$ is $\mathbb{Z}$-graded where $A_n = Dv_n$, $v_n = X^n$ and $v_{-n} = Y^n$ for $n < 0$, and $v_0 = 1$. It follows from the above relations that $v_n v_m = (n, m)v_{n+m} = v_{n+m}\langle n, m\rangle$ for some $(n, m) \in D$. If $n > 0$ and $m > 0$ then

$$\begin{aligned} n \geq m: \qquad & (n, -m) = \sigma^n(a) \cdots \sigma^{n-m+1}(a), \\ & (-n, m) = \sigma^{-n+1}(a) \cdots \sigma^{-n+m}(a), \\ n \leq m: \qquad & (n, -m) = \sigma^n(a) \cdots \sigma(a), \\ & (-n, m) = \sigma^{-n+1}(a) \cdots a, \end{aligned}$$

in other cases $(n, m) = 1$. Clearly, $\langle n, m\rangle = \sigma^{-n-m}((n, m))$.

Example. The Weyl algebra $A_1 = K[h][x, \partial; \sigma, a = h]$ is a GWA where $h = \partial x$ and $\sigma(h) = h - 1$.

Generators and defining relations of the algebra $\mathcal{D}(A)$. The set $S_x = \{x^i \mid i \geq 0\}$ (resp., $S_{x^m} = \{x^{im} \mid i \geq 0\}$) is a multiplicative set of P (resp., P and A). Clearly,

$$K[x, x^{-1}] = S_x^{-1} P = S_{x^m}^{-1} P = S_{x^m}^{-1} A. \tag{2}$$

The polynomial algebra P is a left A_1-module which is isomorphic to the factor module $A_1/A_1\partial$ where the action of A_1 is given by the rule: For all $p \in P$, $x * p = xp$ and $\partial * p = p := \frac{dp}{dx}$. The left A_1-module $P = \bigoplus_{i\geq 0} Kx^i$ is a $\mathbb{Z}$-graded (even $\mathbb{N}$-graded) A_1-module and $h * x^i = (i+1)x^i$ for all $i \geq 0$ where $h = \partial x$.

Theorem 2.1. *Let K be a field of characteristic zero, $A = K + \sum_{i\geq m} Kx^i$ $(m \geq 2)$ be a subalgebra of the polynomial algebra $P = K[x]$. Then*

1. *The ring of differential operators $\mathcal{D}(A)$ on A is a $\mathbb{Z}$-graded subalgebra $\mathcal{D}(A) = \bigoplus_{i\in\mathbb{Z}} \mathcal{D}(A)_{[i]}$ of the $\mathbb{Z}$-graded algebra $A_{1,x}$ where $\mathcal{D}(A)_{[i]} = D\delta_i$ and*

$$\delta_i = \begin{cases} x^i & \text{if } i \geq m, \\ (h-i-1)x^i & \text{if } i = 1, \dots, m-1, \text{ and} \\ 1 & \text{if } i = 0, \end{cases}$$

$$\delta_{-i} = \begin{cases} (h+i-1) \cdot \prod_{j=m-i+1}^{m} (h-j)x^{-i} & \text{if } i = 1, \dots, m-1, \\ (h+i-1) \cdot \prod_{1\neq j=m-i+1}^{m} (h-j)x^{-i} & \text{if } i \geq m. \end{cases}$$

In particular, $\delta_{-m} = (h+m-1)(h-2)\cdots(h-m)x^{-m}$, and for all $i \in \mathbb{Z}$, $\delta_i = \varphi_i x^i$ where the polynomial $\varphi_i \in D = K[h]$ is the coefficient of x^i in the equalities above.

2. *For all $i, j \geq m$, $\delta_{-i}\delta_{-j} = \delta_{-i-j}$ and $\delta_i\delta_j = \delta_{i+j}$.*

3. *$\mathcal{D}(A) = \bigoplus_{\substack{j\geq 0 \\ m\leq i\leq 2m-1}} D\delta_{-i}\delta_{-m}^j \oplus \bigoplus_{|i|<m} D\delta_i \oplus \bigoplus_{\substack{j\geq 0 \\ m\leq i\leq 2m-1}} D\delta_i\delta_m^j$, and $\delta_{-i}\delta_{-m}^j = \delta_{-m}^j\delta_{-i}$ and $\delta_i\delta_m^j = \delta_m^j\delta_i$ for all $j \geq 0$ and $m \leq i \leq 2m-1$.*

4. *The algebra $\mathcal{D}(A)$ is generated algebra by the elements $\{h, \delta_i \mid i = \pm 1, \pm 2, \dots, \pm(2m-1)\}$ that satisfy the finite set of defining relations: For all $i, j = \pm 1, \dots, \pm(2m-1)$, $[h, \delta_i] = i\delta_i$ and*

$$\delta_i\delta_j = \begin{cases} \varphi_i\sigma^i(\varphi_j)\varphi_{i+j}^{-1}\delta_{i+j} & \text{if } |i+j| < 2m, \\ \varphi_i\sigma^i(\varphi_j)\varphi_{i+j-m}^{-1}\delta_{i+j-m}\delta_m & \text{if } 2m \leq i+j < 3m, \\ \varphi_i\sigma^i(\varphi_j)\varphi_{i+j-2m}^{-1}\delta_{i+j-2m}\delta_m^2 & \text{if } 3m \leq i+j < 4m, \\ \varphi_i\sigma^i(\varphi_j)\varphi_{i+j+m}^{-1}\delta_{i+j+m}\delta_{-m} & \text{if } -3m < i+j \leq -2m, \\ \varphi_i\sigma^i(\varphi_j)\varphi_{i+j+2m}^{-1}\delta_{i+j+2m}\delta_{-m}^2 & \text{if } -4m < i+j \leq -3m. \end{cases}$$

Proof. 1. The set $S_x = \{x^i \mid i \geq 0\}$ (resp., $S_{x^m} = \{x^{mi} \mid i \geq 0\}$) is an Ore set of the Weyl algebra A_1 (resp., of A_1 and $\mathcal{D}(A)$) and

$$\begin{aligned} A_{1,x} := S_x^{-1}A_1 = S_{x^m}^{-1}A_1 = S_{x^m}^{-1}\mathcal{D}(P) \simeq \mathcal{D}(S_{x^m}^{-1}P) \\ \overset{(2)}{=} \mathcal{D}(S_{x^m}^{-1}A) \simeq S_{x^m}^{-1}\mathcal{D}(A). \end{aligned} \tag{3}$$

Recall that the Weyl algebra $A_1 = D[x, \partial; \sigma, a = h]$ is GWA when $D = K[h]$, $\sigma(h) = h - 1$ and $h := \partial x$. In particular, the Weyl algebra $A_1 = \bigoplus_{i \in \mathbb{Z}} Dv_i$ is a $\mathbb{Z}$-graded algebra where $v_0 := 1$, $v_i = x^i$ and $v_{-i} = \partial^i$ for $i \geq 1$.

Since the elements of the Ore set S_x are homogeneous elements of the algebra A_1, the localized algebra $A_{1,x} = S_x^{-1}A_1$ is also a $\mathbb{Z}$-graded algebra $A_{1,x} = \bigoplus_{i \in \mathbb{Z}} Dx^i$ (since $\partial = hx^{-1}$). By (3), $\mathcal{D}(A) = \{\delta \in A_{1,x} \mid \delta * A \subseteq A\}$.

Since the algebra A is a $\mathbb{Z}$-graded subalgebra of the polynomial algebra P, the algebra $\mathcal{D}(A)$ is also $\mathbb{Z}$-graded,

$$\begin{aligned} \mathcal{D}(A) = \bigoplus_{i \in \mathbb{Z}} \mathcal{D}(A)_{[i]} \text{ where } \mathcal{D}(A)_{[i]} = \mathcal{D}(A) \cap Dx^i \\ = \{\delta \in Dx^i \mid \delta * A \subseteq A\}. \end{aligned} \tag{4}$$

Now, using the fact that $h * x^i = (i+1)x^i$ for all $i \in \mathbb{Z}$, we obtain the explicit expressions for the graded components $\mathcal{D}(A)_{[i]}$ as in the theorem.

2. Statement 2 follows at once from the definition of the elements δ_{-i} and $\delta_i = x^i$ ($i \geq m$) and the fact that $x^{-j}h = (h+j)x^{-j}$ for all $j \geq 0$.

3. By statement 2, for all $i \geq 1$ and $j = 0, 1, \dots, m-1$, $\delta_{-im-j} = \delta_{-m}^{i}\delta_j$ and $x^{im+j} = (x^m)^i \cdot x^j$. Now, statement 3 follows from statement 1.

4. By statements 1 and 2, the relations in statement 4 hold. Then the relations of statement 4 are defining relations of the algebra $\mathcal{D}(A)$ since they imply the first equality in statement 3 where the direct sums are replaced by sums. □

The subalgebra $\mathcal{A}$ of $\mathcal{D}(A)$ which is a simple GWA. For an automorphism τ of an algebra R, $R^{\tau} := \{r \in R \,|\, \tau(r) = r\}$ is the *algebra of τ-constants/invariants*. The subalgebra $\mathcal{A}$ of $\mathcal{D}(A)$ (Proposition 2.2.(2)) plays a key role in proving that the algebra $\mathcal{D}(A)$ is a simple Noetherian domain (Proposition 2.2.(1)).

Theorem 2.2. *Let K be a field of characteristic zero, $A = K + \sum_{i \geq m} Kx^i$ ($m \geq 2$) be a subalgebra of the polynomial algebra $P = K[x]$. Then*

1. *The algebra $\mathcal{D}(A)$ is a central simple Noetherian domain.*
2. *The subalgebra $\mathcal{A}$ of $\mathcal{D}(A)$ which is generated by the elements h, $X := x^m$ and $Y := \delta_{-m}$ is a GWA $\mathcal{A} = D[X, Y; \sigma^m, a = (h+m-1) \cdot (h-2)(h-3)\cdots(h-m)]$ which is a central simple Noetherian domain where $\sigma(h) = h-1$.*
3. *The algebra $\mathcal{D}(A)$ is a finitely generated left and right $\mathcal{A}$-module, $\mathcal{D}(A) = \sum_{|i|<2m} \mathcal{A}\delta_i = \sum_{|i|<2m} \delta_i \mathcal{A}$.*

Proof. 2. The elements h, $X = x^m$ and $Y = \delta_{-m}$ satisfy the defining relations for the GWA $D[X, Y; \sigma^m, a]$. Then using the fact that the algebra $\mathcal{A}$ is a homogeneous subalgebra of the $\mathbb{Z}$-graded algebra $\mathcal{D}(A)$, we see that $\mathcal{A} = \bigoplus_{i \geq 1} DY^i \oplus D \oplus \bigoplus_{i \geq 1} DX^i$, and so $\mathcal{A} = D[X, Y, \sigma^m, a]$ since $a = YX = (h+m-1) \cdot (h-2)(h-3)\cdots(h-m)x^{-m}x^m = (h+m-1) \cdot (h-2)(h-3)\cdots(h-m)$, by Theorem 2.1.(1). By [5, Corollary 3.2], the GWA $\mathcal{A}$ is a simple algebra since the difference of any two distinct roots of the polynomial $a = (h+m-1) \cdot (h-2)(h-3)\cdots(h-m)$ is not divisible by m. By [5, Proposition 1.3], the GWA $\mathcal{A}$ is a Noetherian domain. Clearly, $Z(\mathcal{A}) = D^{\sigma} = K$ since $\sigma(h) = h-1$ and the field K has characteristic zero.

3. By using the definition, the algebra $\mathcal{A}$ is generated by the elements h, x^m and δ_{-m}. Now, statement 3 follows from Theorem 2.1.(2,3).

1. (i) *The algebra $\mathcal{D}(A)$ is a Noetherian domain*: Since the polynomial algebra $D = K[h]$ is a Noetherian algebra, the GWA $\mathcal{A}$ is also a Noetherian domain [5, Proposition 1.3]. The algebra $\mathcal{D}(A)$ is a finitely generated left and right $\mathcal{A}$-module. Hence, the algebra $\mathcal{D}(A)$ is a Noetherian left and right $\mathcal{A}$-module. Therefore, the algebra $\mathcal{D}(A)$ is a Noetherian algebra.

(ii) *The algebra $\mathcal{D}(A)$ is simple*: Let I be a nonzero ideal of the algebra $\mathcal{D}(A)$. Then $\mathfrak{a} := I \cap D \neq 0$ is a nonzero ideal of the algebra D since the algebra $\mathcal{D}(A)$ is a domain which is a direct sum $\mathcal{D}(A) = \bigoplus_{i\in\mathbb{Z}} D\delta_i$ of eigenspaces of the inner derivation ad_h of the algebra $\mathcal{D}(A)$ (Theorem 2.1.(1)). The subalgebra $\mathcal{A}$ of $\mathcal{D}(A)$ is a simple algebra (statement 2) that contains the algebra D. Hence, $0 \neq \mathfrak{a} \subseteq I \cap \mathcal{A}$ is a nonzero ideal of the algebra $\mathcal{A}$, i.e. $1 \in I \cap \mathcal{A} \subseteq I$, and so $I = \mathcal{D}(A)$. Therefore, the algebra $\mathcal{D}(A)$ is a simple algebra.

(iii) *The algebra $\mathcal{D}(A)$ is central*: By (3), the algebra $\mathcal{D}(A)$ is a central algebra:

$$K \subseteq Z(\mathcal{D}(A)) \subseteq Z(S_{x^m}^{-1}\mathcal{D}(A)) = Z(A_{1,x}) = K.$$

□

The set $S_{x^m} = \{x^{im} \mid i \geq 0\}$ is a denominator set of the algebras $\mathcal{A}$ and $A_1 = \mathcal{D}(P)$. The set $S_x = \{x^i \mid i \geq 0\}$ is a denominator set of the Weyl algebra $A_1 = \mathcal{D}(P)$. We have the following inclusions of algebras

$$A_1 \subset A_{1,x^m} = A_{1,x} = D[x, x^{-1}; \sigma], \quad D = K[h], \quad \sigma(h) = h - 1, \tag{5}$$

$$\mathcal{D}(A) \subset \mathcal{D}(A)_{x^m} \simeq A_{1,x} = D[x, x^{-1}; \sigma], \tag{6}$$

$$\mathcal{A} \subset \mathcal{A}_{x^m} = D[x^m, x^{-m}; \sigma^m] \subset A_{1,x^m} = A_{1,x} = D[x, x^{-1}; \sigma], \tag{7}$$

where the subscripts 'x^m' and 'x' denote the (left and right) localizations at the denominator sets S_{x_m} and S_x, respectively. The rings $D[x^{-1}, x; \sigma]$ and $D[x^m, x^{-m}; \sigma^m]$ are skew Laurent polynomial rings.

Recall that the Weyl algebra $\mathcal{D}(P) = A_1$ is the GWA, $A_1 = D[x, \partial; \sigma, h] = \bigoplus_{i\in\mathbb{Z}} Dv_i$, where $v_0 = 1$, $v_i = x^i$ and $v_{-i} = \partial^i$ for all $i \geq 1$. Since $\partial x = h$, we have that $x^{-1} = h^{-1}\partial$. Then, for all $i \geq 1$,

$$x^{-i} = \prod_{k=0}^{i-1} (h + k)^{-1} \partial^i. \tag{8}$$

Now, for $i = 1$,

$$\delta_{-1} = \frac{h(h - m)}{h} \partial = (h - m)\partial. \tag{9}$$

For $i = 2, \ldots, m-1$,

$$\delta_{-i} = \frac{(h+i-1) \prod\limits_{j=m-i+1}^{m} (h-j)}{\prod\limits_{k=0}^{i-1} (h+k)} \partial^i = \frac{\prod\limits_{j=m-i+1}^{m} (h-j)}{\prod\limits_{k=0}^{i-2} (h+k)} \partial^i. \tag{10}$$

For $i \geq m$,

$$\delta_{-i} = \frac{(h+i-1) \prod\limits_{1 \neq j=-i+m+1}^{m} (h-j)}{\prod\limits_{k=0}^{i-1} (h+k)} \partial^i = \frac{\prod\limits_{j=2}^{m} (h-j)}{\prod\limits_{k=i-m}^{i-2} (h+k)} \partial^i. \tag{11}$$

Corollary 2.3. *Let A be as in Theorem 2.1.*

1. $\mathcal{D}(A) \nsubseteq \mathcal{D}(P)$.
2. *Let δ_{-i}, $i \geq 1$, be as in Theorem 2.1. Then $\delta_{-1} = \mathcal{D}(P)$ and $\delta_{-i} \notin \mathcal{D}(P)$ for $i \geq 2$.*

Proof. 1. Statement 1 follows from statement 2.

2. Statement 2 follows from (9), (10) and (11). □

In Theorem 2.5, the algebra $\mathcal{D}(A) \cap \mathcal{D}(P)$ is described and an explicit set of algebra generators is given for it.

The subalgebra $\mathcal{D}(A)_+$ and $\mathcal{D}(A)_-$ of $\mathcal{D}(A)$. Let the algebra A be as in Theorem 2.1. The algebra $\mathcal{D}(A)$ contains two homogeneous subalgebras $\mathcal{D}(A)_+ := \bigoplus_{i \geq 0} D\delta_i$ and $\mathcal{D}(A)_- := \bigoplus_{i \geq 0} D\delta_{-i}$.

Proposition 2.4. *Let A be as in Theorem 2.1.*

1. *The algebras $\mathcal{D}(A)_{\pm}$ are finitely generated Noetherian algebras.*
2. $\mathcal{D}(A)_+ \subseteq \mathcal{D}(P)$ *but* $\mathcal{D}(A)_- \nsubseteq \mathcal{D}(P)$.
3. *The algebra $\mathcal{D}(A)_+$ is a finitely generated, left and right module over its subalgebra $D[x^m; \sigma^m]$ and the set $\{1, \delta_1, \ldots, \delta_{2m-1}\}$ is a module generating set.*
4. *The algebra $\mathcal{D}(A)_-$ is a finitely generated, left and right module over its subalgebra $D[\delta_{-m}; \sigma^{-m}]$ and the set $\{1, \delta_{-1}, \ldots, \delta_{-2m+1}\}$ is a module generating set.*

Proof. 2. The inclusion $\mathcal{D}(A)_+ \subseteq \mathcal{D}(P)$ is obvious. By Corollary 2.3.(2), $\mathcal{D}(A)_- \not\subseteq \mathcal{D}(P)$.

3. Statement 3 follows at once from the explicit expressions for the elements δ_i $(i \geq 0)$ and the fact that $\mathcal{D}(A)_+ = \bigoplus_{i\geq 0} D\delta_i$.

4. Statement 4 follows from Theorem 2.1.(2,3).

1. The skew polynomial rings $D[x^m; \sigma^m]$ and $D[\delta_{-m}; \sigma^{-m}]$ are Noetherian algebras (since D is so). Now, statement 1 follows at once from statement 4. □

The algebra $\mathcal{A}_1$. Recall that the algebras $\mathcal{D}(A)$ and $\mathcal{D}(P)$ are homogeneous subalgebras of the $\mathbb{Z}$-graded algebra $A_{1,x}$. So, the intersection $\mathcal{A}_1 := \mathcal{D}(A) \cap \mathcal{D}(P) = \mathcal{D}(A) \cap A_1$ is a homogeneous subalgebra of the algebras $\mathcal{D}(A)$, A_1 and $A_{1,x}$. Clearly, $\mathcal{A}_1 = \{\delta \in \mathcal{D}(P) \mid \delta * A \subseteq A\}$.

Theorem 2.5. *Let the algebra A be as in Theorem 2.1.*

1. $\mathcal{A}_1 = \bigoplus_{i\in\mathbb{Z}} Dw_i$ *where* $w_0 := 1$, $w_i = \delta_i$, $w_{-i} = a_i\partial^i$ *for* $i \geq 1$, *and*

$$a_i = \begin{cases} \prod_{j=m-i+1}^{m} (h-j) & \text{if } i = 1, \dots, m-2, \\ \prod_{j=2}^{m} (h-j) & \text{if } i \geq m-1. \end{cases}$$

2. *The algebra $\mathcal{A}_1$ is a finitely generated algebra and the set* $\{w_{-m}, w_{-1} = \delta_{-1}, h, \delta_1, \dots, \delta_{m-1}, x^m\}$ *is an algebra generating set, and* $w_{-1} = \delta_{-1}$.

3. (a) *For all* $i \geq m$, $w_{-i}w_{-1} = hw_{-i-1}$, $w_{-1}w_{-i} = (h-1)w_{-1-i}$ *and* $[w_{-i}, w_{-1}] = w_{-m-1}$.
   (b) *For* $i = 1, \dots, m-1$, $(w_{-1})^i = w_{-i}$.
   (c) $(w_{-1})^m = hw_{-m}$.
   (d) *For all* $i \geq 1$, $[\delta_1, x^i] = ix^{i+1}$.
   (e) $w_{-1}\delta_1 = h(h-1)(h-m)$ *and* $\delta_i w_{-1} = (h-1)(h-2)(h-m-1) = \sigma(h(h-1)(h-m))$.
   (f) *For* $i = 2, \dots, m-1$, $w_{-1}\delta_i = h(h-m)\delta_{i-1}$ *and* $\delta_i w_{-1} = (h-i-1)(h-i-m)\delta_{i-1}$.

Proof. 1. Notice that $\mathcal{D}(A)_+ \subseteq A_1$, and so $\mathcal{D}(A)_+ \subseteq \mathcal{A}_1$. Now statement 1 follows from the fact that the Weyl algebra $A_1 = D[x, \partial; \sigma, h] = \bigoplus_{i\geq 1} D\partial^i \oplus D \oplus \bigoplus_{i\geq 1} Dx^i$ is a GWA and from (9)–(11).

3. Straightforward.

2. By statement 1, the set $G = \{h, w_i \mid i \in \mathbb{Z}\backslash\{0\}\}$ is a generating set for the algebra $\mathcal{A}_1$. By the statements 3(a) and 3(b), the elements $\{w_i \mid i \leq -m-1\}$ and $\{w_{-j} \mid j = 2, \dots, m-1\}$ are redundant in G. Similarly, by the statement 3(d), the elements $\{w_i \mid i \geq m+1\}$ are also redundant in G, and statement 2 follows. □

The generalized Weyl algebras $\mathbb{A}$ and $\mathbb{B}$ such that $\mathbb{A} \subset \mathcal{D}(A) \subset \mathbb{B} \subset T^{-1}\mathbb{A} = T^{-1}\mathcal{D}(A) = T^{-1}\mathbb{B}$. Let $\mathbb{A}$ be the subalgebra of $\mathcal{D}(A)$ which is generated by the elements δ_{-1}, h and δ_1. By Theorem 2.1.(1), $\delta_{-1} = h(h-m)x^{-1}$ and $\delta_1 = (h-2)x$, and so the algebra

$$\mathbb{A} = D[\delta_1, \delta_{-1}; \sigma, h(h-1)(h-m)], \;\; D = K[h], \;\; \sigma(h) = h-1, \qquad (12)$$

is a GWA such that $\mathbb{A} \subset \mathcal{A}_1$ since $\delta_{-1}, h, \delta_1 \in \mathcal{A}_1$ (Theorem 2.5.(2)). In particular, the algebra $\mathbb{A} = \bigoplus_{i>0} D\delta_{-1}^i \oplus \bigoplus_{i\geq 0} D\delta_1^i$ is a free left/right D-module, where the set $\{\delta_{\pm 1}^i \mid i \geq 0\}$ is a free basis over D.

The multiplicative submonoid $T = \langle h - i \mid i \in \mathbb{Z}\rangle$ of D is a (left and right) denominator set of the algebras $\mathbb{A}$, $\mathcal{A}_1$, $\mathcal{D}(A)$ and A_1 such that

$$\begin{gathered} T^{-1}\mathbb{A} \simeq T^{-1}\mathcal{A}_1 \simeq T^{-1}\mathcal{D}(A) \simeq T^{-1}A_1 =: \mathbb{B} = T^{-1}D[x, x^{-1}; \sigma], \\ T^{-1}D = K[h, (h-i)^{-1}]_{i\in\mathbb{Z}}, \; \sigma(h) = h-1. \end{gathered} \qquad (13)$$

This follows from the explicit descriptions of the free bases over D of the algebras $\mathbb{A}$, $\mathcal{A}_1$, $\mathcal{D}(A)$ and $A_1 = \bigoplus_{i>0} D\partial^i \oplus \bigoplus_{i\geq 0} Dx^i$ (Theorem 2.1, Theorem 2.5). Notice that the algebra $\mathbb{B} = T^{-1}D[x, x^{-1}; \sigma, 1]$ is a GWA where the ring $T^{-1}D$ is a Dedekind ring.

Similarly, the multiplicative set $D\backslash\{0\}$ is a (left and right) denominator set of the algebras $\mathbb{A}$, $\mathcal{A}_1$, $\mathcal{D}(A)$ and A_1 such that

$$\begin{gathered} D^{-1}\mathbb{A} \simeq D^{-1}\mathcal{A}_1 \simeq D^{-1}\mathcal{D}(A) \simeq D^{-1}A_1 =: B = K(h)[x, x^{-1}; \sigma], \\ \sigma(h) = h-1, \end{gathered} \qquad (14)$$

where $D^{-1}\mathbb{A}$ denotes the localization $(D\backslash\{0\})^{-1}\mathbb{A}$ of the algebra $\mathbb{A}$ at $D\backslash\{0\}$, and $K(h)$ is the field of rational functions in the variable h over the field K. We have the following diagram of algebras where the vertical lines denote containments of the algebras:

$$\begin{array}{c}
B = D^{-1}\mathbb{A} = D^{-1}\mathcal{A}_1 = D^{-1}\mathcal{D}(A) = D^{-1}A_1 = K(h)[x, x^{-1}; \sigma] \\
| \\
\mathbb{B} = T^{-1}\mathbb{A} = T^{-1}\mathcal{A}_1 = T^{-1}\mathcal{D}(A) = T^{-1}A_1 = T^{-1}D[x, x^{-1}] \\
\diagup \qquad \diagdown \\
\mathcal{D}(A) \qquad\qquad A_1 \\
| \quad \diagdown \qquad \diagup \\
\mathcal{A} \qquad \mathcal{A}_1 = \mathcal{D}(A) \cap A_1 \\
| \\
\mathbb{A}
\end{array}$$

We will see that the properties of the algebra $\mathcal{D}(A)$ are a mixture of properties of the GWAs $\mathbb{A}$ and $\mathbb{B}$. Theorem 2.6 and Theorem 2.7 are about some properties of the algebras $\mathbb{A}$ and $\mathbb{B}$.

Theorem 2.6. *1. The algebra $\mathbb{A}$ is a finitely generated, central, non-simple Noetherian domain with* $\mathrm{GK}\,(\mathbb{A}) = 2$.

2. ([12, Theorem 1.6]) $\mathrm{gldim}(\mathbb{A}) = 2$.

3. ([4, Theorem 2]) *All nonzero left ideals of the algebra $\mathbb{A}$ are co-finite ($\dim_K(\mathbb{A}/I) < \infty$).*

4. ([4, Theorem 2]) $\mathrm{Kdim}(\mathbb{A}) = 1$.

5. ([4, Theorem 4]) *In $\mathbb{A}$ there are only finitely many nonzero ideals.*

6. ([4, Theorem 1]) *Up to isomorphism, there only tow simple finite dimensional $\mathbb{A}$-modules: $L_1 = \mathbb{A}/\mathbb{A}(\delta_{-1}, h-1, \delta_1)$, $\dim_K(L_1) = 1$ and $L_{m-1} = \mathbb{A}/\mathbb{A}(\delta_{-1}^{m-1}, h-m, \delta_1)$, $\dim_K(L_{m-1}) = m-1$.*

7. ([5, Theorem 3.3]) *The category of finite dimensional modules is not semisimple.*

8. ([5, Theorem 6]) *For all simple $\mathbb{A}$-modules M and N, the vector spaces $\mathrm{Ext}^i_{\mathbb{A}}(M, N)$ and $\mathrm{Tor}^{\mathbb{A}}_i(M, N)$ are finite dimensional for all i.*

9. ([5, Theorem 4]) *Let M be a simple $\mathbb{A}$-module and $q \in \mathbb{A}\backslash K$, then the kernel and cokernel of the linear map $q_M : M \to M$, $m \mapsto qm$ are finite dimensional.*

Theorem 2.7. *1. The algebra $\mathbb{B}$ is a finitely generated, central, simple Noetherian domain with* $\mathrm{GK}\,(\mathbb{B}) = 2$*.*

2. ([12, Theorem 1.6]) $\mathrm{gldim}(\mathbb{B}) = 1$*.*

3. ([4, Theorem 2]) *All nonzero left ideals of the algebra $\mathbb{B}$ are co-finite ($\dim_K(\mathbb{B}/I) < \infty$).*

4. ([4, Theorem 2]) $\mathrm{Kdim}(\mathbb{B}) = 1$*.*

5. ([4, Theorem 1, Theorem 5]) *All simple $\mathbb{B}$-modules are infinite dimensional.*

Every proper factor module of $\mathcal{D}(A)$ has finite length and the Krull dimension of $\mathcal{D}(A)$. Recall that the algebra $\mathcal{D}(A)$ is a finitely generated over its subalgebra $\mathcal{A}$. Proposition shows that the subalgebra $\mathcal{A}$ of $\mathcal{D}(A)$ is large in the sense that it meets every nonzero left ideal of the algebra $\mathcal{D}(A)$.

Proposition 2.8. *For all nonzero left ideals I of the algebra $\mathcal{D}(A)$, $\mathcal{A} \cap I \neq 0$.*

Proof. The Gelfand-Kirillov dimensions of the domains $\mathcal{D}(A)$ and $\mathcal{A}$ is 2. By Theorem 2.2, the algebra $\mathcal{D}(A)$ is a finitely generated module over its subalgebra $\mathcal{A}$. Hence, $2 = \mathrm{GK}\,(\mathcal{A}) \leq \mathrm{GK}_{\mathcal{A}}(\mathcal{D}(A)) \leq \mathrm{GK}\,(\mathcal{A}) = 2$, and so $\mathrm{GK}_{\mathcal{A}}(\mathcal{D}(A)) = 2$. Then, by [21, Proposition 8.3.5],

$$\mathrm{GK}_{\mathcal{A}}(\mathcal{D}(A)/I) < \mathrm{GK}_{\mathcal{A}}(\mathcal{D}(A)) - 1 = 2 - 1 = 1.$$

Hence, $\mathcal{A} \cap I \neq 0$ since $\mathrm{GK}\,(\mathcal{A}) = 2 > 1 = \mathrm{GK}_{\mathcal{A}}(\mathcal{D}(A)/I)$. □

Theorem 2.9. *For all nonzero left ideals I of the algebra $\mathcal{D}(A)$, the $\mathcal{D}(A)$-module $\mathcal{D}(A)/I$ has finite length.*

Proof. By [5, Theorem 2.1], for all nonzero left ideals I' of the algebra $\mathcal{A}$, the $\mathcal{A}$-module $\mathcal{A}/I'$ has finite length. By Theorem 2.2, the algebra $\mathcal{D}(A)$ is a finitely generated $\mathcal{A}$-module. Now, the theorem follows from Proposition 2.8. □

Theorem 2.10. *The Krull dimension of the algebra $\mathcal{D}(A)$ is 1.*

Proof. The theorem follows at once from Theorem 2.9. □

3. Classification of simple $\mathcal{D}(A)$-modules

The aim of this section is to classify simple $\mathcal{D}(A)$-modules where $A = K + \sum_{i\geq m} Kx^i$ (Theorem 3.8 and Theorem 3.12). They are partitioned in two (disjoint) sets: D-torsion and D-torsion free. The simple $\mathcal{D}(A)$-modules in each of the two sets are classified (Theorem 3.8 and Theorem 3.12).

At the beginning of the section we recall a classification of simple modules over a generalized Weyl $A = D(\sigma, a) = D[x, y; \sigma, a]$ where D is a (commutative) Dedekind domain with some extra condition on the automorphism that is satisfied for our GWAs. In all the papers we cite below these algebras are denoted by 'A', we hope that this notation will not lead to confusion.

For an algebra A, we denote by $\widehat{A}$ the set of isomorphism classes of simple A-modules. For an A-module M, we denote by $[M]$ its isomorphism class. If P is an isomorphism invariant property of simple modules (e.g., 'being weight') then $\widehat{A}\,(P)$ stands for the set of all isomorphism classes of simple A-modules that satisfy P.

Classification of simple A-modules where $A = D(\sigma, a)$ and D is a Dedekind ring. Let $A = D(\sigma, a) = D[x, y; \sigma, a]$ be a GWA such that D is a Dedekind ring, $a \neq 0$, and the automorphism σ of D satisfies the condition:

(*) $\sigma^i(\mathfrak{p}) \neq \mathfrak{p}$ *for all* $i \in \mathbb{Z} \setminus \{0\}$ *and all maximal ideals* $\mathfrak{p}$ *of* D.

Example. The Weyl algebra $A_1 = K[h][x, \partial; \sigma, h]$ is an example of the GWA A.

Example. $A = K[h](\sigma, a)$ where $\sigma(h) = h - 1$ and K is a field of characteristic zero. In particular, the algebras $\mathbb{A}$ is of this type, see (12). A classification of simple $K[h](\sigma, a)$-modules is given in [4, 5].

Example. The GWA $\mathbb{B}$ is an example of the GWA A, see (13).

The set $S := D\backslash\{0\}$ is an Ore set of the domain A. So, a simple A-module M is either *D-torsion* ($S^{-1}M = 0$) or *D-torsion free* ($S^{-1}M \neq 0$). In the second case, the $S^{-1}A$-module $S^{-1}M$ is simple.

Let us recall a classification of simple A-modules for the algebra $A = D(\sigma, a)$, see [4–6] for details. Clearly,

$$\widehat{A} = \widehat{A}\,(D\text{-torsion}) \coprod \widehat{A}\,(D\text{-torsion free}). \tag{15}$$

The set $\widehat{A}\,(D\text{-torsion}) = \widehat{A}\,(\text{weight})$. The group $\langle\sigma\rangle \simeq \mathbb{Z}$ acts freely on the set $\mathrm{Max}\,(D)$ of maximal ideals of the Dedekind ring D. For each maximal ideal $\mathfrak{p}$ of D, $\mathcal{O}(\mathfrak{p}) = \{\sigma^i(\mathfrak{p})\,|\,i \in \mathbb{Z}\}$ is its orbit. We use the bijection $\mathbb{Z} \to \mathcal{O}(\mathfrak{p})$, $i \mapsto \sigma^i(\mathfrak{p})$, to define the order $\leqslant$ on each orbit $\mathcal{O}(\mathfrak{p})$: $\sigma^i(\mathfrak{p}) \leqslant \sigma^j(\mathfrak{p})$ iff $i \leqslant j$. A maximal ideal of D is called *marked* if it contains the element a. There are only finitely many marked ideals. An orbit $\mathcal{O}$ is called *degenerated* if it contains a marked ideal. Marked ideals, say $\mathfrak{p}_1 < \cdots < \mathfrak{p}_s$, of a degenerated orbit $\mathcal{O}$ partition it into $s+1$ parts,

$$\Gamma_1 = (-\infty, \mathfrak{p}_1],\ \ \Gamma_2 = (\mathfrak{p}_1, \mathfrak{p}_2], \ldots, \Gamma_s = (\mathfrak{p}_{s-1}, \mathfrak{p}_s],\ \ \Gamma_{s+1} = (\mathfrak{p}_s, \infty). \quad (16)$$

Two ideals $\mathfrak{p}, \mathfrak{q} \in \mathrm{Max}\,(D)$ are called *equivalent* $\mathfrak{p} \sim \mathfrak{q}$ if they belong either to a non-degenerated orbit or to some Γ_i. We denote by $\mathrm{Max}\,(D)/\!\sim$ the set of equivalence classes in $\mathrm{Max}\,(D)$.

An A-module V is called *weight* if $V = \bigoplus_{\mathfrak{p}\in \mathrm{Max}\,(D)} V_\mathfrak{p}$ where $V_\mathfrak{p} = \{v \in V \,|\, \mathfrak{p}v = 0\}$ = the sum of all simple D-submodules of V which are isomorphic to $D/\mathfrak{p}$. The set $\mathrm{Supp}\,(V) = \{\mathfrak{p} \in \mathrm{Max}\,(D) \,|\, V_\mathfrak{p} \neq 0\}$ is called the *support* of V, elements of $\mathrm{Supp}\,(V)$ are called *weights* and $V_\mathfrak{p}$ is called the *component* of V of weight $\mathfrak{p}$. Clearly, an A-module is weight iff it is a semisimple D-module. Clearly,

$$\widehat{A}\,(D\text{-torsion}) = \widehat{A}\,(\text{weight}), \quad (17)$$

i.e., a simple A-module is D-torsion iff it is weight.

Theorem 3.1 ([4–6], Classification of simple D-torsion/weight A-modules)**.** *The map* $\mathrm{Max}\,(D)/\!\sim\ \to \widehat{A}\,(D\text{-torsion})$*,* $\Gamma \mapsto [L(\Gamma)]$*, is a bijection with the inverse* $[M] \mapsto \mathrm{Supp}\,(M)$ *where*

1. *If* Γ *is a non-degenerated orbit then* $L(\Gamma) = A/A\mathfrak{p}$ *where* $\mathfrak{p} \in \Gamma$*.*
2. *If* $\Gamma = (-\infty, \mathfrak{p}]$ *then* $L(\Gamma) = A/A(\mathfrak{p}, x)$*.*
3. *If* $\Gamma = (\sigma^{-n}(\mathfrak{p}), \mathfrak{p}]$ *for some* $n \geqslant 1$ *then* $L(\Gamma) = A/A(y^n, \mathfrak{p}, x)$*. The* D*-length of* $L(\Gamma)$ *is* n*.*
4. *If* $\Gamma = (\mathfrak{p}, \infty)$ *then* $L(\Gamma) = A/A(\sigma(\mathfrak{p}), y)$*.*

The set $\widehat{A}\,(D\text{-torsionfree})$. For elements $\alpha, \beta \in D$, we write $\alpha < \beta$ if $\mathfrak{p} < \mathfrak{q}$ for all $\mathfrak{p}, \mathfrak{q} \in \mathrm{Max}\,(D)$ such that $\mathcal{O}(\mathfrak{p}) = \mathcal{O}(\mathfrak{q})$, $\alpha \in \mathfrak{p}$ and $\beta \in \mathfrak{q}$. (We write also $\alpha < \beta$ if there are no such ideals $\mathfrak{p}$ and $\mathfrak{q}$). Recall that the GWA $A = \bigoplus_{i\in\mathbb{Z}} A_i$ is a $\mathbb{Z}$-graded algebra where $A_i = Dv_i = v_iD$, $v_0 = 1$, $v_i = x^i$ and $v_{-i} = y^i$ for all $i \geq 1$.

Definition, [4–6]. An element $b = v_{-m}\beta_{-m}+v_{-m+1}\beta_{-m+1}+\cdots+\beta_0 \in A$ (where $m \geqslant 1$, all $\beta_i \in D$ and $\beta_{-m}, \beta_0 \neq 0$) is called a *normal* element if $\beta_0 < \beta_{-m}$ and $\beta_0 < a$.

The set $S := D \setminus \{0\}$ is an Ore set of the domain A. Let $k := S^{-1}D$ be the field of fractions of D. The algebra $B := S^{-1}A = k[x, x^{-1}; \sigma]$ is a skew Laurent polynomial ring which is a (left and right) principle ideal domain. So, any simple B-module is of type B/Bb for some irreducible element b of B. Two simple B-modules are isomorphic, $B/Bb \simeq B/Bc$, iff the elements b and c are *similar* (i.e., there exists an element $d \in B$ such that 1 is the greatest common right divisor of c and d, and bd is a least common left multiple of c and d).

Theorem 3.2 ([4–6], CLASSIFICATION OF SIMPLE D-TORSIONFREE A-MODULES)**.** $\widehat{A}\,(D\text{-torsionfree}) = \{[M_b := A/A \cap Bb] \,|\, b$ *is a normal irreducible element of* $B\}$. *The* A*-modules* M_b *and* $M_{b'}$ *are isomorphic iff the elements* b *and* b' *are similar.*

For all nonzero elements $\alpha, \beta \in D$, the B-modules $S^{-1}M_b$ and $S^{-1}M_{\beta b\alpha^{-1}}$ are isomorphic. If an element $b = v_{-m}\beta_{-m} + \cdots + \beta_0$ is irreducible in B but not necessarily normal the next lemma shows that there are explicit elements α and β such that the element $\beta b\alpha^{-1}$ is normal and irreducible in B.

Lemma 3.3 ([4, Lemma 13], NORMALIZATION PROCEDURE)**.** *Given an element* $b = v_{-m}\beta_{-m} + \cdots + \beta_0 \in A$ *where* $m \geqslant 1$*, all* $\beta_i \in D$ *and* $\beta_{-m}, \beta_0 \neq 0$*. Fix a natural number* $s \in \mathbb{N}$ *such that* $\sigma^{-s}(\beta_0) < \beta_{-m}$*,* $\sigma^{-s}(\beta_0) < \beta_0$ *and* $\sigma^{-s}(\beta_0) < a$*. Let* $\alpha = \prod_{i=0}^{s} \sigma^{-i}(\beta_0)$ *and* $\beta = \prod_{i=1}^{s+m} \sigma^{-i}(\beta_0)$*. Then the element* $\beta b\alpha^{-1}$ *is a normal element which is called a normalization of* b *and denoted* b^{norm} *(we can always assume that* s *is the least possible).*

The algebra $A = K + \sum_{i\geq m} Kx^i$ is a simple weight $\mathcal{D}(A)$-module. Clearly, $A = \sum_{i\in E} Kx^i$ where $E := \{0, m, m+1, \ldots\}$. Recall that by the very definition of the algebra $\mathcal{D}(A)$ of differential operators on A, the algebra A is a left $\mathcal{D}(A)$-module and the action of an element $\delta \in \mathcal{D}(A)$ on an element $a \in A$ is denoted either by $\delta * (a)$ or $\delta(a)$. For all $i \in E$, $h * x^i = (i + 1)x^i$. This implies that the $\mathcal{D}(A)$-module A is a weight $\mathcal{D}(A)$-module with $\mathrm{Supp}(A) = \{(h - i - 1) \,|\, i \in E\}$. In particular, the

$\mathcal{D}(A)$-module A is D-torsion. If follows from the equalities

$$\begin{gathered}\delta_1 = (h-2)x,\ \delta_{-1} = h(h-m)x^{-1},\ \delta_{-1}\delta_1 = h(h-1)(h-m) \\ \text{and } \delta_1\delta_{-1} = (h-1)(h-2)(h-m-1)\end{gathered} \tag{18}$$

that the maps

$$\begin{aligned}\delta_1 &: Kx^i \to Kx^{i+1}, \quad p \mapsto \delta_1 * p, \quad i \geq m, \\ \delta_{-1} &: Kx^{i+1} \to Kx^i, \quad p \mapsto \delta_{-1} * p, \quad i \geq m,\end{aligned}$$

are bijections. Similarly, it follows from the equalities $\delta_{-m}\delta_m = (h+m-1)(h-2)\cdots(h-m)$ and $\delta_m\delta_{-m} = (h-1)(h-m-2)\cdots(h-2m)$ that the maps

$$\begin{aligned}\delta_m &: K \to Kx^m, \quad p \mapsto \delta_m * p, \\ \delta_{-m} &: Kx^m \to K, \quad p \mapsto \delta_{-m} * p,\end{aligned}$$

are bijections. Therefore, the algebra A is a simple weight $\mathcal{D}(A)$-module with $\mathrm{Supp}(A) = \{(h-i-1) \mid i \in E\}$.

Classification of simple weight $\mathcal{D}(A)$-modules with support that belongs to the orbit $\mathcal{O}(h)$. The ideal $(h) = Dh$ is a maximal ideal of the polynomial algebra $D = K[h]$ with $D/(h) = K$. Let $\mathcal{O}(h) = \mathcal{O}((h)) = \{\sigma^i(h) = (h-i) \mid i \in \mathbb{Z}\}$ be its σ-orbit. We will see that (up to isomorphism) there are only two simple weight $\mathcal{D}(A)$-modules with support in $\mathcal{O}(h)$: the algebra A and a 'complementary' module A' which we are going to define. Furthermore, $\mathrm{supp}(A') = \mathcal{O}(h) \backslash \mathrm{Supp}(A)$.

The polynomial algebra $K[x]$ has the canonical structure of the left A_1-module. Namely, $K[x] \simeq A_1/A_1\partial$; $x * p = xp$ and $\partial * p = \frac{dp}{dx}$ for all $p \in P$. The *Laurent polynomial algebra* $L = K[x, x^{-1}] = \bigoplus_{i\in\mathbb{Z}} Kx^i$, which is the localization of the polynomial algebra $K[x]$ at $S_x = \{x^i \mid i \geq 0\}$, is a left $A_{1,x}$-module. By (4), the Laurent polynomial algebra L is a left module over the algebras $A_{1,x} \simeq S_{x^m}^{-1}\mathcal{D}(A)$ and $\mathcal{D}(A)$. One can easily verifies using Theorem 2.1, that the subalgebra A is a $\mathcal{D}(A)$-submodule of L. Consider the $\mathcal{D}(A)$-module,

$$A' := L/A = \bigoplus_{i\in E'} Kx^i,\ \ E' := \mathbb{Z}\backslash E = \{\dots, -2, -1, 1, 2, \dots, m-1\}. \tag{19}$$

By (18), the maps

$$\begin{aligned}\delta_1 &: Kx^i \to Kx^{i+1}, \quad p \mapsto \delta_1 * p, \quad i \in E'\backslash\{-1, m-1\}, \\ \delta_{-1} &: Kx^{i+1} \to Kx^i, \quad p \mapsto \delta_{-1} * p, \quad i \in E'\backslash\{-1, m-1\},\end{aligned}$$

are bijections. Since $\delta_2 = (h-3)x^2$ and $\delta_{-2} = (h+1)(h-m+1)(h-m)x^{-2}$, we have that

$$\begin{aligned} \delta_{-2}\delta_2 &= (h+1)(h-m+1)(h-m)(h-1) \text{ and} \\ \delta_2\delta_{-2} &= (h-3)(h-1)(h-m-1)(h-m-2), \end{aligned} \tag{20}$$

and so the maps

$$\begin{aligned} \delta_2 &: Kx^{-1} \to Kx, \quad p \mapsto \delta_2 * p, \\ \delta_{-2} &: Kx \to Kx^{-1}, \quad p \mapsto \delta_{-2} * p, \end{aligned}$$

are bijections. Therefore, the $\mathcal{D}(A)$-module A' is a simple weight $\mathcal{D}(A)$-module with $\mathrm{Supp}(A') = \mathcal{O}(h)\backslash\mathrm{Supp}(A) = \{(h-i-1) \mid i \in E'\}$.

Lemma 3.4. *The $\mathcal{D}(A)$-modules A and A' are the only two (up to isomorphism) simple weight $\mathcal{D}(A)$-modules with support in the orbit $\mathcal{O}(h)$.*

Proof. Recall that the $\mathcal{D}(A)$-modules A and A' are non-isomorphic simple weight $\mathcal{D}(A)$-modules with support in the orbit $\mathcal{O}(h)$. Now, the lemma follows at once from the fact that every simple weight module is uniquely determined by its support and that $\mathcal{O}(h) = \mathrm{Supp}(A) \coprod \mathrm{Supp}(A')$. □

Let us collect properties of the $\mathcal{D}(A)$-modules A and A' in the next two lemmas.

Lemma 3.5. *1. The algebra $A = \bigoplus_{i\in E} Kx^i$ is a simple weight S_{x^m}-torsion $\mathcal{D}(A)$-module with* $\mathrm{Supp}(A) = \{(h-i-1) \mid i \in E\}$ *where $E = \{0, m, m+1, \dots\}$, and* $\mathrm{End}_{\mathcal{D}(A)}(A) = K$.

2. ${}_{\mathcal{D}(A)}A \simeq \mathcal{D}(A)/\mathcal{D}(A)(h-1, \delta_{-1}) = \bigoplus_{i\in E} K\delta_i\overline{1}$ *where* $\overline{1} := 1 + \mathcal{D}(A)(h-1, \delta_{-1})$.

Proof. 1. The weight spaces of the weight $\mathcal{D}(A)$-module A are 1-dimensional, hence $\mathrm{End}_{\mathcal{D}(A)}(A) = K$. The rest of statement 1 have been proven above.

2. The $\mathcal{D}(A)$-module $W = \mathcal{D}(A)/\mathcal{D}(A)(h-1) \simeq \bigoplus_{i\in\mathbb{Z}} K\delta_i 1^*$ is weight with $\mathrm{Supp}(W) = \mathbb{Z}$ where $1^* = 1 + \mathcal{D}(A)(h-1)$. The map $W \to A$, $1^* \mapsto \overline{1}$ is a $\mathcal{D}(A)$-module epimorphism. Hence, ${}_{\mathcal{D}(A)}A \simeq \mathcal{D}(A)/\mathcal{D}(A)(h-1, \delta_{-1})$, by Lemma 3.4. □

Lemma 3.6. *1. The algebra* $A' = \bigoplus_{i\in E'} Kx^i$ *is a simple weight* S_{x^m}*-torsion* $\mathcal{D}(A)$*-module with* $\mathrm{Supp}(A') = \mathcal{O}(h)\backslash\mathrm{Supp}(A) = \{(h-i-1)\,|\, i\in E'\}$ *where* $E' = \mathbb{Z}\backslash E = \{\dots,-2,-1,1,2,\dots,m-1\}$, *and* $\mathrm{End}_{\mathcal{D}(A)}(A') = K$.

2. ${}_{\mathcal{D}(A)}A' \simeq \mathcal{D}(A)/\mathcal{D}(A)(h,\delta_1) = \bigoplus_{i\in E'} K\delta_i\overline{1}'$ *where* $\overline{1}' := 1+\mathcal{D}(A)(h,\delta_1)$.

Proof. 1. The weight spaces of the weight $\mathcal{D}(A)$-module A' are 1-dimensional, hence $\mathrm{End}_{\mathcal{D}(A)}(A') = K$. The rest of statement 1 have been proven above.

2. The $\mathcal{D}(A)$-module $W' = \mathcal{D}(A)/\mathcal{D}(A)h \simeq \bigoplus_{i\in\mathbb{Z}} K\delta_i 1^o$ is weight with $\mathrm{Supp}(W) = \mathbb{Z}$ where $1^o = 1+\mathcal{D}(A)h$. The map $W'\to A'$, $1^o\mapsto \overline{1}'$ is a $\mathcal{D}(A)$-module epimorphism. Hence, ${}_{\mathcal{D}(A)}A' \simeq \mathcal{D}(A)/\mathcal{D}(A)(h,\delta_1)$, by Lemma 3.4. □

Classification of simple D-torsion $\mathcal{D}(A)$-modules. Recall that $\mathbb{A}\subset\mathcal{D}(A)\subset\mathbb{B} = T^{-1}\mathbb{A} = T^{-1}\mathcal{D}(A)$. So, every $\mathbb{B}$-module is automatically is an $\mathbb{A}$-module and $\mathcal{D}(A)$-module. The group $\langle\sigma\rangle$ acts on the set $\mathrm{Max}(D)$ of maximal ideal of the algebra $D = K[h]$. The field K has characteristic zero and $\sigma(h) = h-1$. So, every orbit $\mathcal{O}(\mathfrak{p}) = \{\sigma^i(\mathfrak{p})\,|\, i\in\mathbb{Z}\}$ contains infinite number of elements where $\mathfrak{p}\in\mathrm{Max}(D)$. We denote by $\mathrm{Max}(D)/\langle\sigma\rangle$ is the set of all σ-orbits in $\mathrm{Max}(D)$.

The algebra $\mathbb{B} = T^{-1}D[x,x^{-1};\sigma,1]$ is a GWA where $T^{-1}D$ is a Dedekind ring and the automorphism σ satisfies the condition (*) above. Notice that $\mathrm{Max}(T^{-1}D) = \{T^{-1}\mathfrak{p}\,|\,\mathfrak{p}\in\mathrm{Max}(D)\backslash\mathcal{O}(h)\}$ where $\mathcal{O}(h)$ is the σ-orbit of the maximal ideal (h) of the algebra D, and the map $\mathrm{Max}(D)\backslash\mathcal{O}(h)\to\mathrm{Max}(T^{-1}D)$, $\mathfrak{p}\mapsto T^{-1}\mathfrak{p}$ is a bijection.

For each orbit $\mathcal{O}\in\mathrm{Max}(D)/\langle\sigma\rangle\backslash\{\mathcal{O}(h)\}$, we fix its element, say $\mathfrak{p}_{\mathcal{O}}$. So, $\mathcal{O}(\mathfrak{p}_{\mathcal{O}}) = \mathcal{O}$.

Proposition 3.7. *1.* $\widehat{\mathbb{B}}(T^{-1}D\text{-torsion}) = \{\mathbb{B}/\mathbb{B}\mathfrak{p}_{\mathcal{O}}\,|\,\mathfrak{p}_{\mathcal{O}}\in\mathrm{Max}(D)/\langle\sigma\rangle\backslash\{\mathcal{O}(h)\}\}$.

2. The restriction map $\widehat{\mathbb{B}}(T^{-1}D\text{-torsion})\to\widehat{\mathcal{D}(A)}(D\text{-torsion})$, $M\to {}_{\mathcal{D}(A)}M$ *is an injection.*

Proof. 1. Statement 1 follows at once from Theorem 3.1 and the fact that the defining element of the GWA $\mathbb{B}$ is 1, and so every orbit of the automorphism σ in $\mathrm{Max}(T^{-1}D)$ is not degenerated.

2. Given $[M]\in\widehat{\mathbb{B}}(T^{-1}D\text{-torsion})$. By statement 1,

$$M = \mathbb{B}/\mathbb{B}\mathfrak{p} = \bigoplus_{i\in\mathbb{Z}} x^{-i}T^{-1}D/T^{-1}D\mathfrak{p} \simeq \bigoplus_{i\in\mathbb{Z}} x^{-i}D/D\mathfrak{p}$$

is a direct sum of non-isomorphic simple D-modules for some $\mathfrak{p} = \mathfrak{p}_{\mathcal{O}} \in \mathrm{Max}(D)\backslash\mathcal{O}(h)$. By Theorem 3.1 in case of the GWA $\mathbb{A}$, the weight $\mathbb{A}$-module M is simple, hence the $\mathcal{D}(A)$-module M is simple since $\mathbb{A} \subset \mathcal{D}(A)$. □

In view of Proposition 3.7.(2), we can write $\widehat{\mathbb{B}}(T^{-1}D\text{-torsion}) \subseteq \widehat{\mathcal{D}(A)}$ (D-torsion).

Theorem 3.8 (CLASSIFICATION OF SIMPLE D-TORSION $\mathcal{D}(A)$-MODULES)**.**
1) $\widehat{\mathcal{D}(A)}(D\text{-torsion}) = \{A, A'\} \coprod \widehat{\mathbb{B}}(T^{-1}D - \text{torsion})$.
2) $\widehat{\mathcal{D}(A)}(D\text{-torsion}) = \{A, A'\} \coprod \{\mathbb{B}/\mathbb{B}\mathfrak{p}_{\mathcal{O}} \mid \mathfrak{p}_{\mathcal{O}} \in \mathrm{Max}(D)\backslash\mathcal{O}(h))\}$, $\mathrm{Supp}(\mathbb{B}/\mathbb{B}\mathfrak{p}_{\mathcal{O}}) = \mathcal{O}$.
3) For all $[M] \in \widehat{\mathcal{D}(A)}(D\text{-torsion})$, $\mathrm{Supp}(M) = \infty$ and $\dim_K(M) = \infty$.

Proof. 1. Notice that $\mathrm{Max}(D) = \mathcal{O}(h) \coprod \mathrm{Max}(T^{-1}D)$ where the inclusion $\mathrm{Max}(T^{-1}D) \subset \mathrm{Max}(D)$ is due to the injection $\mathrm{Max}(T^{-1}D) \to \mathrm{Max}(D)$, $\mathfrak{m} \mapsto D\cap\mathfrak{m}$. Recall that every simple D-torsion $\mathcal{D}(A)$-module is a simple weight $\mathcal{D}(A)$-module, and vice versa, see (17). Now, statement 1 follows from Lemma 3.4 and Proposition 3.7.

2. Statement 2 follows from statement 1 and Proposition 3.7.

3. Statement 3 follows from statement 2. □

In order to describe the set of simple D-torsion free $\mathcal{D}(A)$-modules we need to know a classification of simple weight $\mathbb{A}$-modules (Theorem 3.9) and how simple weight $\mathcal{D}(A)$-modules with support from the orbit $\mathcal{O}(h)$ decompose under restriction to the subalgebra $\mathbb{A}$ of $\mathcal{D}(A)$ (Lemma 3.11).

The set $\widehat{\mathbb{A}}\,(D\text{-torsion}) = \widehat{\mathbb{A}}\,(\text{weight})$. Recall that the algebra $\mathbb{A}$ is a generalized Weyl algebra $\mathbb{A} = D[\delta_1, \delta_{-1}; \sigma, a = h(h-1)(h-m)]$ where $D = K[h]$ and $\sigma(h) = h - 1$. The orbit $\mathcal{O}(h)$ is the only degenerated orbit and the maximal ideals $(h) < (h-1) < (h-m)$ are the only marked maximal ideals. They partition the orbit $\mathcal{O}(h)$ into subsets (see (16)):

$$\Gamma_- = (-\infty, (h)], \quad \Gamma_1 = ((h), (h-1)], \quad \Gamma_{m-1} = ((h-1), (h-m)],$$
$$\Gamma_+ = ((h-m), \infty).$$

Theorem 3.9 (CLASSIFICATION OF SIMPLE D-TORSION/WEIGHT $\mathbb{A}$-MODULES)**.** *The map* $\mathrm{Max}\,(D)/\!\sim\; \to \widehat{\mathbb{A}}\,(D\text{-torsion})$, $\Gamma \mapsto [L(\Gamma)]$, *is a bijection with the inverse* $[M] \mapsto \mathrm{Supp}\,(M)$ *where*

1. *If* $\Gamma \in \mathrm{Max}(D)/\langle\sigma\rangle\backslash\{\mathcal{O}(h)\}$ *is a non-degenerated orbit then* $L(\Gamma) = \mathbb{A}/\mathbb{A}\mathfrak{p}$ *where* $\mathfrak{p} \in \Gamma$.

2. *If* $\Gamma = \Gamma_- = (-\infty, (h)]$ *then* $L_- := L(\Gamma_-) = \mathbb{A}/\mathbb{A}(h, \delta_1)$.

3. *If* $\Gamma = \Gamma_1, \Gamma_{m-1}$ *then* $L_1 := L(\Gamma_1) = \mathbb{A}/\mathbb{A}(\delta_{-1}, h-1, \delta_1)$ *and* $L_{m-1} := L(\Gamma_{m-1}) = \mathbb{A}/\mathbb{A}(\delta_{-1}^{m-1}, h-m, \delta_1)$. *These two modules are the only finite dimensional simple* $\mathbb{A}$*-modules;* $\dim_K L(\Gamma_1) = 1$ *and* $\dim_K L(\Gamma_{m-1}) = m-1$.

4. *If* $\Gamma = \Gamma_+$ *then* $L_+ := L(\Gamma_+) = \mathbb{A}/\mathbb{A}(h-m-1, \delta_{-1})$.

Proof. This is a particular case of Theorem 3.1. □

Recall that $\mathbb{A} \subset \mathcal{D}(A) \subset \mathbb{B}$. So every $\mathbb{B}$-module is also an $\mathbb{A}$-module and a $\mathcal{D}(A)$-module (by restriction). Corollary 3.10 shows that the algebras $\mathbb{A}$, $\mathcal{D}(A)$ and $\mathbb{B}$ have the same simple D-torsion modules provided their supports do not belong to the orbit $\mathcal{O}(h)$. For the algebras $R = \mathbb{A}$, $\mathcal{D}(A)$, $\mathbb{B}$, we denote by $\widehat{R}(D\text{-torsion}\,|\,\mathcal{O})$ the set of simple D-torsion R-modules with support disjoint from $\mathcal{O}(h)$.

Corollary 3.10. $\widehat{\mathbb{A}}(D\text{-torsion}\,|\,\mathcal{O}) = \widehat{\mathcal{D}(A)}(D\text{-torsion}\,|\,\mathcal{O}) = \widehat{\mathbb{B}}(D\text{-torsion}\,|\,\mathcal{O})$ $= \{\mathbb{B}/\mathbb{B}\mathfrak{p}_{\mathcal{O}}\,|\,\mathfrak{p}_{\mathcal{O}} \in \mathrm{Max}(D)\backslash\mathcal{O}(h)\}$ *and* $\mathrm{Supp}(\mathbb{B}/\mathbb{B}\mathfrak{p}_{\mathcal{O}}) = \mathcal{O}$.

Proof. The corollary follows from the classifications of simple D-torsion modules for the algebras $\mathbb{A}$, $\mathcal{D}(A)$ and $\mathbb{B}$ (Theorem 3.8 and Theorem 3.9). □

By Lemma 3.4, the $\mathcal{D}(A)$-modules A and A' are the only two (up to isomorphism) simple weight $\mathcal{D}(A)$-modules with support in the orbit $\mathcal{O}(h)$. Lemma 3.11 shows that these modules are semisimple $\mathbb{A}$-modules of length 2.

Lemma 3.11. 1. ${}_{\mathbb{A}}A = L_1 \oplus L_+$ *is a direct sum of simple weight* $\mathbb{A}$*-modules where the* $\mathbb{A}$*-modules* L_1 *and* L_+ *are defined in Theorem 3.9.(3,4).*

2. ${}_{\mathbb{A}}A' = L_- \oplus L_{m-1}$ *is a direct sum of simple weight* $\mathbb{A}$*-modules where the* $\mathbb{A}$*-modules* L_1 *and* L_+ *are defined in Theorem 3.9.(2,3).*

Proof. 1. Recall that ${}_{\mathcal{D}(A)}A = K + \sum_{i\geq m} Kx^i$. Then ${}_{\mathbb{A}}K \simeq L_1$ and ${}_{\mathbb{A}}(\sum_{i\geq m} Kx^i)$ $\simeq L_+$. Hence, ${}_{\mathbb{A}}A = L_1 \oplus L_+$ since $\delta_1 * K = 0$, $\delta_{-1} * K = 0$ and $\delta_{-1} * x^m = 0$.

2. Similarly, ${}_{\mathcal{D}(A)}A' = (\sum_{i\leq -1} Kx^i)\oplus(\sum_{1\leq i\leq m-1} Kx^i)$. Then ${}_{\mathbb{A}}(\sum_{i\leq -1} Kx^i) \simeq L_-$ and ${}_{\mathbb{A}}(\sum_{1\leq i\leq m-1} Kx^i) \simeq L_{m-1}$. Hence, ${}_{\mathbb{A}}A' = L_{m-1}\oplus L_-$ since $\delta_1 * x^{-1} = 0$, $\delta_{-1} * x = 0$ and $\delta_1 * x^{m-1} = 0$. □

Classification of simple D-torsion free $\mathcal{D}(A)$-modules. Recall that the algebra $\mathbb{A}$ is a GWA $\mathbb{A} = D[\delta_1, \delta_{-1}; \sigma, a = h(h-1)(h-m)]$. In order to stress that we consider 'normal' elements for the GWA $\mathbb{A}$ we say '$\mathbb{A}$-*normal*', see Theorem 3.12, i.e. an element $b = \delta_{-1}^m\beta_{-m} + \delta_{-1}^{m-1}\beta_{-m+1} + \cdots + \beta_0 \in \mathbb{A}$ (where $m \geqslant 1$, all $\beta_i \in D$ and $\beta_{-m}, \beta_0 \neq 0$) is called an $\mathbb{A}$-*normal* element if $\beta_0 < \beta_{-m}$ and $\beta_0 < a$.

Theorem 3.12 (CLASSIFICATION OF SIMPLE D-TORSION FREE $\mathcal{D}(A)$-MODULES)**.** $\widehat{\mathcal{D}(A)}$ (D-torsion free) $= \{[M_b := \mathcal{D}(A)/\mathcal{D}(A)\cap Bb] \mid b$ *is an* $\mathbb{A}$-*normal irreducible element of* $B\}$. *The* $\mathcal{D}(A)$-*modules* M_b *and* $M_{b'}$ *are isomorphic iff the elements* b *and* b' *are similar.*

Proof. Let $\mathcal{R}$ be the RHS of the equality in the theorem.

(i) $\mathcal{R} \subseteq \widehat{\mathcal{D}(A)}$ (D-torsion free): Given $M_b := [\mathcal{D}(A)/\mathcal{D}(A)\cap Bb] \in \mathcal{R}$ where b is an $\mathbb{A}$-normal irreducible element of B. We have to prove that $M_b \in \widehat{\mathcal{D}(A)}$ (D-torsion free). By the very definition the $\mathcal{D}(A)$-module M_b is D-torsion free (since $M_b \subseteq B/Bb$). By Theorem 2.9, the $\mathcal{D}(A)$-module M_b has finite length. It remains to show that the $\mathcal{D}(A)$-module M_b is simple. Suppose that this is not the case, i.e. the left ideal $\mathcal{D}(A)\cap Bb$ of the algebra $\mathcal{D}(A)$ is not a maximal left ideal, we seek a contradiction. Then there is an element $\alpha \in D\backslash K$ such that the left ideal $\mathcal{D}(A)\cap Bb$ is properly contained in the left ideal $D\alpha + \mathcal{D}(A)\cap Bb \neq \mathcal{D}(A)$. Hence, let W be a simple weight $\mathcal{D}(A)$-factor module of the $\mathcal{D}(A)$-module $\mathcal{D}(A)/(D\alpha + \mathcal{D}(A)\cap Bb)$. In particular the action of the element $b \in \mathbb{A} \subseteq \mathcal{D}(A)$ has nonzero kernel. By Corollary 3.10 and Lemma 3.11, the weight $\mathbb{A}$-module W is either simple or a direct sum of two simple weight $\mathbb{A}$-modules. Hence, the action of the element b is not injective on a simple $\mathbb{A}$-submodule of W, this contradicts to [5, Lemma 3.7] since the element b is $\mathbb{A}$-normal, a contradiction.

(ii) $\mathcal{R} \supseteq \widehat{\mathcal{D}(A)}$ (D-torsion free): Let $M \in \widehat{\mathcal{D}(A)}$ (D-torsion free). We have to show that $M \simeq M_b$ for some $\mathbb{A}$-normal irreducible element b of B. The B-module $D^{-1}M$ is simple. Hence, $M \simeq M_b$ for some irreducible element of the algebra B. Since $D^{-1}\mathbb{A} = B$ we may assume that $b = \delta_{-1}^m\beta_{-m} + \delta_{-1}^{m-1}\beta_{-m+1} + \cdots + \beta_0$ with all $\beta_i \in D$, $\beta_{-m} \neq 0$ and $\beta_0 \neq 0$.

By Lemma 3.3, we may assume that the element b is $\mathbb{A}$-normal since the B-modules B/Bb and $B/B\beta b\alpha = B/Bb\alpha$ are isomorphic (via the map $u \mapsto u\alpha$), and the statement (ii) follows. □

4. The algebras $\mathcal{D}(A(m))$ where $m \in \mathbb{N}^n$

In this section, properties of the algebras $\mathcal{D}(A(m))$ of differential operators are studied where $m \in \mathbb{N}^n$. Proofs of Theorems 1.2–1.4 are given. The key idea of the proofs is to use properties of the generalized Weyl algebras $\mathcal{A}(m)$ of rank n.

Generalized Weyl algebras of rank n, [2–9]. Let D be a ring, $\sigma = (\sigma_1, \ldots, \sigma_n)$ an n-tuple of commuting automorphisms of D, $a = (a_1, \ldots, a_n)$ an n-tuple of elements of the centre $Z(D)$ of D such that $\sigma_i(a_j) = a_j$ for all $i \neq j$. The **generalized Weyl algebra** $A = D(\sigma, a) = D[x, y; \sigma, a]$ of rank n is a ring generated by D and $2n$ indeterminates $x_1, \ldots, x_n, y_1, \ldots, y_n$ subject to the defining relations:

$$y_i x_i = a_i, \quad x_i y_i = \sigma_i(a_i), \quad x_i d = \sigma_i(d) x_i,$$
$$\text{and } \ y_i d = \sigma_i^{-1}(d) y_i \ \text{ for all } \ d \in D,$$
$$[x_i, x_j] = [x_i, y_j] = [y_i, y_j] = 0 \ \text{ for all } \ i \neq j,$$

where $[x, y] = xy - yx$. We say that a and σ are the sets of *defining* elements and automorphisms of the GWA A, respectively.

The GWA $A = \bigoplus_{\alpha \in \mathbb{Z}^n} A_\alpha$ is a $\mathbb{Z}^n$-graded algebra ($A_\alpha A_\beta \subseteq A_{\alpha+\beta}$ for all elements $\alpha, \beta \in \mathbb{Z}^n$) where $A_\alpha = Dv_\alpha = v_\alpha D$, $v_\alpha = v_{\alpha_1}(1) \otimes \cdots \otimes v_{\alpha_n}(n)$, $v_m(i) := x_i^m$ and $v_{-m}(i) := y_i^m$ for all $m \geq 1$, and $v_0(i) := 1$.

Example. Let $D_i[x_i, y_i; \sigma_i, a_i]$ be GWAs of rank 1 over a field K where $i = 1, \ldots, n$. Then their tensor product over the field K,

$$\bigotimes_{i=1}^n D_i[x_i, y_i; \sigma_i, a_i] = D[x, y; \sigma, a],$$

is a GWA of rank n where the $D = \bigotimes_{i=1}^n D_i$, $\sigma = (\sigma_1, \ldots, \sigma_n)$ and $a = (a_1, \ldots, a_n)$. The $\mathbb{Z}^n$-grading of the GWA $D[x, y; \sigma, a]$ of rank n is the tensor product of $\mathbb{Z}$-gradings of the tensor components/GWAs of rank 1.

Example. The n'th Weyl algebra $A_n = A_n(K)$ is a generalized Weyl algebra $A = D_n[x, y; \sigma; a]$ of rank n where $D_n = K[h_1, ..., h_n]$ is a polynomial algebra in n variables with coefficients in K, $\sigma = (\sigma_1, \dots, \sigma_n)$ where $\sigma_i(h_j) = h_j - \delta_{ij}$, δ_{ij} is the Kronecker delta function and $a = (h_1, \dots, h_n)$. The map

$$A_n \to A, \quad x_i \mapsto x_i, \quad \partial_i \mapsto y_i, \quad i = 1, \dots, n,$$

is an algebra isomorphism (notice that $\partial_i x_i \mapsto h_i$). In particular, the GWA $A_n = \bigoplus_{\alpha \in \mathbb{Z}^n} D_n v_\alpha$ is a $\mathbb{Z}^n$-graded algebra where $v_\alpha = v_{\alpha_1}(1) \otimes \cdots \otimes v_{\alpha_n}(n)$, $v_m(i) := x_i^m$ and $v_{-m}(i) := \partial_i^m$ for all $m \geq 1$, and $v_0(i) := 1$.

Generators and defining relations for the algebra $\mathcal{D}(A(m))$. Proof of Theorem 1.1. 1. The set $S_{n,x} := \{\prod_{i=1}^n x_i^{n_i} \mid n_i \geq 0\}$ (resp., $S_{n,x^m} := \{\prod_{i=1}^n x_i^{m_i n_i} \mid n_i \geq 0\}$) is a multiplicative set of the polynomial algebra $P_n = K[x_1, \dots, x_n]$ (resp., P_n and $A(m)$). Clearly,

$$K[x, x^{-1}] := K[x_1^{\pm 1}, \dots, x_n^{\pm n}] = S_{n,x}^{-1} P_n = S_{n,x^m}^{-1} P_n = S_{n,x^m}^{-1} A(m). \quad (21)$$

The set $S_{n,x}$ (resp., S_{n,x^m}) is an Ore set of the Weyl algebra A_n (resp., of A_n and $\mathcal{D}(A(m))$) and

$$\begin{aligned} A_{n,x} := S_{n,x}^{-1} A_n = S_{n,x^m}^{-1} A_n &= S_{n,x^m}^{-1} \mathcal{D}(P_n) \simeq \mathcal{D}(S_{n,x^m}^{-1} P_n) \\ &\overset{(21)}{=} \mathcal{D}(S_{n,x^m}^{-1} A(m)) \simeq S_{n,x^m}^{-1} \mathcal{D}(A(m)). \end{aligned} \quad (22)$$

Recall that the Weyl algebra $A_n = D_n[x, \partial; \sigma, a]$ is a GWA of rank n, see above. In particular, the Weyl algebra $A_n = \bigoplus_{\alpha \in \mathbb{Z}^n} D_n v_\alpha$ is a $\mathbb{Z}^n$-graded algebra.

Since the elements of the Ore set $S_{n,x}$ are homogeneous elements of the algebra A_n, the localized algebra $A_{n,x}$ is also a $\mathbb{Z}^n$-graded algebra

$$A_{n,x} = \bigoplus_{\alpha \in \mathbb{Z}^n} D_n x^\alpha = D_n[x_1^{\pm 1}, \dots, x_n^{\pm 1}; \sigma_1, \dots, \sigma_n] \quad (23)$$

which is a skew Laurent polynomial algebra where $D_n = K[h_1, \dots, h_n]$, $h_i = \partial_i x_i$ and $x^\alpha = \prod_{i=1}^n x_i^{\alpha_i}$. By (22), $\mathcal{D}(A(m)) = \{\delta \in A_{n,x} \mid \delta * A(m) \subseteq A(m)\}$.

Since the algebra $A(m)$ is a $\mathbb{Z}^n$-graded subalgebra of the polynomial algebra P_n, the algebra $\mathcal{D}(A(m))$ is also $\mathbb{Z}^n$-graded,

$$\begin{aligned}\mathcal{D}(A(m)) = \bigoplus_{\alpha\in\mathbb{Z}^n} \mathcal{D}(A(m))_{[\alpha]} \;\text{ where }\; \mathcal{D}(A(m))_{[\alpha]} \\ = \mathcal{D}(A(m)) \cap D_n x^\alpha = \{\delta \in D_n x^\alpha \mid \delta * A(m) \subseteq A(m)\}.\end{aligned} \tag{24}$$

Now, using the fact that $h_i * x^\alpha = (\alpha_i+1)x^\alpha$ and for all $i = 1,\dots,n$ and $\alpha \in \mathbb{Z}^n$, we obtain the explicit expressions for the graded components, $\mathcal{D}(A(m))_{[\alpha]} = \bigotimes_{i=1}^n \mathcal{D}(A(m_i))_{[\alpha_i]}$, i.e. $\mathcal{D}(A(m)) = \bigotimes_{i=1}^n \mathcal{D}(A(m_i))$.

2. Statement 2 follows from statement 1 and Theorem 2.1.(4). □

The algebra $\mathcal{D}(A(m))$ is a $\mathbb{Z}^n$-graded algebra. Recall that $\mathcal{D}(A(m)) = \bigotimes_{i=1}^n \mathcal{D}(A(m_i))$ (Theorem 1.1.(1)). If $m_i \geq 2$ then the algebra $\mathcal{D}(A(m_i)) = \bigoplus_{j\in\mathbb{Z}} K[h_i]\delta_j(i)$ is a $\mathbb{Z}$-graded algebra where the elements $\delta_j(i) = \delta_j$ are defined in Theorem 2.2.(1). If $m_i = 1$ then the algebra $\mathcal{D}(A(1))$ is the Weyl algebra $A_1 = \bigoplus_{j\in\mathbb{Z}} K[h_i]\delta_j(i)$ which is a $\mathbb{Z}$-graded algebra since it is a GWA where $\delta_j(i) = \delta_1(i)^j = x_i^j$ and $\delta_{-j}(i) = \delta_{-1}(i)^j = \partial_i^j$ for $j \geq 0$. Since $\mathcal{D}(A(m)) = \bigotimes_{i=1}^n \mathcal{D}(A(m_i))$ and every tensor component is a $\mathbb{Z}$-graded algebra the algebra $\mathcal{D}(A(m))$ is a $\mathbb{Z}^n$-graded algebra

$$\mathcal{D}(A(m)) = \bigotimes_{\alpha\in\mathbb{Z}^n} D_n\delta_\alpha, \quad D_n = K[h_1,\dots,h_n], \quad \delta_\alpha = \prod_{i=1}^n \delta_{\alpha_i}(i). \tag{25}$$

Notice that $D_n\delta_\alpha = \delta_\alpha D_n$ since $\delta_\alpha d = \sigma^\alpha(d)\delta_\alpha$ where $\sigma^\alpha = \prod_{i=1}^n \sigma_i$, $\sigma_i(h_j) = h_j - \delta_{ij}$. The $\mathbb{Z}^n$-grading on the algebra $\mathcal{D}(A(m))$ in (25) coincides with the induced $\mathbb{Z}^n$-grading that is determined by the embedding $\mathcal{D}(A(m)) \subseteq A_{n,x}$ and the $\mathbb{Z}^n$-grading of the algebra $A_{n,x}$ in (23).

The generalized Weyl algebras $\mathbb{A}_n$ and $\mathbb{B}_n$ such that $\mathbb{A}_n \subset \mathcal{D}(A(m)) \subset \mathbb{B}_n \subset T_n^{-1}\mathbb{A}_n = T_n^{-1}\mathcal{D}(A(m)) = T_n^{-1}\mathbb{B}_n$. Recall that $\mathcal{D}(A(m)) = \bigotimes_{i=1}^n \mathcal{D}(A(m_i))$. For each number $i = 1,\dots,n$, let $\mathbb{A}(i)$ be the subalgebra of $\mathcal{D}(A(m_i))$ which is generated by the elements $\delta_{-1}(i)$, h_i and $\delta_1(i)$. By Theorem 2.1.(1), $\delta_{-1}(i) = h_i(h_i - m)x_i^{-1}$ and $\delta_1(i) = (h_i-2)x_i$, and so the algebra

$$\begin{gathered}\mathbb{A}(i) = D(i)[\delta_1(i), \delta_{-1}(i);\ \sigma_i, h_i(h_i-1)(h_i-m_i)], \\ D(i) = K[h_i], \ \ \sigma_i(h_i) = h_i - 1,\end{gathered} \tag{26}$$

is a GWA such that $\mathbb{A}(i) \subset \mathcal{A}_1(i) := \mathcal{D}(A(m_i)) \cap A_1(i)$ where $A_1(i) = K\langle x_i, \partial_i \,|\, \partial_i x_i - x_i \partial_i = 1\rangle$ is the (first) Weyl algebra since $\delta_{-1}(i), h_i, \delta_1(i) \in \mathcal{A}_1(i)$ (Theorem 2.5.(2)). Let

$$\mathbb{A}_n := \bigotimes_{i=1}^n \mathbb{A}(i) \ \text{ and } \ \mathcal{A}_n := \bigotimes_{i=1}^n \mathcal{A}_1(i).$$

Then $\mathbb{A}_n \subseteq \mathcal{A}_n$.

The multiplicative submonoid $T(i) = \langle h_i - j \,|\, j \in \mathbb{Z}\rangle$ of $D(i)$ is a (left and right) denominator set of the algebras $\mathbb{A}(i)$, $\mathcal{A}_1(i)$, $\mathcal{D}(A(m_i))$ and $A_1(i)$ such that

$$\begin{aligned}T(i)^{-1}\mathbb{A}(i) &\simeq T(i)^{-1}\mathcal{A}_1(i) \simeq T(i)^{-1}\mathcal{D}(A(m_i)) \\ &\simeq T(i)^{-1}A_1(i) =: \mathbb{B}(i) = T(i)^{-1}D(i)[x_i, x_i^{-1}; \sigma_i]\end{aligned} \tag{27}$$

where $T(i)^{-1}D(i) = K[h_i, (h_i - j)^{-1}]_{j\in\mathbb{Z}}$ and $\sigma_i(h_i) = h_i - 1$. Let

$$\mathbb{B}_n := \bigotimes_{i=1}^n \mathbb{B}(i) \ \text{ and } \ \mathcal{A}(m) := \bigotimes_{i=1}^n \mathcal{A}(m_i)$$

where $\mathcal{A}(m_i)$ is a subalgebra of $\mathcal{D}(A(m_i))$ which is generated by the elements h_i, $X_i := x_i^{m_i}$ and $Y_i := \delta_{-m_i}(i)$. The algebra $\mathcal{A}(m_i)$ is a GWA of rank 1,

$$\mathcal{A}(m_i) = K[h_i][X_i, Y_i; \sigma_i^{m_i}, a_i = (h_i + m_i - 1)\cdot(h_i - 2)(h_i - 3)\cdots(h_i - m_i)],$$

which is a central simple Noetherian domain where $\sigma_i(h_i) = h_i - 1$, see Theorem 2.2.(2).

We have the following diagram of algebras where the vertical lines denote containments of the algebras where $T_n := T(1)\cdots T(n)$ is a denominator set of the corresponding algebras:

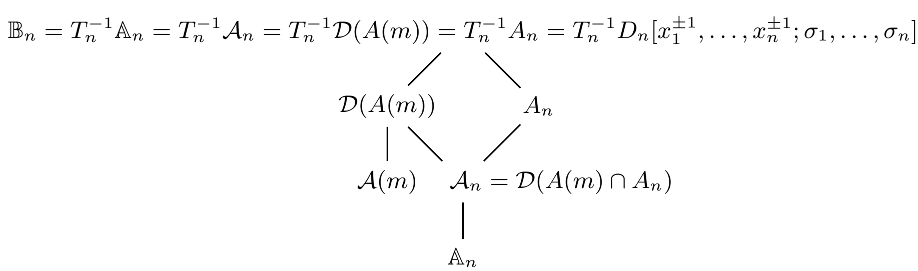

Figure 1

Proposition 4.1. *Let $m = (m_1, \dots, m_n) \in \mathbb{N}^n$. Then*

1. *The subalgebra $\mathcal{A}(m)$ of $\mathcal{D}(A(m))$ is a GWA of rank n which is a central simple Noetherian domain of Gelfand-Kirillov dimension $2n$.*

2. *The algebra $\mathcal{D}(A(m))$ is a finitely generated left and right $\mathcal{A}(m)$-module,*

$$\mathcal{D}(A(m)) = \sum_{\{\alpha\in\mathbb{Z}^n : |\alpha_1|<2m_1,\dots,|\alpha_n|<2m_n\}} \mathcal{A}(m)\delta_\alpha = \sum_{\{\alpha\in\mathbb{Z}^n : |\alpha_1|<2m_1,\dots,|\alpha_n|<2m_n\}} \delta_\alpha \mathcal{A}(m).$$

Proof. 2. Statement 2 follows from the fact that $\mathcal{D}(A(m)) = \bigotimes_{i=1}^{n} \mathcal{D}(A(m_i))$ and Theorem 2.2.(3).

1. By [5, Proposition 1.3], the GWA $\mathcal{A}(m)$ a Noetherian domain. By [10, Theorem 4.5], the GWA $\mathcal{A}(m)$ a simple algebra. The algebra $\mathcal{A}(m)$ is central since the algebra $A_{n,x}$ is so and

$$\mathcal{A}(m) \subset S^{-1}_{n,x^m}\mathcal{A}(m) \simeq D_n[x_1^{\pm m_i}, \dots, x_n^{\pm m_n}; \sigma_1^{m_1}, \dots, \sigma_n^{m_n}] \simeq A_{n,x} \ \ (x_i^{m_i} \mapsto x_i, \ \ h_i \mapsto m_i h_i).$$

The GWA $\mathcal{A}(m) = \bigotimes_{i=1}^{n} \mathcal{A}(m_i)$ is a tensor product of simple GWAs (see Theorem 2.2.(2)). By [10, Corollary 4.8.(2)], the Gelfand-Kirillov dimension of the algebra $\mathcal{A}(m)$ is $2n$. Now, by statement 2 and [21, Proposition 8.2.9.(ii)], $\mathrm{GK}\,(\mathcal{D}(A(m))) = \mathrm{GK}\,(\mathcal{A}(m)) = 2n$. □

Proof of Theorem 1.2.

(i) *The algebra $\mathcal{D}(A(m))$ is central*: $K \subseteq Z(\mathcal{D}(A(m))) \overset{(24)}{\subseteq} Z(A_{n,x}) = K$, and so the algebra $\mathcal{D}(A(m))$ is central.

(ii) *The algebra $\mathcal{D}(A(m))$ is Noetherian with Gelfand-Kirillov dimension $2n$*: By Proposition 4.1, the subalgebra $\mathcal{A}(m)$ of $\mathcal{D}(A(m))$ is a Noetherian algebra of Gelfand-Kirillov dimension $2n$ such that the algebra $\mathcal{D}(A(m))$ is a finitely generated left and right $\mathcal{A}(m)$-module. Hence, the algebra $\mathcal{D}(A(m))$ is also Noetherian and by [21, Proposition 8.2.9.(ii)], $\mathrm{GK}\,(\mathcal{D}(A(m))) = \mathrm{GK}\,(\mathcal{A}(m)) = 2n$.

(iii) *The algebra $\mathcal{D}(A(m))$ is simple and $\mathbb{Z}^n$-graded*: By (25), the algebra $\mathcal{D}(A(m))$ is a $\mathbb{Z}^n$-graded algebra and the $\mathbb{Z}^n$-graded components

$D_n\delta_\alpha$ ($\alpha \in \mathbb{Z}^n$) of the algebra $\mathcal{D}(A(m))$ are the common eigenspaces of the commuting inner derivations $\mathrm{ad}_{h_1}, \ldots, \mathrm{ad}_{h_n}$ of the algebra $\mathcal{D}(A(m))$. Therefore every nonzero ideal of the algebra $\mathcal{D}(A(m))$ is a homogeneous ideal and as a result has nontrivial intersection with the subalgebra D_n of $\mathcal{D}(A(m))$. Since $D_n \subseteq \mathcal{A}(m)$ and the algebra $\mathcal{A}(m)$ is simple (Proposition 4.1.(1)), all nonzero ideals of the algebra $\mathcal{D}(A(m))$ are equal to $\mathcal{D}(A(m))$, and so the algebra $\mathcal{D}(A(m))$ is a simple algebra. □

The Krull dimension of the algebras $\mathcal{D}(A(m))$. Proof of Theorem 1.4.

By [10, Corollary 4.8.(5)], the Krull dimension of the GWA $\mathcal{A}(m)$ is n. By Proposition 4.1, the algebra $\mathcal{D}(A(m))$ is a finitely generated left and right $\mathcal{A}(m)$-module. Hence, the Krull dimension of the algebra $\mathcal{D}(A(m))$ is smaller or equal to the Krull dimension of the algebra $\mathcal{A}(m)$ which is n. The polynomial algebra D_n is the zero graded component of the $\mathbb{Z}^n$-graded algebra $\mathcal{D}(A(m))$. Hence, them map $I \mapsto \mathcal{D}(A(m)) \otimes_{D_n} I$ (resp., $I \mapsto I \otimes_{D_n} \mathcal{D}(A(m)))$ from the set of ideals of the algebra D_n to the set of left (resp., right) ideals of the algebra $\mathcal{D}(A(m))$ is an injection. Hence, the Krull dimension of D_n, which is n, is smaller or equal to the Krull dimension of $\mathcal{D}(A(m))$. Therefore, the Krull dimension of the algebra $\mathcal{D}(A(m))$ is n. □

An analogue of the Inequality of Bernstein for the algebras $\mathcal{D}(A(m))$. By [10, Corollary 4.8.(4)], an analogue of the Inequality of Bernstein holds for the algebra $\mathcal{A}(m)$: *For all nonzero finitely generated $\mathcal{A}(m)$-modules M, $\mathrm{GK}\,(M) \geq n$.*

Proof of Theorem 1.3. By Proposition 4.1, the algebra $\mathcal{D}(A(m))$ is a finitely generated left and right $\mathcal{A}(m)$-module. Hence, each finitely generated $\mathcal{D}(A(m))$-module M is also a nonzero finitely generated $\mathcal{A}(m)$-module. Now,

$$\mathrm{GK}_{\,\mathcal{D}(A(m))}(M) \geq \mathrm{GK}_{\,\mathcal{A}(m)}(M) \geq n,$$

and Theorem 1.3 follows. □

The global dimension of the algebras $\mathcal{D}(A(m))$. Recall that Morita equivalent algebras have the same global dimension and the global dimension of the Weyl algebra A_n is n (in characteristic zero).

Proof of Theorem 1.5. By [22, Theorem, p. 29], in the case $n = 1$, the algebra $\mathcal{D}(A(m_1))$ is Morita equivalent to the Weyl algebra A_1. Hence, for an arbitrary $n \geq 1$, the algebra

$$\mathcal{D}(A(m)) = \mathcal{D}(\overset{n}{\underset{i=1}{\otimes}} A(m_i)) \simeq \overset{n}{\underset{i=1}{\otimes}} \mathcal{D}(A(m_i)) \ \text{(Theorem 1.2.(1))},$$

where $m \in \mathbb{N}^n$, is Morita equivalent to the Weyl algebra $A_n = A_1^{\otimes n}$. Therefore, the global dimension of the algebra $\mathcal{D}(A(m))$ is equal to the global dimension of the Weyl algebra A_n which is n. $\square$

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