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RESEARCH ARTICLE

p-Conjecture for tame automorphisms of $\mathbb{C}^3$

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Abstract. The famous Jung-van der Kulk [4, 11] theorem says that any polynomial automorphism of $\mathbb{C}^2$ can be decomposed into a finite number of affine automorphisms and triangular automorphisms, i.e. that any polynomial automorphism of $\mathbb{C}^2$ is a tame automorphism. In [5] there is a conjecture saying that for any tame automorphism of $\mathbb{C}^3$, if (p, d_2, d_3) is a multidegree of this automorphism, where p is a prime number and $p \le d_2 \le d_3$, then $p|d_2$ or $d_3 \in p\mathbb{N} + d_2\mathbb{N}$. Up to now this conjecture is unsolved. In this note, we study this conjecture and give some results that are partial results in the direction of solving the conjecture. We also give some complimentary results.

1. Introduction

Let $\mathbb{C}$ be a field of complex numbers (although someone can think about any field k of characteristic zero, since any given, in this note, results and used notions are correct, or correctly defined, also in this context). For any polynomial mapping $F = (F_1, \dots, F_m) : \mathbb{C}^n \to \mathbb{C}^m$ by multidegree of F we mean the following sequence of integers $\operatorname{mdeg} F = (\deg F_1, \dots, \deg F_m)$, where $\deg P$ for any polynomial $P \in \mathbb{C}[X_1, \dots, X_n]$ denotes the usual degree of the polynomial P.

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Let us mention that in the Scottish Book [16, Problem 79] Mazur and Orlicz posed the following question: "If $F = (F_1, \ldots, F_n) : \mathbb{C}^n \to \mathbb{C}^n$ is a one-to-one polynomial map whose inverse is also a polynomial map, is each F_i of degree one?" In other words, they asked whether every polynomial automorphism of $\mathbb{C}^n$ has multidegree $(1, \ldots, 1)$. The answer to this question is obviously "no" (for $n > 1$ and obviously "yes" for $n = 1$), and in the Scottish Book itself one can find the following example: let $1 \leq i \leq n$ and $a = a(X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n) \in \mathbb{C}[X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n]$ with $\deg a > 1$. Then

$$E : \mathbb{C}^n \ni (x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_{i-1}, x_i + a, x_{i+1}, \ldots, x_n) \in \mathbb{C}^n$$

is a polynomial automorphism with multidegree $(1, \ldots, 1, \deg a, 1, \ldots, 1)$. A map as above is called an *elementary polynomial map* or shortly an *elementary map.* Taking finite compositions of such elementary maps and elements of the affine subgroup $\mathrm{Aff}(\mathbb{C}^n)$, i.e. the group of polynomial automorphisms $F = (F_1, \ldots, F_n) : \mathbb{C}^n \to \mathbb{C}^n$ such that $\deg F_i = 1$ for all i (i.e. such that $\operatorname{mdeg} F = (1, \ldots, 1)$), we get automorphisms called *tame.* The other (equivalent) definition of a tame automorphism is as follows: an automorphism of $\mathbb{C}^n$ is called *tame* if it is a composition of finite number of affine automorphisms and triangular automorphisms, where *triangular* automorphisms are automorphisms of the form

$$T : \mathbb{C}^n \ni \left\{ \begin{array}{c} x_1 \\ x_2 \\ \vdots \\ x_n \end{array} \right\} \mapsto \left\{ \begin{array}{l} x_1 \\ x_2 + f_2(x_1) \\ \vdots \\ x_n + f_n(x_1, \ldots, x_{n-1}) \end{array} \right\} \in \mathbb{C}^n.$$

By the famous theorem of Jung [4] and Van der Kulk [11], it is known that if $n = m = 2$ and $F \in \mathrm{Aut}(\mathbb{C}^2)$, where $\mathrm{Aut}(\mathbb{C}^n)$ denotes the group of polynomial automorhisms of $\mathbb{C}^n$, then for $(d_1, d_2) = \operatorname{mdeg} F$ we have $d_1 | d_2$ or $d_2 | d_1$. The theorem of Jung and van der Kulk says even more, that is, that any polynomial automorphism of $\mathbb{C}^2$ is actually a tame automorphism, in other words $\mathrm{Aut}(\mathbb{C}^2) = \mathrm{Tame}(\mathbb{C}^2)$.

On the other hand, if d_1, d_2 are positive integers such that $d_1 | d_2$ then $F = \Phi_2 \circ \Phi_1$, where

$$\begin{aligned} \Phi_1 &: \mathbb{C}^2 \ni (x, y) \mapsto (x + y^{d_1}, y) \in \mathbb{C}^2, \\ \Phi_2 &: \mathbb{C}^2 \ni (u, w) \mapsto (u, w + u^{\frac{d_2}{d_1}}) \in \mathbb{C}^2, \end{aligned}$$

is an automorphism of $\mathbb{C}^2$ with $\operatorname{mdeg} F = (d_1, d_2)$. Similarly if $d_2|d_1$ we can give an appropriate automorphism of $\mathbb{C}^2$. Thus a sequence of positive integers (d_1, d_2) is the multidegree of some polynomial automorphism of $\mathbb{C}^2$ if and only if $d_1|d_2$ or $d_2|d_1$.

The other important consequence of the Jung-van der Kulk theorem is that in dimension two there is no wild automorphisms, where an automorphism $F \in \operatorname{Aut}(\mathbb{C}^n)$ is called *wild* if $F \notin \operatorname{Tame}(\mathbb{C}^n)$. One of the remarkable, and for the very long time unsolved, problem was whether there are wild automorphisms in dimension $n > 2$. The first known example of hypothetical wild automorphism was proposed by Nagata [17] in 1972. Now, this example is called Nagata automorphism:

$$\sigma : \mathbb{C}^3 \ni (x,y,z) \mapsto (x+2y(y^2+zx)-z(y^2+zx)^2, y-z(y^2+zx), z) \in \mathbb{C}^3.$$

It took more than thirty years to prove that the Nagata automorphism is indeed a wild automorphism. This remarkable result was obtained by Shestakov and Umirbaev [18] in 2004. It should be noted that the problem of existence of wild automorphisms in dimension $n > 3$ is still unsolved. In particular, the existence of wild automorphism in dimension $n = 3$ do not imply the existence of such automorphism in higher dimensions. To illustrate this phenomenon, let us notice that the Nagata example is stably tame automorphism. More precisely, the automorphism of $\mathbb{C}^4 = \mathbb{C}^3 \times \mathbb{C}$ obtained from Nagata example $\sigma : \mathbb{C}^3 \to \mathbb{C}^3$ as follows

$$\widetilde{\sigma} : \mathbb{C}^3 \times \mathbb{C} \ni ((x,y,z),w) \mapsto (N(x,y,z),w) \in \mathbb{C}^3 \times \mathbb{C}$$

is a tame automorphism of $\mathbb{C}^4$. To prove this result, Martha Smith used locally nilpotent derivations of the polynomial ring $\mathbb{C}[X,Y,Z,W]$ and the construction of the exponential map of such a derivations (for more details see [20]).

The multidegree of a polynomial automorphisms seems to be useful tool for recognition of wild automorphism. For example the second author of this note together with Jakub Zygadło proved [9], using the multidegree, that for any $s = 1, 2, \ldots$ the automorphism $N^s = N \circ \cdots \circ N$ (composition, s times, of N with itself) is a wild automorphism of $\mathbb{C}^3$, where N is the following slight modification of the Nagata example

$$N : \mathbb{C}^3 \ni (x,y,z) \mapsto (z, y-z(y^2+zx), x+2y(y^2+zx)-z(y^2+zx)^2) \in \mathbb{C}^3.$$

Since N is the composition of the Nagata automorphism with the following affine automorphism $\mathbb{C}^3 \ni (x,y,z) \mapsto (z,y,x) \in \mathbb{C}^3$, it follows, by the result of Shestakov and Umirbaev, that $N = N^1$ is a wild automorphism. The proof of the wildness of N^s for $s \geq 2$ goes as follows. One can calculate that $\operatorname{mdeg} N^s = (4s-3, 4s-1, 4s+1)$ and observe that $\gcd(4s-3, 4s-1) = 1$ and $4s+1 \notin (4s-3)\mathbb{N} + (4s-1)\mathbb{N}$. Thus the wildness of N^s for $s \geq 2$ is a consequense of the following result.

Theorem 1 ([9, Thm. 2.1])**.** *Let $d_3 \geq d_2 > d_1 \geq 3$ be positive integers. If d_1 and d_2 are odd numbers such that $\gcd(d_1, d_2) = 1$, then (d_1, d_2, d_3) is a multidegree of tame automorphism of $\mathbb{C}^3$ if and only if $d_3 \in d_1\mathbb{N} + d_2\mathbb{N}$, i.e. if and only if d_3 is a linear combination of d_1 and d_2 with coefficients in $\mathbb{N}$.*

Let us notice, also, that for the above mentioned slight modification N of the Nagata automorphism σ, we have that N is wild but $\operatorname{mdeg} N = (1,3,5) \in \operatorname{mdeg}\left(\operatorname{Tame}(\mathbb{C}^3)\right)$, where we here mdeg consider as a map $\operatorname{Aut}(\mathbb{C}^3) \ni F \mapsto \operatorname{mdeg} F \in \mathbb{N}^3$. Indeed, the mapping $F : \mathbb{C}^3 \ni (x,y,z) \mapsto (x, y+x^3, z+x^5) \in \mathbb{C}^3$ is a tame automorphism with $\operatorname{mdeg} F = (1,3,5)$. The existence of tame automorphism with multidegree $(1,3,5)$ can also be obtained from the following result.

Proposition 1 ([5, Prop. 2.2])**.** *If for a sequence of integers $1 \leq d_1 \leq \ldots \leq d_n$ there is $i \in \{1, \ldots, n\}$ such that*

$$d_i = \sum_{j=1}^{i-1} k_j d_j \qquad \text{with } k_j \in \mathbb{N},$$

then there exists a tame automorphism F of $\mathbb{C}^n$ with $\operatorname{mdeg} F = (d_1, \ldots, d_n)$.

And its consequence

Corollary 1 ([5, Cor. 2.3])**.** *If for a sequence of integers $1 \leq d_1 \leq \ldots \leq d_n$ we have $d_1 \leq n-1$, then there exists a tame automorphism F of $\mathbb{C}^n$ with $\operatorname{mdeg} F = (d_1, \ldots, d_n)$.*

Fortunately, for Nagata automorphism N we can use the weighted multidegrees to show that N is wild [1].

The first result about multidegrees of polynomial automorphisms of $\mathbb{C}^3$ or multidegrees of tame automorphisms of $\mathbb{C}^3$ was given in [5], and says that $(3,4,5) \notin \operatorname{mdeg}(\operatorname{Tame}(\mathbb{C}^3))$, in other words that there is no tame automorphism of $\mathbb{C}^3$ with multidegree $(3,4,5)$. For more information of multidegrees of polynomial automorphisms the reader is referred to [1–3, 6–8, 10, 14, 15, 21].

2. *p*-conjecture

We start this section with the following conjecture which was stated in [5] for $p \geq 3$ but of course it can be considered also for $p = 2$.

Conjecture 2 ([5, Conj. 5.1]). For any prime number $p \geq 2$ and $d_3 \geq d_2 \geq p$ the following is true $(p, d_2, d_3) \in \operatorname{mdeg}(\operatorname{Tame}(\mathbb{C}^3))$ if and only if $p|d_2$ or $d_3 \in p\mathbb{N} + d_2\mathbb{N}$.

Up to now, this conjecture is proved for $p = 2$ [5, Example 3.1], $p = 3$ [7, Thm. 1.1], $p = 5$ [8, Cor. 7.8] and for any other prime numbers p but with some additional restrictions for d_2 :

Theorem 3 ([8, Thm. 7.1]). *Let $2 \leq p \leq d_2 \leq d_3$ be integers, and let p be a prime. If*
(1) $\frac{d_3}{d_2} \neq \frac{3}{2}$ or
(2) $\frac{d_3}{d_2} = \frac{3}{2}$ and $\frac{d_2}{2} > p - 2$,
then $(p, d_2, d_3) \in \operatorname{mdeg}\left(\operatorname{Tame}\left(\mathbb{C}^3\right)\right)$ if and only if $p|d_2$ or $d_3 \in p\mathbb{N} + d_2\mathbb{N}$.

Theorem 4 ([8, Thm. 7.9]). *Let $p \geq 5$ be a prime such that $p \leq 35$. Then $(p, 2(p-2), 3(p-3)) \notin \operatorname{mdeg}\left(\operatorname{Tame}\left(\mathbb{C}^3\right)\right)$.*

For example, by the above theorems, we have that for $p = 7$ only unknown case is the triple $(7, 8, 12)$. Similarly, for $p = 11$, the only unknown cases are $(11, 12, 18)$, $(11, 14, 21)$ and $(11, 16, 24)$.

Besides of stating Conjecture 2 in [5] it was observed that one can not expect the similar result in the case d_1 is not a prime number. Namely, it was shown that

Proposition 2 ([5, Prop. 5.2]). *For any number $d_3 \geq 6$, we have $(4, 6, d_3) \in \operatorname{mdeg}(\operatorname{Tame}(\mathbb{C}^3))$.*

In the next section we give some generalization of the above proposition (see Theorems 10 and 11).

Up to the end of this section, we will prove that there is 'only' one step in order to prove that there is no tame automorphism F of $\mathbb{C}^3$ with multidegree equal to $(7, 8, 12)$ (and so there is 'only' one step to prove the Conjecture 2 in the case $p = 7$). Namely, we have the following fact.

Theorem 5. *If there exists a tame automorphism $F = (F_1, F_2, F_3)$ of $\mathbb{C}^3$ with $\operatorname{mdeg} F = (7, 8, 12)$, then there are polynomials $P, Q \in \mathbb{C}[x, y, z]$ of the form*

$$P = y + P_2 + \cdots + P_8, \qquad Q = z + Q_2 + \cdots + Q_{12}, \qquad P_8, Q_{12} \neq 0, \quad (1)$$

where P_i, Q_i are homogeneous of degree i, that satisfy the condition

$$\deg[P, Q] = 3, \tag{2}$$

where, by definition, $\deg[f, g]$ *is equal to*

$$2+\max\left\{\deg\left(\frac{\partial f}{\partial x}\frac{\partial g}{\partial y}-\frac{\partial f}{\partial y}\frac{\partial g}{\partial x}\right), \deg\left(\frac{\partial f}{\partial x}\frac{\partial g}{\partial z}-\frac{\partial f}{\partial z}\frac{\partial g}{\partial x}\right), \deg\left(\frac{\partial f}{\partial y}\frac{\partial g}{\partial z}-\frac{\partial f}{\partial z}\frac{\partial g}{\partial y}\right)\right\}. \tag{3}$$

The above theorem means that the 'only' step in proving that there is no tame automorphism F of $\mathbb{C}^3$ with $\operatorname{mdeg} F = (7, 8, 12)$ is to prove that there is no pair of polynomials $P, Q \in \mathbb{C}[x, y, z]$ such in the proposition. Unfortunately, this step is not easy and up to now we do not know how this step should be done, but we hope that someone else can do it in the future.

Before we prove the above proposition, we recall some results that we will need in the proof. The first one is the following.

Theorem 6 ([8, Thm. 3.15])**.** *Let $(d_1, d_2, d_3) \neq (1, 1, 1)$, $d_1 \leq d_2 \leq d_3$, be a sequence of positive integers. To prove that there is no tame automorphism F of $\mathbb{C}^3$ with* $\operatorname{mdeg} F = (d_1, d_2, d_3)$ *it is enough to show that a (hypothetical) automorphism F of $\mathbb{C}^3$ with* $\operatorname{mdeg} F = (d_1, d_2, d_3)$ *admits neither a reduction of type III nor an elementary reduction. Moreover, if we additionally assume that $\frac{d_3}{d_2} = \frac{3}{2}$ or $3 \nmid d_1$, then it is enough to show that no (hypothetical) automorphism of $\mathbb{C}^3$ with multidegree (d_1, d_2, d_3) admits an elementary reduction. In both cases we can restrict our attention to automorphisms $F : \mathbb{C}^3 \to \mathbb{C}^3$ such that $F(0, 0, 0) = (0, 0, 0)$.*

The second one is the following.

Theorem 7 ([8, Thm. 3.18])**.** *For every sequence of positive integers $(d_1, \ldots, d_n) \neq (1, \ldots, 1)$, if there is a tame automorphism $F : \mathbb{C}^n \to \mathbb{C}^n$ such that F admits an elementary reduction, $F(0, \ldots, 0) = (0, \ldots, 0)$ and* $\operatorname{mdeg} F = (d_1, \ldots, d_n)$*, then there is also a tame automorphism $\widetilde{F} : \mathbb{C}^n \to \mathbb{C}^n$ such that $\widetilde{F}$ admits an elementary reduction,* $\operatorname{mdeg} \widetilde{F} = (d_1, \ldots, d_n)$*, $\widetilde{F}(0, \ldots, 0) = (0, \ldots, 0)$ and the linear part of $\widetilde{F}$ is equal to* $\mathrm{id}_{\mathbb{C}^n}$.

Now, we can make the first step in the proof of Theorem 5. Using the above two theorems, if we assume (to make a prove by a contradiction) that there is a tame automorphisms $F = (F_1, F_2, F_3)$ of $\mathbb{C}^3$ with $\operatorname{mdeg} F = (7, 8, 12)$, then we can assume, without lose of generality, that

F admits an elementary reduction and

$$F_1 = x + F_{1,2} + \cdots + F_{1,7}, \qquad F_{1,7} \neq 0, \tag{4}$$
$$F_2 = y + F_{2,2} + \cdots + F_{2,8}, \qquad F_{2,8} \neq 0, \tag{5}$$
$$F_3 = z + F_{3,2} + \cdots + F_{3,12}, \qquad F_{3,12} \neq 0, \tag{6}$$

where $F_{i,j}$ denotes the homogeneous component of degree j of the polynomial F_i.

To make a next step in the proof, let us recall that $F = (F_1, F_2, F_3)$ admits an elementary reduction means that there exists a polynomial G in two variables over $\mathbb{C}$ such that at least one of the following inequalities hold

$$\deg\left(F_1 - G(F_2, F_3)\right) < \deg F_1, \tag{7}$$
$$\deg\left(F_2 - G(F_1, F_3)\right) < \deg F_2, \tag{8}$$
$$\deg\left(F_3 - G(F_1, F_2)\right) < \deg F_3. \tag{9}$$

To exclude the possibility of the last two inequalities, we use the following result due to Umirbaev and Shestakov.

Theorem 8 ([18, Thm. 2]). *Let $f, g \in \mathbb{C}[X_1, \ldots, X_n]$ be a p-reduced pair, and let $G(X, Y) \in k[X, Y]$ with $\deg_Y G(X, Y) = pq + r, 0 \leq r < p$. Then*

$$\deg G(f, g) \geq q\left(p \deg g - \deg g - \deg f + \deg[f, g]\right) + r \deg g.$$

The notion of p-reduced pair of polynomials is also due to Umirbaev and Shestakov.

Definition 1 ([18, Def. 1]). A pair $f, g \in \mathbb{C}[X_1, \ldots, X_n]$ is called *-reduced if
(i) f, g are algebraically independent;
(ii) $\overline{f}, \overline{g}$ are algebraically dependent, where $\overline{h}$ means the homogenous part of h of maximal degree;
(iii) $\overline{f} \notin \mathbb{C}[\overline{g}]$ and $\overline{g} \notin \mathbb{C}[\overline{f}]$.
Moreover, we say that f, g is a p-reduced pair if f, g is a *-reduced pair with $\deg f < \deg g$ and $p = \frac{\deg f}{\gcd(\deg f, \deg g)}$.

Some generalization of the above inequalty can be found in [12, 13].

One can easily check that the inequality in Theorem 8 is also true if $\overline{f}$ and $\overline{g}$ are algebraically independent.

Let us, also, recall that for $f, g \in \mathbb{C}[X_1, \ldots, X_n]$, by definition, we have

$$\deg[f,g] = 2 + \max_{1\le i<j\le n} \deg\left(\frac{\partial f}{\partial X_i}\frac{\partial g}{\partial X_j} - \frac{\partial f}{\partial X_j}\frac{\partial g}{\partial X_i}\right). \tag{10}$$

In particular, if f and g are algebraically independent we have $\deg[f,g] \ge 2$.

Now, assume that our hypothetical tame automorphism $F=(F_1,F_2,F_3)$ of $\mathbb{C}^3$ with $\operatorname{mdeg} F = (7,8,12)$ admits an elementary reduction of the form $(F_1, F_2 - G(F_1,F_3), F_3)$, i.e. for some polynomial G in two variables over $\mathbb{C}$, we have $\deg(F_2 - G(F_1,F_3)) < \deg F_2$. This means, in particular, that

$$8 = \deg F_2 = \deg G(F_1,F_3). \tag{11}$$

Since $p = \frac{\deg F_1}{\gcd(\deg F_1, \deg F_3)} = 7$, it follows from Theorem 8, that we have

$$\begin{aligned} \deg G(F_1,F_3) &\ge q\,(p \deg F_3 - \deg F_3 - \deg F_1 + \deg[F_1,F_3]) + r \deg F_3 \qquad (12)\\ &= q\,(7\cdot 12 - 12 - 7 + \deg[F_1,F_3]) + 12r, \end{aligned}$$

where $\deg_y G(x,y) = pq + r$ with $0 \le r < p$. Since $7\cdot 12 - 12 - 7 + \deg[F_1,F_3] > 8$ and $12 > 8$, it follows from (11) and (12) that $q = r = 0$. But, this means that G is actually a polynomial in one variable, and $G(F_1,F_3) = G(F_1)$. This, of course, contradicts with the facts that $\deg F_1 = 7$ and $\deg G(F_1) = \deg G(F_1,F_3) = 8$.

In a similar way one can check that a hypothetical tame automorphism $F = (F_1,F_2,F_3)$ of $\mathbb{C}^3$ with $\operatorname{mdeg} F = (7,8,12)$ can not admit an elementary reduction of the form $(F_1,F_2,F_3 - G(F_1,F_2))$. The only difference to the above case is that we obtain $q = 0$ and $r \le 1$, which means that $G(F_1,F_2)$ is of the form $g_0(F_1) + F_2 g_1(F_1)$ for some polynomials g_0, g_1 in one variable over $\mathbb{C}$. But, this means that $\deg G(F_1,F_2) \in 7\mathbb{N} \cup (8 + 7\mathbb{N})$ which is a contradiction with $\deg G(F_1,F_2) = \deg F_3 = 12$.

Up to now, we have showed that if there exists a hypothetical tame automorphism $F = (F_1,F_2,F_3)$ of $\mathbb{C}^3$ with $\operatorname{mdeg} F = (7,8,12)$, then such an automorphism admits an elementary reduction of the form $(F_1 - G(F_2,F_3), F_2, F_3)$ and satisfies (4)–(6). If this is the case, we have

$$7 = \deg F_1 = \deg G(F_2,F_3) \tag{13}$$

and for $p = \frac{\deg F_2}{\gcd(\deg F_2, \deg F_3)} = 2$, by virtue of the inequality from Theorem 8, we have

$$\begin{aligned} \deg G(F_2,F_3) &\ge q\,(p \deg F_3 - \deg F_3 - \deg F_2 + \deg[F_2,F_3]) + r \deg F_3 \qquad (14)\\ &= q\,(2\cdot 12 - 12 - 8 + \deg[F_2,F_3]) + 12r, \end{aligned}$$

where $\deg_y G(x,y) = pq + r$ with $0 \le r < p$. Since $12 > 7$, it follows that $r = 0$. Similarly, if we would have $\deg[F_2, F_3] \ge 4$, we would have $q = 0$ because in this situation we would have $2 \cdot 12 - 12 - 8 + \deg[F_2, F_3] > 7$. Thus, if such an automorphism F exists it must satisfy, additionally, that $\deg[F_2, F_3] \le 3$.

Since F_2 and F_3, as components of an automorphism F, are algebraically independent, we have two cases: $\deg[F_2,F_3] = 2$ or $\deg[F_2,F_3] = 3$. To exclude the first one, we use the following result.

Theorem 9 ([8, Thm. 3.21])**.** *Let $f, g \in \mathbb{C}[X_1, \dots, X_n]$ be such that*

$$f = X_1 + f_2 + \cdots + f_l, \qquad g = X_2 + g_2 + \cdots + g_m,$$

where f_i, g_i are homogeneous forms of degree i. If $\deg\,[f,g] = d \le \min\{l,m\}$, $d \ge 2$, and f_i, g_i for $i = 1, \dots, d-1$ do not involve X_r, where $r > 2$, then f and g do not involve X_r.

Applying the above theorem for $d = 2$, $n = 3$, $\mathbb{C}[X_1, \dots, X_n] = \mathbb{C}[x,y,z]$, $X_1 = y$, $X_2 = z$ and

$$\begin{aligned} f &= F_2 = y + F_{2,2} + \cdots + F_{2,8} \in \mathbb{C}[x,y,z], \\ g &= F_3 = z + F_{3,2} + \cdots + F_{3,12} \in \mathbb{C}[x,y,z], \end{aligned}$$

we obtain that in the case $\deg[F_2, F_3] = 2$ we have $F_2, F_3 \in \mathbb{C}[y,z]$. Since (F_1, F_2, F_3) is an automorphism of $\mathbb{C}^3$, we have that (F_2, F_3) is an automorphism of $\mathbb{C}^2$, but $\deg F_2 = 8 \nmid 12 = \deg F_3$ which gives a contradiction with the Jung-van der Kulk theorem.

Thus, we have proved that the only possible (more precisely, the only not excluded case) is the case when $\deg[F_2, F_3] = 3$. This completes the proof of the theorem.

3. Some complimentary results

In this section we will show the following two theorems which are complimentary to Conjecture 2 and generalize Proposition 2.

Theorem 10. *For any composite number $d_1 = ab$ with $a, b > 1$ there are infinitely many pairs of integers $(d_2, d_3) \in \mathbb{N}_+^2$ such that $d_3 > d_2 > d_1$, $d_1 \nmid d_2$, $\gcd(d_1, d_2) = a$, $d_3 \notin d_1\mathbb{N} + d_2\mathbb{N}$ and $(d_1, d_2, d_3) \in \operatorname{mdeg}(\operatorname{Tame}(\mathbb{C}^3))$.*

Theorem 11. *For any non-coprime positive integers $d_2 > d_1 > 1$ with $d_1 \nmid d_2$ the following set*

$$\operatorname{mdeg}\left(\operatorname{Tame}(\mathbb{C}^3)\right) \cap \{(d_1, d_2, d_3)\,|\, d_3 \notin d_1\mathbb{N} + d_2\mathbb{N}\}$$

is infinite.

Proof. (*of Theorem 10*) Let us take any positive integer k and define d_2 as $(kb+1)a$. Then, we have $d_1 \nmid d_2$ and $\gcd(d_1, d_2) = \gcd(ab, (kb+1)a) = \gcd(ab, kab+a) = \gcd(ab, a) = a$. Notice that $d_1(kb+1) = \operatorname{lcm}(d_1, d_2) = d_2 b$ and consider the following polynomial

$$\begin{aligned} &\left(X + Z^r + Z^{d_1}\right)^{kb+1} - \left(Y + Z^{d_2}\right)^b \\ = &\sum_{l=0}^{kb+1} \binom{kb+1}{l} \left(X + Z^r\right)^l Z^{\operatorname{lcm}(d_1,d_2)-ld_1} - \sum_{l=0}^{b} \binom{b}{l} Y^l Z^{\operatorname{lcm}(d_1,d_2)-ld_2}. \end{aligned}$$

Since, in the above two sums, for $l = 0$ we obtain the same summand $Z^{\operatorname{lcm}(d_1,d_2)}$, we see that:

$$\begin{aligned} &\left(X + Z^r + Z^{d_1}\right)^{kb+1} - \left(Y + Z^{d_2}\right)^b \\ = &\sum_{l=1}^{kb+1} \binom{kb+1}{l} \left(X + Z^r\right)^l Z^{\operatorname{lcm}(d_1,d_2)-ld_1} - \sum_{l=1}^{b} \binom{b}{l} Y^l Z^{\operatorname{lcm}(d_1,d_2)-ld_2}. \end{aligned}$$

Because $d_1, d_2 > 1$ we have

$$\deg\left(\sum_{l=1}^{b} \binom{b}{l} Y^l Z^{\operatorname{lcm}(d_1,d_2)-ld_2}\right) \le \operatorname{lcm}(d_1, d_2) - d_2 + 1$$

and, if we take $r \in \{1, \ldots, d_1 - 1\}$, we also have

$$\begin{aligned} \deg\left(\sum_{l=2}^{kb+1} \binom{kb+1}{l} \left(X + Z^r\right)^l Z^{\operatorname{lcm}(d_1,d_2)-ld_1}\right) &\le \operatorname{lcm}(d_1, d_2) - 2d_1 + 2r \\ &< \operatorname{lcm}(d_1, d_2) - d_1 + r \end{aligned}$$

and

$$\deg\left(\binom{kb+1}{1} \left(X + Z^r\right) Z^{\operatorname{lcm}(d_1,d_2)-d_1}\right) = \operatorname{lcm}(d_1, d_2) - d_1 + r.$$

Since $d_1 < d_2$ and $r \ge 1$, it follows that $\operatorname{lcm}(d_1, d_2) - d_1 + r > \operatorname{lcm}(d_1, d_2) - d_2 + 1$, and finally we obtain that

$$\deg\left(\left(X + Z^r + Z^{d_1}\right)^{kb+1} - \left(Y + Z^{d_2}\right)^b\right) = \operatorname{lcm}(d_1, d_2) - d_1 + r.$$

Now, one can take

$$F_1(x,y,z) = (x + z^r + z^{d_1}, y + z^{d_2}, z)$$

and

$$F_2(u,v,w) = \left(u, v, w + \left(u^{kb+1} - v^b\right) u^n\right)$$

obtaining that:

$$\operatorname{mdeg}(F_2 \circ F_1) = (d_1, d_2, d_3),$$

where

$$d_3 = \operatorname{lcm}(d_1, d_2) - d_1 + r + nd_1.$$

Since $a | \operatorname{lcm}(d_1, d_2) - d_1 + r + nd_1$, it follows that $a \nmid d_3$ whenewer $a \nmid r$, for example if $r \in \{1, \dots, d_1 - 1\} \setminus \{a, 2a, \dots, (b-1)a\}$. Because $a = \gcd(d_1, d_2)$, one can see that $d_3 \notin d_1\mathbb{N} + d_2\mathbb{N}$. Of course, for different pairs of integeres k, n, we obtain different pair of integers (d_2, d_3). □

Proof. (*of Theorem 11*) Let $d = \gcd(d_1, d_2)$ and let $\tilde{d}_1, \tilde{d}_2$ be such that $d_1 = \tilde{d}_1 d$ and $d_2 = \tilde{d}_2 d$. Now, consider the following polynomial

$$\left(X + Z^r + Z^{d_1}\right)^{\tilde{d}_2} - \left(Y + Z^{d_2}\right)^{\tilde{d}_1}$$
$$= \sum_{l=1}^{\tilde{d}_2} \binom{\tilde{d}_2}{l} (X + Z^r)^l Z^{\operatorname{lcm}(d_1,d_2) - ld_1} - \sum_{l=1}^{\tilde{d}_1} \binom{\tilde{d}_1}{l} Y^l Z^{\operatorname{lcm}(d_1,d_2) - ld_2}$$

and the polynomial automorphism $F_2 \circ F_1$, where

$$\begin{aligned} F_1(x,y,z) &= (x + z^r + z^{d_1}, y + z^{d_2}, z), \\ F_2(u,v,w) &= \left(u, v, w + \left(u^{\tilde{d}_2} - v^{\tilde{d}_1}\right) u^n\right). \end{aligned}$$

Since $d_2 > d_1 > 1$, it follows by the similar arguments as in the proof of Theorem 10 that if $r \in \{1, \dots, d_1 - 1\} \setminus \{d, 2d, d_1 - d\}$, then

$$\operatorname{mdeg}(F_2 \circ F_1) = (d_1, d_2, \operatorname{lcm}(d_1, d_2) - d_1 + r + nd_1)$$

and $\operatorname{lcm}(d_1, d_2) - d_1 + r + nd_1 \notin d_1\mathbb{N} + d_2\mathbb{N}$. Thus, we have

$$\begin{aligned} &\{ (d_1, d_2, \operatorname{lcm}(d_1, d_2) - d_1 + r + nd_1) \,|\, n \in \mathbb{N} \} \\ \subset\ &\operatorname{mdeg}\left(\operatorname{Tame}(\mathbb{C}^3)\right) \cap \{ (d_1, d_2, d_3) \,|\, d_3 \notin d_1\mathbb{N} + d_2\mathbb{N} \} \end{aligned}$$

□

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