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On relations between generalized norms in locally finite groups

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Dedicated to Professor Yu. A. Drozd
on the occasion of his 80th birthday

Abstract. In the paper the relations between such generalized norms as the norm of Abelian non-cyclic subgroups and the norm of decomposable subgroups in the class of infinite locally finite groups are studied. The local nilpotency and non-Dedekindness of the norm of Abelian non-cyclic subgroups are considered as the restrictions. It was proved that any infinite locally finite group with mentioned restrictions on the norm of Abelian non-cyclic subgroups is a finite extension of a quasicyclic p-subgroup and does not contain Abelian non-cyclic p'-subgroups. Moreover, in such groups the norm of Abelian non-cyclic subgroups necessarily includes Abelian non-cyclic subgroups and therefore is a non-Hamiltonian $\overline{HA}$-group (i.e., a group with the normality condition for Abelian non-cyclic subgroups), whose structure is known. It was shown that for infinite locally finite groups with the non-Dedekind locally nilpotent norm N_G^A the relation $N_G^A \supseteq N_G^d$ holds. The inclusion is proper for infinite torsion non-primary locally nilpotent groups with the mentioned restrictions on the norm N_G^A, as well as for infinite locally finite groups in which the norm N_G^A is a non-Dedekind non-primary locally nilpotent group.

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Introduction

Let Σ be the system of all subgroups of a group G which have some theoretical group property. The intersection of normalizers of all subgroups from Σ of a group G is called Σ-norm of a group G. Clearly, Σ-norm of a group is the maximal subgroup of a group, which normalizes any subgroup from the system Σ, and contains the centre of a group.

It is evident that all subgroups from the system Σ are normal (under the condition that the system Σ is non-empty) in a group, which coincides with its Σ-norm. Groups with systems Σ of normal subgroups are studied actively since the end of the XIX century. So, under this condition the structure and properties of such groups were described for different natural systems Σ of subgroups (particulary, for the system Σ of all subgroups, all non-cyclic, all Abelian non-cyclic, all infinite Abelian, all infinite cyclic, non-Abelian subgroups, etc.). It would be natural to raise the question of study the properties of groups with proper Σ-norm, which satisfies some restrictions.

Such a task was formulated by R. Baer in the 1930s of the past century [1] for the system Σ of all subgroups of a group. This Σ-norm was called the norm $N(G)$ of a group G and defined as the intersection of the normalizers of all subgroups of a group. Since the norm $N(G)$ normalizes every subgroup of a group, all subgroups are normal in $N(G)$ (i.e. it is Dedekind and either Abelian or Hamiltonian).

Further R. Baer's results were applied to different systems Σ of subgroups and different restrictions on Σ-norms (see, for example, [4–6, 9–16, 20]). Clearly, the norm $N(G)$ of a group is contained in all other Σ-norms, which can be considered as its generalizations. In this paper the relations between two of generalized norms (the norm of decomposable subgroups and the norm of Abelian non-cyclic subgroups) in infinite locally finite groups are considered.

Note that the norm N_G^d of decomposable subgroups of a group G is the intersection of normalizers of all decomposable subgroups of this group or the group itself, if the system of such subgroups is empty [16]. The subgroup of a group G is decomposable if it can be presented as the direct product of two non-trivial factors [18].

Clearly, that in the case $N_G^d = G$ all decomposable subgroups are normal in a group G or the system of such subgroups is empty. Non-Abelian groups with such a property were studied in details in [18] and were called *di*-groups.

The existence of decomposable subgroups in a group is directly connected with the existence of decomposable Abelian subgroups, which are mostly non-cyclic. So, the norm N_G^d of decomposable subgroups of a group G to a certain extent depends on the properties of the norm N_G^A of Abelian non-cyclic subgroups of this group.

The norm N_G^A of Abelian non-cyclic subgroups of a group G is the intersection of normalizers of all Abelian non-cyclic subgroups of this group under the condition that the system of such subgroups is non-empty (see [10, 15]). If the norm N_G^A contains at least one Abelian non-cyclic subgroup, any such a subgroup is normal in N_G^A. Non-Abelian groups with this property were described by F. Lyman [17] and called $\overline{HA}$-groups ($\overline{HA_p}$-groups, if they are p-groups). Therefore, if the norm of Abelian non-cyclic subgroups contains an Abelian non-cyclic subgroup, it is either a Dedekind or a non-Hamiltonian $\overline{HA}$-group.

Relations between these norms were studied in [9, 11–14, 16] for different classes of groups. It worse to note that any torsion locally nilpotent di-group G that contains an Abelian non-cyclic subgroup is a $\overline{HA_p}$-group, so $G = N_G^d = N_G^A$.

It was proved in [16] that in locally finite groups that contain an Abelian non-cyclic subgroup one of the relations holds:

$$N_G^A = N_G^d, \text{ or } N_G^A \supset N_G^d, \text{ or } N_G^A \subset N_G^d.$$

In this case, in any locally finite p-group that contains an Abelian non-cyclic subgroup these norms coincide $N_G^A = N_G^d$ [16, Theorem 1.1]. In the classes of finite non-primary and infinite torsion locally nilpotent non-primary groups the inclusion takes place $N_G^A \supseteq N_G^d$ [16, Theorem 1.2-1.3].

In further studies (see [12]) it was proved that under the condition of the existence of an Abelian non-cyclic subgroup torsion locally nilpotent groups have non-Dedekind norm N_G^d of decomposable subgroups if and only if they are locally finite p-groups and $N_G^d = N_G^A$. Besides, the norms N_G^d and N_G^A coincide in any locally finite group, if the norm N_G^d of decomposable subgroups is non-Dedekind and locally nilpotent (see [14]). However, under the same restriction for the norm of Abelian non-cyclic subgroups the norm N_G^d of decomposable subgroups, in general, is different from N_G^A.

In relation to this, there arises a necessity to study relations between the norm of Abelian non-cyclic subgroups and the norm of decomposable subgroups in infinite non-primary locally finite groups under the condition that the norm of Abelian non-cyclic subgroups is locally nilpotent and non-Dedekind, and to study the properties of these groups.

It will be proved further that any infinite locally finite group G with non-Dedekind locally nilpotent norm N_G^A of Abelian non-cyclic subgroups is a finite extension of a quasicyclic subgroup and $N_G^A \supseteq N_G^d$.

1. Properties of infinite locally finite groups with locally nilpotent non-Dedekind norm of Abelian non-cyclic subgroups

Let's consider the properties of infinite locally finite groups with locally nilpotent non-Dedekind norm of Abelian non-cyclic subgroups. The following statement are needed for the sequel.

Lemma 1 ([11]). *If a group G contains an Abelian non-cyclic subgroup M such that*

$$M \cap N_G^A = E,$$

then the subgroup N_G^A is Dedekind.

Clearly, if a group G contains an Abelian non-cyclic subgroup and coincides with the norm N_G^A of Abelian non-cyclic subgroups, then all Abelian non-cyclic subgroups are normal. So, such a group is either Dedekind or a non-Hamiltonian $\overline{HA}$-group. The properties and structure of torsion non-primary locally nilpotent $\overline{HA}$-groups are described in the following proposition (see [17]).

Proposition 1. *A torsion locally nilpotent non-Hamiltonian group G is a $\overline{HA}$-group if and only if*

$$G = G_p \times B,$$

where G_p is a Sylow p-subgroup of a group G, which is a non-Hamiltonian $\overline{HA}_p$-group, B is a finite Dedekind p'-group, all Abelian subgroups of which are cyclic.

Therefore, if the norm N_G^A of Abelian non-cyclic subgroups is non-Dedekind, contains an Abelian non-cyclic subgroup and is a locally nilpotent $\overline{HA}$-group, then it has the structure described in Proposition 1. But, the subgroup N_G^A may contain no Abelian non-cyclic subgroups at all and degenerates in the unit subgroup. The example of such a group is the infinite torsion Frobenius group (see [2, Example 3.4]).

Let's prove that in the class of infinite locally finite groups with non-Dedekind locally nilpotent norm N_G^A the condition of the existence of Abelian non-cyclic subgroups in a group is equivalent to the condition of the existence of such subgroups in the norm N_G^A.

Lemma 2. *An infinite locally finite group G with non-Dedekind locally nilpotent norm N_G^A of Abelian non-cyclic subgroups contains Abelian non-cyclic subgroups if and only if the norm N_G^A contains such subgroups.*

Proof. The sufficiency of the conditions of the theorem is obvious, so we will prove only their necessity.

Let a group G contain Abelian non-cyclic subgroups and have non-Dedekind locally nilpotent norm N_G^A. Then N_G^A is locally nilpotent and the direct product of its Sylow p-subgroups by Proposition 1.4 [3]. Let's suppose, contrary to the theorem conditions, N_G^A does not contain Abelian non-cyclic subgroups. By the non-Dedekindness of the norm N_G^A, Lemma 3 [11] and the assumption, N_G^A is the direct product of a finite quaternion 2-group Q of order greater than 8 and a cyclic Sylow $2'$-subgroup $(N_G^A)_{2'} = \langle h \rangle$:

$$N_G^A = Q \times \langle h \rangle,$$

where $Q = \langle a \rangle \langle b \rangle$, $|a| = 2^n$, $n \geq 3$, $|b| = 4$, $a^{2^{n-1}} = b^2$, $b^{-1}ab = a^{-1}$, $|h| = m$, $(m, 2) = 1$.

By the infiniteness of a group G and the Kargapolov-Hall-Kulatilaka theorem (see [7, 8]), it contains an infinite Abelian subgroup A. If A does not satisfy the minimal condition for Abelian subgroups, then we can consider that it is the direct product of infinitely many subgroups of prime order. Let $N_G^A \cap A = A_1$. Then $|A_1| < \infty$ and

$$A = A_1 \times A_2,$$

where $|A_2| = \infty$ and $N_G^A \cap A_2 = E$.

By Lemma 1 the norm N_G^A is Dedekind, which contradicts the conditions. Thus, A is a group with minimal condition for Abelian subgroups and is the direct product of finitely many quasicyclic subgroups and a finite Abelian group.

Let's denote divisible part of a group A by P. Then P is the direct product of quasicyclic subgroups. By the conditions $\left|N_G^A\right| < \infty$, $N_G^A \lhd G$, we get $\left|G : C_G(N_G^A)\right| < \infty$ and $P \subset C_G(N_G^A)$. Thus, P is contained in the center of the group $G_1 = P \cdot N_G^A$. By Lemma 1 [11], we get $G_1 = N_{G_1}^A$ and G_1 is a non-Hamiltonian $\overline{HA}$-group. So, on account of the description of non-Hamiltonian $\overline{HA}$-groups (see [17]), we conclude that P is a quasicyclic group. By Lemma 1 $N_G^A \cap P \neq E$ and N_G^A contains a generalized quaternion group, which contradicts the results [17].

Thus, a group does not contain infinite Abelian subgroups, which is impossible. In other words, the assumption is false and a group contains

Abelian non-cyclic subgroups, if the norm N_G^A contains such subgroups. Lemma is proved. $\square$

Taking into account Lemma 2 and the definition of the norm N_G^A of Abelian non-cyclic subgroups, the study of the properties of infinite locally finite groups with non-Dedekind locally nilpotent norm N_G^A will provide under the condition that N_G^A contains at least one Abelian non-cyclic subgroup, i.e. N_G^A is non-Hamiltonian locally nilpotent $\overline{HA}$-group.

Let's prove some statements which characterize the impact of the norm of Abelian non-cyclic subgroups on the properties of infinite locally finite groups under the condition of non-Dedekindness and locally nilpotency of the norm N_G^A. These statements generalize the results [11], which were proved earlier for infinite torsion locally nilpotent groups.

Theorem 1. *If a torsion non-primary group G has non-Dedekind locally nilpotent norm N_G^A of Abelian non-cyclic subgroups with non-Dedekind Sylow p-subgroup $(N_G^A)_p$, then all Abelian p'-subgroups of a group G are cyclic. If under these conditions a group G is locally finite, then all its Sylow p'-subgroups are finite and do not contain Abelian non-cyclic subgroups. In particular, Sylow q-subgroups (q is an odd prime, $q \in \pi(G)$, $q \neq p$) are cyclic, Sylow 2-subgroups ($p \neq 2$) are either cyclic or finite quaternion 2-groups.*

Proof. Since the norm N_G^A is a non-Hamiltonian locally nilpotent $\overline{HA}$-group, by Proposition 1

$$N_G^A = (N_G^A)_p \times B,$$

where $(N_G^A)_p$ is Sylow p-subgroup of the norm, which is a non-Hamiltonian $\overline{HA}_p$-group, B is a finite Dedekind group, all Abelian subgroups of which are cyclic and $(|B|, p) = 1$.

Let $G_{p'}$ be an arbitrary Sylow p'-subgroup of a group G. Let's prove that all Abelian subgroups of the group $G_{p'}$ are cyclic. Indeed, let $A \leq G_{p'}$ be an Abelian non-cyclic subgroup. Since $(N_G^A)_p$ is characteristic and the subgroup A is N_G^A-admissible,

$$[\langle x\rangle, A] \subseteq (N_G^A)_p \cap A = E$$

for an arbitrary element $x \in (N_G^A)_p$. Taking into account that $\langle x, A\rangle = \langle x\rangle \times A$ is Abelian non-cyclic, and moreover, N_G^A-admissible subgroup, we obtain that

$$\langle x, A\rangle \cap (N_G^A)_p = \langle x\rangle \lhd (N_G^A)_p.$$

But in this case $(N_G^A)_p$ is Dedekind, contrary to the theorem conditions. Therefore, all Abelian p'-subgroups of a group G are cyclic.

Let G be a locally finite group. Since $G_{p'}$ does not contain infinite Abelian subgroups, by the Kargapolov-Hall-Kulatilaka theorem (see [7, 8]) $G_{p'}$ is a finite group and by the proved above all its Abelian subgroups are cyclic. It follows also that all Sylow q-subgroups of a group G ($q \in \pi(G)$, $q \neq p$) are cyclic for an odd prime q, Sylow 2-subgroups (when $p \neq 2$) are either cyclic or finite quaternion 2-groups. The theorem is proved. □

Corollary 1. *If the norm of Abelian non-cyclic subgroups of a non-primary locally finite group G is a locally nilpotent non-Dedekind group and $2 \notin \pi(G)$, then G has non-cyclic Sylow p-subgroups only for a unique prime $p \in \pi(G)$.*

Lemma 3. *An infinite torsion group G with locally nilpotent non-Dedekind norm N_G^A satisfies the minimal condition for Abelian subgroups.*

Proof. Let a group G and its norm N_G^A of Abelian non-cyclic subgroups satisfy the lemma conditions. Then N_G^A is a non-Dedekind locally nilpotent $\overline{HA}$-group. By the description of such groups [11, Proposition 1-2], the norm N_G^A is either finite or a finite extension of a quasicyclic p-subgroup for a prime $p \in \pi(G)$.

Suppose that G does not satisfy the minimal condition for Abelian subgroups. Then it contains an Abelian subgroup A, which can be presented as the direct product of infinitely many subgroups of prime order. Let

$$A_1 = N_G^A \cap A.$$

Then $|A_1| < \infty$ and

$$A = A_1 \times A_2,$$

where $|A_2| = \infty$ and $N_G^A \cap A_2 = E$. By Lemma 1 [11] the norm N_G^A must be Dedekind, which contradicts the lemma condition. Therefore, G is a group with the minimal condition for Abelian subgroups, which is desired conclusion. The lemma is proved. □

Taking into account that in the class of locally finite groups the minimal condition for Abelian subgroups is equivalent to the minimal condition for all subgroups (see for instance [19]), we get the following result.

Corollary 2. *Any infinite locally finite group G that has non-Dedekind locally nilpotent norm N_G^A is a Chernikov group.*

Lemma 4. *Let G be an infinite locally finite group that has locally nilpotent norm N_G^A with Non-Hamiltonian Sylow p-subgroup $(N_G^A)_p$. Then G is a finite extension of a quasicyclic p-subgroup.*

Proof. Let a group G satisfy the lemma conditions. Then by Corollary 2 it is Chernikov group and a finite extension of a divisible Abelian subgroup P. Since all Sylow q-subgroups of a group G $(q \neq p)$ are either cyclic or quaternion 2-groups by Theorem 1, P is the direct product of finitely many quasicyclic p-subgroups.

Let $P \supseteq (A_1 \times A_2)$, where A_1 and A_2 are quasicyclic p-subgroups. Since

$$N_G^A \lhd G_1 = (A_1 \times A_2) \cdot N_G^A,$$

by Theorem 1.16 [3] the center of the group G_1 contains such divisible Abelian subgroup A that $|A \cap N_G^A| < \infty$ and

$$G_1 = A \cdot N_G^A.$$

Therefore, G_1 is a locally nilpotent group with infinite center. By Lemma 1 [11] G_1 is a $\overline{HA}$-group, so by the description of such groups (see [17]), we conclude that $P = A$ is a quasicyclic p-subgroup, which is the maximal divisible subgroup of a group G. The lemma is proved. □

Corollary 3. *Any infinite locally finite group G with infinite locally nilpotent non-Dedekind norm N_G^A is a finite extension of this norm.*

2. Relations between the norm of Abelian non-cyclic subgroups and the norm of decomposable subgroups in infinite locally finite groups

Let's consider relations between the norm N_G^A of Abelian non-cyclic subgroups and the norm N_G^d of decomposable subgroups in infinite locally finite groups under the condition that the norm N_G^A is locally nilpotent and non-Dedekind. In [14] it was proved that in the class of locally finite groups these norms coincide, if the norm N_G^d of decomposable subgroups is non-Dedekind and locally nilpotent.

The following example confirms that the non-Dedekindness and locally nilpotency of the norm N_G^A of Abelian non-cyclic subgroups in locally finite groups do not guarantee the non-Dedekindness of the norm N_G^d of decomposable subgroups.

Example 1. Let $G = A\langle b\rangle \times \langle h\rangle$, where A is quasicyclic 2-subgroup, $b^2 = a_1 \in A$, $|a_1| = 2$, $b^{-1}ab = a^{-1}$ for any element $a \in A$, $|h| = 3$.

The norm of Abelian non-cyclic subgroups of this group is non-primary locally nilpotent and coincides with G. At the same time, the norm of decomposable subgroups is finite Abelian, $N_G^d = \langle a_2\rangle \times \langle h\rangle = N_G(\langle b, h\rangle) \cap N_G(\langle a_i b, h\rangle)$, $a_i \in A$, so, $N_G^A \neq N_G^d$.

The following lemma defines one of the sufficient conditions under which the norm N_G^d of decomposable subgroups of a torsion group is Dedekind.

Lemma 5. *If the center $Z(G)$ of a torsion group G contains a non-primary cyclic subgroup, then the norm N_G^d of decomposable subgroups is Dedekind.*

Proof. Let G be a torsion group with non-primary center $Z(G)$ and $\langle xy\rangle \subset Z(G)$, where $|x| = p$, $|y| = q$, $p \neq q$ are primes. Let a be an arbitrary element of the norm N_G^d. If $(|a|, pq) = 1$ or $(|a|, pq) = q$, then the subgroup $\langle a\rangle \times \langle x\rangle$ is decomposable and hence N_G^d-admissible. Accordingly, its characteristic Hall subgroup $\langle a\rangle$ is normal in N_G^d. If $(|a|, pq) = p$, then the subgroup $\langle a\rangle \times \langle y\rangle$ is decomposable, it is N_G^d-admissible and again $\langle a\rangle \lhd N_G^d$.

Let now $|a| \vdots pq$. Then $\langle a\rangle$ can be presented as the product $\langle a\rangle = \langle a_1\rangle \times \langle a_2\rangle$, where $(|a_1|, p) = 1$ and $(|a_2|, q) = 1$. Since the subgroup $\langle x, a_1\rangle = \langle x\rangle \times \langle a_1\rangle$ is decomposable, then it and its characteristic subgroup $\langle a_1\rangle$ are N_G^d-admissible.

Similarly, subgroups $\langle y, a_2\rangle = \langle y\rangle \times \langle a_2\rangle$ and $\langle a_2\rangle$ are also N_G^d-admissible. Therefore, $\langle a\rangle = \langle a_1\rangle \times \langle a_2\rangle$ is N_G^d-admissible and hence $\langle a\rangle \lhd N_G^d$. By the arbitrariness of element a selection, the norm N_G^d is Dedekind. The lemma is proved. □

Corollary 4. *If a locally finite group G has non-Dedekind norm N_G^d of decomposable subgroups, then the center $Z(G)$ of a group G is a p-group (in particular, the identity subgroup).*

Corollary 5. *The norm N_G^d of decomposable subgroups of a torsion non-primary locally nilpotent group G is Dedekind.*

Proof. By the condition, the norm N_G^d is locally nilpotent. If it is non-Dedekind, then by Lemma 2.2 [14] a group G does not contain non-primary cyclic subgroups, contrary to the condition. □

Theorem 2. *If an infinite torsion non-primary locally nilpotent group G has non-Dedekind norm N_G^A of Abelian non-cyclic subgroups, then the norm N_G^d of decomposable subgroups coincides with the norm $N(G)$ of a group and is a proper subgroup of the norm N_G^A:*

$$N_G^d = N(G) \subset N_G^A.$$

Proof. Let a group G and its norm N_G^A of Abelian non-cyclic subgroups satisfy the theorem conditions. Then by Corollary 5 the norm N_G^d of decomposable subgroups is Dedekind. Since in the class of infinite non-primary locally nilpotent groups $N_G^A \supseteq N_G^d$ [16, Theorem 1.3] and the norm N_G^A is non-Dedekind, the inclusion is proper.

Let's prove that $N_G^d = N\left(G\right)$. Let a be an arbitrary element of a group G. If $\langle a\rangle$ is non-primary subgroup, then it is decomposable and hence is N_G^d-admissible. Let $\langle a\rangle$ be a cyclic p-group. Since a group G is non-primary and locally nilpotent, then there exists an element $x \in G$, $(|x|, p) = 1$, which is permutable with a. Then $\langle a\rangle \times \langle x\rangle$ is non-primary, decomposable and N_G^d-admissible. Accordingly, its normal Sylow p-subgroup $\langle a\rangle$ is also N_G^d-admissible. Therefore, N_G^d normalizes any subgroup of a group, so $N_G^d \subseteq N(G)$. Considering that there is a reverse inclusion, we obtain $N_G^d = N(G)$. □

The following example confirms, that the norm of decomposable subgroups can be Hamiltonian in infinite torsion non-primary locally nilpotent groups.

Example 2. Let $G = ((A \times \langle b\rangle) \leftthreetimes \langle c\rangle) \times H$, where A is the quasicyclic 3-subgroup, $|b| = |c| = 3$, $|a_1| = 3$, $[A] = E$, $[b, c] = a_1 \in A$, $H = \langle h_1, h_2\rangle$ is the quaternion group of order 8.

In this group the norm of decomposable subgroups is infinite Hamiltonian:

$$N_G^d = N_G(\langle b, h_1\rangle) \cap N_G(\langle c, h_2\rangle) = A \times H = N(G).$$

Note that the norm N_G^A of Abelian non-cyclic subgroups coincides with the group G and $N_G^d \subset N_G^A$.

Let's study the relations between the norm of Abelian non-cyclic subgroups and the norm of decomposable subgroups in the case, when the norm of Abelian non-cyclic subgroups is a non-Dedekind non-primary locally nilpotent group.

Theorem 3. *If the norm N_G^A of Abelian non-cyclic subgroups of an infinite locally finite group G is a non-Dedekind non-primary locally nilpotent group, then the norm N_G^d of decomposable subgroups is Dedekind and $N_G^d \subset N_G^A$.*

Proof. Let the norm N_G^A of Abelian non-cyclic subgroups of an infinite locally finite group G satisfy the theorem conditions. Since an infinite locally finite group contains an infinite Abelian subgroup, it contains an Abelian non-cyclic subgroup. By Lemma 2 its norm N_G^A also contains an Abelian non-cyclic subgroup, so it is a non-Hamiltonian $\overline{HA}$-group of the type specified in Proposition 1.

By Theorem 1.4 [16] in an locally finite group with an Abelian non-cyclic subgroup the one of the following relations $N_G^A \supset N_G^d$ or $N_G^A \subseteq N_G^d$ takes place. First, let's consider the case $N_G^A \supset N_G^d$. Since the norm N_G^A is locally nilpotent and non-primary, the norm N_G^d is also locally nilpotent. Moreover, if it is non-Dedekind, then by Lemma 2.2 [14] a group does not contain non-primary cyclic subgroups, which contradicts the conditions the norm N_G^A of Abelian non-cyclic subgroups satisfies. Therefore, in this case N_G^d is Dedekind. Now, taking into account that the norm N_G^A is non-Dedekind and contains the norm N_G^d, we conclude that the inclusion is proper.

Now, let $N_G^A \subseteq N_G^d$. Then N_G^d is also non-Dedekind. Moreover, if it is locally nilpotent, then by Theorem 2.1 [14] $N_G^A = N_G^d$ and N_G^d is non-primary locally nilpotent, which contradicts Lemma 2.2 [14]. Therefore, N_G^d isn't a locally nilpotent non-primary *di*-group. Since $N_G^A \subseteq N_G^d$ and, by the proved above N_G^A is a locally nilpotent non-primary $\overline{HA}$-group, $N_G^A = G_p \times B$, where G_p is a Sylow p-subgroup of the norm N_G^A, which is non-Hamiltonian $\overline{HA}_p$-group, B is a finite Dedekind p'-group, all Abelian subgroups of which are cyclic.

Let a be an arbitrary element of G_p and $b \in B$. Since $(|a|, |b|) = 1$, the subgroup $\langle a \rangle \times \langle b \rangle$ is decomposable and normal in N_G^d. Then $\langle a \rangle \lhd N_G^A$ and $\langle a \rangle \lhd G_p$. This means that any subgroup is normal in G_p, so it is Dedekind, contrary to the condition. Therefore, this case is impossible. The theorem is proved. □

Let's study the relations between the norm N_G^A of Abelian non-cyclic subgroups and the norm N_G^d of decomposable subgroups in infinite locally finite and non-locally nilpotent groups under the condition, that the norm N_G^A is locally nilpotent and non-Dedekind.

The following example confirms the existence of infinite torsion non-locally nilpotent groups when the norm N_G^A satisfies mentioned conditions.

Example 3. $G = ((A \times \langle b \rangle) \leftthreetimes \langle c \rangle) \times H$, where A is the quasicyclic 5-subgroup, $|b| = |c| = 5$, $[A, \langle c \rangle] = E$, $[b, c] = a_1 \in A$, $|a_1| = 5$, $H = \langle d \rangle \leftthreetimes \langle h \rangle$, $|d| = 3$, $|h| = 4$, $h^{-1}dh = d^{-1}$.

It is easy to show that in this group the norm of Abelian non-cyclic subgroups is a group of the following type:

$$N_G^A = ((A \times \langle b \rangle) \leftthreetimes \langle c \rangle) \times \left\langle h^2 \right\rangle .$$

The group G is non-nilpotent, but its norm N_G^A of Abelian non-cyclic subgroups is nilpotent and non-primary.

At the same time, the norm N_G^d of decomposable subgroups coincides with the center $Z(G) = A \times \langle h^2 \rangle$ and $N_G^A \supset N_G^d$.

Theorem 4. *If G is an infinite locally finite non-locally nilpotent group with the non-Dedekind locally nilpotent norm N_G^A of Abelian non-cyclic subgroups, then $N_G^A \supseteq N_G^d$. Moreover, the equality $N_G^d = N_G^A$ is achieved.*

Proof. Let the condition $N_G^A \supseteq N_G^d$ do not take place in a group G, which satisfies the theorem conditions. By Theorem 1.4 [16] in a locally finite group with an Abelian non-cyclic subgroup one of the following relations holds:

$$N_G^A \supseteq N_G^d \text{ or } N_G^A \subset N_G^d.$$

So $N_G^A \subset N_G^d$, moreover, the inclusion is proper. We get that a group G contains a quasicyclic subgroup P, that is not N_G^d-admissible. On the other hand, by Lemma 4 a group G is a finite extension of a quasicyclic subgroup, therefore, $P \lhd G$. We get the contradiction, so $N_G^A \supseteq N_G^d$.

The equality is achieved, in particular, for groups with the non-Dedekind and locally nilpotent norm of decomposable subgroups. The example of such a group is below. The theorem is proved. □

Example 4. $G = (A \times \langle b \rangle) \leftthreetimes \langle c \rangle \leftthreetimes \langle h \rangle$, where A is a quasicyclic 11-subgroup, $|b| = |c| = 11$, $|h| = 5$, $[A, \langle c \rangle] = E$, $[b, c] = a_1 \in A$, $|a_1| = 11$, $h^{-1}a_1h = a_1^4$, $h^{-1}a_mh = a_m^{\alpha_m}$, $\alpha_m^5 \equiv 1 \pmod{11^m}$, $\alpha_m \not\equiv 1 \pmod{11^m}$ for any element $a_m \in A$, $|a_m| = 11^m$, $m > 1$, $h^{-1}bh = b^3$, $h^{-1}ch = c^5$.

This group is non-locally nilpotent. All its Abelian non-cyclic subgroups are contained in the Sylow 11-subgroup and normal in it. The group G has the identity center and does not contain non-primary Abelian subgroups. The element h is not contained in the normalizer of Abelian non-cyclic subgroup $\langle a_1\rangle \times \langle bc\rangle$. So

$$N_G^d = N_G^A = (A \times \langle b\rangle) \leftthreetimes \langle c\rangle\,.$$

Combining assertions of Theorem 2, 3 and 4, we get the following result, which characterizes the relations between the norm N_G^A of Abelian non-cyclic subgroups and the norm N_G^d of decomposable subgroups in infinite locally finite groups under the condition that the norm of Abelian non-cyclic subgroups is non-Dedekind and locally nilpotent.

Theorem 5. *In any infinite locally finite group G with the non-Dedekind locally nilpotent norm N_G^A of Abelian non-cyclic subgroups the inclusion $N_G^A \supseteq N_G^d$ takes place, where N_G^d is the norm of decomposable subgroups.*

The following example confirms, that the condition of the locally nilpotency and non-Dedekindness of the norm N_G^A is essential, because it can be $N_G^A \neq N_G^d$ and $N_G^A \subset N_G^d$ in the class of infinite locally finite and non-locally nilpotent groups without it.

Example 5 (see [2], example 3.4)**.** $G = A \leftthreetimes B$ is the Frobenius group, where A is an infinite elementary Abelian p-group ($p \neq 3$), B is the quasicyclic 3-group.

In this group $N_G^d = A$. Since $N_G(B) = B$, $N_G(a^{-1}Ba) = a^{-1}Ba$ and $a^{-1}Ba \cap B = E$ for $a \neq 1$, $N_G^A = E$. Therefore, $N_G^A \subset N_G^d$, moreover, the inclusion is proper.

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