© Algebra and Discrete Mathematics RESEARCH ARTICLE
Volume **39** (2025). Number 2, pp. 27–58
DOI:10.12958/adm2345

Fundamental theorem of $(A,\mathcal{G},H)$-comodules

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Communicated by R. Wisbauer

Abstract. Let k be a field, H a Hopf algebra with a bijective antipode, $\mathcal{G}$ an H-comodule Lie algebra and A a commutative $(\mathcal{G}, H)$-comodule algebra. We assume that there is an H-colinear algebra map from H to $A^{\mathcal{G}}$. We generalize the Fundamental Theorem of (A, H)-Hopf modules to $(A, \mathcal{G}, H)$-comodules, and we deduce relative projectivity in the category of $(A, \mathcal{G}, H)$-comodules. In many applications, A could be a commutative G-graded $\mathcal{G}$-module algebra, where G is an abelian group and $\mathcal{G}$ is a G-graded Lie algebra; or a rational $(\mathcal{G}, G)$-module algebra, where G is an affine algebraic group and $\mathcal{G}$ is a rational G-module Lie algebra.

Introduction

Throughout this paper, k will be a field and all algebras and linear spaces will be over k. Let H be a Hopf algebra and A a right H-comodule algebra (A, ρ_A). A right (A, H)-Hopf module is a right A-module and a right H-comodule $\rho_M : M \to M \otimes H$ such that the A-action and the H-coaction are compatible in a natural way, that is,

$$\rho_M(ma) = \rho_M(m)\rho_A(a) \quad \forall a \in A, m \in M.$$

A homomorphism of (A, H)-Hopf modules is a right A-linear map which is also a right H-colinear map. For an (A, H)-Hopf module M, the sub-

2020 Mathematics Subject Classification: *Primary: 16T05. Secondary: 17B60.*

Key words and phrases: *Lie algebra, module over a Lie algebra, Hopf algebra, Hopf module, H-comodule Lie algebra, $(\mathcal{G}, H)$-comodule algebra, $(A, \mathcal{G}, H)$-comodule.*

space M^{coH} of coinvariants of M is a right A^{coH}-module, $M^{coH}\otimes_{A^{coH}} A$ is a right (A,H)-Hopf module. The Fundamental Theorem of (A,H)-Hopf modules [3, Theorem 3] states that if there is an H-colinear algebra map $\phi : H \to A$, then the k-linear map

$$M^{coH}\otimes_{A^{coH}} A \to M; m\otimes_{A^{coH}} a \mapsto ma$$

is an isomorphism of (A,H)-Hopf modules. Let H be a Hopf algebra with a bijective antipode and $\mathcal{G}$ a right H-comodule Lie algebra and A a $(\mathcal{G},H)$-comodule algebra. We introduce the notion of left-right $(A,\mathcal{G},H)$-comodule: a precise definition is given in Section 1. A homomorphism of left-right $(A,\mathcal{G},H)$-comodules is a left A-linear map which is also $\mathcal{G}$-linear and right H-colinear.

We denote by ${}_{A,\mathcal{G}}\mathcal{M}^H$ the category of left-right $(A,\mathcal{G},H)$-comodules with morphisms the homomorphisms of left-right $(A,\mathcal{G},H)$-comodules. Assume that there is an H-colinear algebra map ϕ from H to the subalgebra $A^{\mathcal{G}}$ of $\mathcal{G}$-invariants of A. Let M be a left-right $(A,\mathcal{G},H)$-comodule. We have a projection map $p_M : M \to M^{coH}$ which enables us to equip M with a new $\mathcal{G}$-action $\diamond'$ for which M^{coH} is a $\mathcal{G}$-submodule of M. If this $\mathcal{G}$-action is trivial on M^{coH} and A^{coH}, we show that M^{coH} is equal to the vector subspace $M^{\mathcal{G}coH}$ of simultaneously $\mathcal{G}$-invariant and H-coinvariant elements of M, and we prove the Fundamental Theorem for left-right $(A,\mathcal{G},H)$-comodules; more precisely, we show that the k-linear map

$$\alpha : A\otimes_B M^{coH} \to M; a\otimes_B m \mapsto am$$

is an isomorphism of $(A,\mathcal{G},H)$-comodules with inverse

$$\beta : M \to A\otimes_B M^{coH}; m \mapsto \phi(m_1)\otimes_B p_M(m_0),$$

where $B = A^{coH}$. We also show (not assuming the existence of a map ϕ) that the functor

$$F = A\otimes_{A^{\mathcal{G}coH}} - : {}_{A^{\mathcal{G}coH}}\mathcal{M} \to {}_{A,\mathcal{G}}\mathcal{M}^H; M \mapsto A\otimes_{A^{\mathcal{G}coH}} M$$

is left adjoint to the functor

$$G = (-)^{AcoH} : {}_{A,\mathcal{G}}\mathcal{M}^H \to {}_{A^{\mathcal{G}coH}}\mathcal{M}; M \mapsto M^{\mathcal{G}coH}.$$

Using this result and the Fundamental Theorem, we prove that the functor G is dual Maschke, that is, every left-right $(A,\mathcal{G},H)$-comodule is G-relative projective. We refer to [2] for details on Maschke functors. Here are some examples of $(\mathcal{G},H)$-comodule algebras:

(i) a G-graded $\mathcal{G}$-module algebra [9], where G is an abelian group and $\mathcal{G}$ is a G-graded Lie algebra is a $(\mathcal{G}, kG)$-comodule algebra, where kG is the group algebra of G (Example 1);

(ii) a commutative rational $(\mathcal{G}, G)$-module algebra [10], where G is an affine algebraic group and $\mathcal{G}$ is a rational G-module Lie algebra is a $(\mathcal{G}, k[G])$-comodule algebra, where $k[G]$ is the affine coordinate ring of G (Example 2);

(iii) an H-comodule Poisson algebra A [10] is a $(\mathcal{G}, H)$-comodule Lie algebra, where H is a Hopf algebra with a bijective antipode and $\mathcal{G}$ is equal to A as a Lie algebra (Example 4).

We refer to [20] for the Fundamental Theorem of Hopf modules in the category of Yetter-Drinfeld modules, to [11] for the Fundamental Theorem for quasi-Hopf H-bimodules, to [1] for the Fundamental Theorem of two-sided Hopf modules over quasi-Hopf algebras, to [10] for the Fundamental Theorem of Poisson (A, H)-Hopf modules which is the motivation of the present work, to [14] for the Fundamental Theorem of Poisson Hopf modules for weak Hopf algebra and to [15] for the Fundamental Theorem of Poisson Hopf modules for Hopf group coalgebras.

1. The Category of $(A,\mathcal{G},H)$-comodules

A Hopf algebra is an algebra H that possesses a multiplication $m_H : H \otimes H \to H$, a comultiplication $\Delta_H : H \to H \otimes H$, an antipode $\mathcal{S}_H : H \to H$ and a counit $\epsilon_H : H \to k$, satisfying the defining relations

$$(\Delta_H \otimes id_H) \circ \Delta_H = (id_H \otimes \Delta_H) \circ \Delta_H,$$

$$(\epsilon_H \otimes id_H) \circ \Delta_H = (id_H \otimes \epsilon_H) \circ \Delta_H = id_H, \quad \text{and}$$

$$m_H \circ [(S_H \otimes id_H) \otimes \Delta_H] = m_H \circ [(id_H \otimes S_H) \otimes \Delta_H] = \epsilon_H.$$

For background on Hopf algebras and coactions of Hopf algebras on rings, we refer to [19] and [18]. We will use Sweedler-Heyneman notation, writing:

$$\Delta_H(h) = h_1 \otimes h_2 \text{ for all } h \in H.$$

By an H-comodule, we will mean a right H-comodule. We will consider H as an H-comodule via Δ_H. When M is an H-comodule with structure map ρ_M, the vector subspace

$$M^{coH} = \{m \in M; \rho_M(m) = m \otimes 1_H\}$$

of M is called the subspace of H-coinvariants of M.

A k-algebra A is an H-*comodule algebra* if A is an H-comodule satisfying

$$(aa')_0\otimes(aa')_1=(a_0a'_0)\otimes(a_1a'_1),(1_A)_0\otimes(1_A)_1=1_A\otimes 1_H\ \forall a,a'\in A,\quad (1)$$

that is, ρ_A is an algebra homomorphism.

If A is an H-comodule algebra, then the subspace A^{coH} of H-coinvariants of A is a subalgebra of A called the subalgebra of H-coinvariants of A. When kG is the group algebra of an arbitrary group G, kG-comodule algebras are just G-graded algebras.

Definition 1. Let A be an H-comodule algebra. A vector space M is an (A,H)-Hopf module if M is an A-module and an H-comodule such that

$$(am)_0\otimes(am)_1=(a_0m_0)\otimes(a_1m_1)\quad\forall a\in A, m\in M.\qquad (2)$$

It is easy to see that A is an (A,H)-Hopf module whenever A is an H-comodule algebra. If A is an H-comodule algebra and T is a sub-H-comodule algebra of A, then A is a (T,H)-Hopf module. An (A,H)-Hopf submodule of an (A,H)-Hopf module M is an A-submodule and an H-subcomodule of M.

Let $\mathcal{G}$ be a Lie algebra with Lie bracket $[,]:\mathcal{G}\otimes\mathcal{G}\to\mathcal{G}$. The elements of $\mathcal{G}$ will be denoted $X,Y,\ldots$. A vector space M is a $\mathcal{G}$-module if there is a k-linear map $\diamond:\mathcal{G}\otimes M\to M$ satisfying

$$[X,Y]\diamond m=X\diamond(Y\diamond m)-Y\diamond(X\diamond m)\quad\forall X,Y\in\mathcal{G}, m\in M.\qquad (3)$$

If M and N are two $\mathcal{G}$-modules, a k-linear map f from M to N is $\mathcal{G}$-linear if

$$f(X\diamond m)=X\diamond f(m)\quad\forall m\in M, X\in\mathcal{G}.$$

A subalgebra of a Lie algebra $\mathcal{G}$ is a vector subspace of $\mathcal{G}$ which is stable under the Lie bracket.

A vector space A is a left $\mathcal{G}$-module algebra if A is a $\mathcal{G}$-module (the action is denoted $\diamond$) such that

$$X\diamond(aa')=(X\diamond a)a'+a(X\diamond a')\quad\forall a,a'\in A, X\in\mathcal{G}.\qquad (4)$$

The relation (4) means that $\mathcal{G}$ acts on A by derivations.

Note that A is a left $\mathcal{G}$-module algebra means that we have a Lie algebra homomorphism from $\mathcal{G}$ to the Lie algebra $Der_k(A)$ of k-derivations of A: the bracket of $Der_k(A)$ is

$$[D, D'] = D \circ D' - D' \circ D \quad \forall D, D' \in Der_k(A).$$

A left $\mathcal{G}$-module algebra is exactly a left $U(\mathcal{G})$-module algebra, where $U(\mathcal{G})$ is the universal enveloping algebra of $\mathcal{G}$. We know that $U(\mathcal{G})$ is a (cocommutative) Hopf algebra. Thus we have the smash product $A\#U(\mathcal{G})$ of A with $U(\mathcal{G})$. A vector space M is a left $A\#U(\mathcal{G})$-module or an $(A, \mathcal{G})$-module if and only if M is a left A-module and a $\mathcal{G}$-module (the action is denoted again $\diamond$) such that

$$X \diamond (am) = (X \diamond a)m + a(X \diamond m) \quad \forall a \in A, m \in M, X \in \mathcal{G}. \tag{5}$$

In the following definition, the coaction of H on $\mathcal{G} \otimes \mathcal{G}$ is the diagonal coaction.

Definition 2. A Lie algebra $\mathcal{G}$ in the category $\mathcal{M}^H$ (or an H-comodule Lie algebra) is a Lie algebra $\mathcal{G}$, (with Lie bracket $[,] : \mathcal{G} \otimes \mathcal{G} \to \mathcal{G}$) which is also an H-comodule such that the structure map $[,]$ is an H-colinear map, that is,

$$[X, Y]_0 \otimes [X, Y]_1 = [X_0, Y_0] \otimes X_1 Y_1 \quad \forall X, Y \in \mathcal{G}. \tag{6}$$

We will call a Lie algebra in $\mathcal{M}^H$ an H-comodule Lie algebra. We refer to [8] for the definition of an H-comodule Lie algebra. Because of the antisymmetry of the bracket, we have

$$[X_0, Y_0] \otimes X_1 Y_1 = [X_0, Y_0] \otimes Y_1 X_1 \quad \forall X, Y \in \mathcal{G}.$$

Let $\mathcal{G}$ be an H-comodule Lie algebra. An H-subcomodule Lie algebra of $\mathcal{G}$ is a Lie subalgebra of $\mathcal{G}$ which is also an H-subcomodule of $\mathcal{G}$.

The Lie-center of $\mathcal{G}$ is defined to be

$$Z(\mathcal{G}) = \{X \in \mathcal{G}; [X, \mathcal{G}] = 0\}.$$

Lemma 1. *Let $\mathcal{G}$ be an H-comodule Lie algebra. Let $\mathcal{H}$ be an H-subcomodule of $\mathcal{G}$, and $\mathcal{L}$ the centralizer of $\mathcal{H}$ in $\mathcal{G}$, that is,*

$$\mathcal{L} = \{X \in \mathcal{G}; [X, \mathcal{H}] = 0\}.$$

Then $\mathcal{L}$ is an H-subcomodule Lie algebra of $\mathcal{G}$. If $\mathcal{H} = \mathcal{G}$, then $\mathcal{L}$ is the Lie-center of $\mathcal{G}$.

Proof. Let $Y \in \mathcal{L}$, $X \in \mathcal{H}$, $\rho(Y) = Y_0 \otimes Y_1$ and $\rho(X) = X_0 \otimes X_1$. Using the relation (6), we have

$$[Y_0, X] \otimes Y_1 = [Y, X_0]_0 \otimes ([Y, X_0]_1 S_H(X_1)).$$

Each $[Y, X_0]$ is equal to 0 since $Y \in \mathcal{L}$ and $X_0 \in \mathcal{H}$. It follows that the right term is equal to 0. We deduce that $[Y_0, X] \otimes Y_1 = 0$. Now taking the summands $\{Y_1\}$ to be linearly independant, we have $[Y_0, X] = 0$ for each summand Y_0. So $\mathcal{L}$ is an H-subcomodule of $\mathcal{G}$. It is well known that $\mathcal{L}$ is a Lie subalgebra of $\mathcal{G}$. □

The reader will find another proof of the above lemma in [8, Lemma 7], where $\mathcal{G}$ is finite dimensional over a field of characteristic 0.

Definition 3 ([9])**.** Let $\mathcal{G}$ be an H-comodule Lie algebra. A vector space M is a $(\mathcal{G}, H)$-comodule if M is a $\mathcal{G}$-module which is also an H-comodule such that

$$(X \diamond m)_0 \otimes (X \diamond m)_1 = (X_0 \diamond m_0) \otimes (X_1 m_1) \quad \forall X \in \mathcal{G}, m \in M. \tag{7}$$

The H-comodule Lie algebra $\mathcal{G}$ considered as a $\mathcal{G}$-module via the adjoint action on itself is a $(\mathcal{G}, H)$-comodule. If k is considered as a trivial $\mathcal{G}$-module and a trivial H-comodule, then k is a $(\mathcal{G}, H)$-comodule which we will call a trivial $(\mathcal{G}, H)$-comodule. A $(\mathcal{G}, H)$-subcomodule of a $(\mathcal{G}, H)$-comodule M is a $\mathcal{G}$-submodule of M which is also an H-subcomodule of M. A $(\mathcal{G}, H)$-comodule homomorphism is a $\mathcal{G}$-linear map which is also H-colinear. We denote by ${}_{\mathcal{G}}\mathcal{M}^H$ the category of $(\mathcal{G}, H)$-comodules with $(\mathcal{G}, H)$-comodule homomorphisms. A $(\mathcal{G}, H)$-comodule is called an $(H, \mathcal{G})$-module in [8]. When H is the group algebra of a group G, a $(\mathcal{G}, H)$-comodule is called a $(G, \mathcal{G})$-module in [8].

Definition 4. Let $\mathcal{G}$ be an H-comodule Lie algebra. A vector space A is a $(\mathcal{G}, H)$-comodule algebra if A is a $(\mathcal{G}, H)$-comodule which is also a $\mathcal{G}$-module algebra and an H-comodule algebra, that is, a $\mathcal{G}$-module algebra, an H-comodule algebra and the relation (7) is satisfied.

The base field k is a $(\mathcal{G}, H)$-comodule algebra with a trivial Lie bracket and a trivial H-coaction: we will call it a trivial $(\mathcal{G}, H)$-comodule algebra. A $(\mathcal{G}, H)$-subcomodule algebra of a $(\mathcal{G}, H)$-comodule algebra A is an H-subcomodule algebra of A which is also a $\mathcal{G}$-submodule of A, that is, an H-subcomodule, a subalgebra and a $\mathcal{G}$ submodule of A.

Definition 5. Let $\mathcal{G}$ be an H-comodule Lie algebra and A a $(\mathcal{G}, H)$-comodule algebra. A vector space M is an $(A, \mathcal{G}, H)$-comodule if M is an $A\#U(\mathcal{G})$-module, an (A, H)-Hopf module and a $(\mathcal{G}, H)$-comodule, or equivalently, M is an A-module, a $\mathcal{G}$-module, an H-comodule and the relations (2), (5), and (7) are satisfied.

If $\mathcal{G}$ is an H-comodule Lie algebra and A is a $(\mathcal{G}, H)$-comodule algebra, then A is an $(A, \mathcal{G}, H)$-comodule.

If M is an $(A, \mathcal{G}, H)$-comodule, an $(A, \mathcal{G}, H)$-subcomodule of M is an A-submodule of M which is also a $\mathcal{G}$-submodule of M, and a right H-subcomodule of M, that is, an A-submodule of M which is also a $(\mathcal{G}, H)$-subcomodule of M.

An $(A, \mathcal{G}, H)$-comodule homomorphism is an A-linear map which is also $\mathcal{G}$-linear and H-colinear, that is, an $A\#U(\mathcal{G})$-linear map which is also an (A, H)-Hopf module map.

We denote by ${}_{A,\mathcal{G}}\mathcal{M}^H$ the category of $(A, \mathcal{G}, H)$-comodules with $(A, \mathcal{G}, H)$-comodule homomorphisms. If k is considered as a trivial $(\mathcal{G}, H)$-comodule algebra, then ${}_{k,\mathcal{G}}\mathcal{M}^H$ is exactly ${}_{\mathcal{G}}\mathcal{M}^H$. If A is a $(\mathcal{G}, H)$-comodule algebra, the categories ${}_{\mathcal{G}}\mathcal{M}^H$, ${}_{A}\mathcal{M}^H$ and ${}_{A\#U(\mathcal{G})}\mathcal{M}$ contain ${}_{A,\mathcal{G}}\mathcal{M}^H$ as a subcategory. This remark will enable us to use some well known results of modules over smash products and of Hopf modules.

For the remainder of this section and in Section 2, H is a Hopf algebra, $\mathcal{G}$ is an H-comodule Lie algebra, and A is a $(\mathcal{G}, H)$-comodule algebra.

If M is an $(A, \mathcal{G}, H)$-comodule, we set

$$M^{\mathcal{G}} = \{m \in M, X \diamond m = 0 \quad \forall X \in \mathcal{G}\}.$$

$$M^{coH} = \{m \in M, m_0 \otimes m_1 = m \otimes 1_H\}, \quad \text{and}$$
$$M^{\mathcal{G}coH} = \{m \in M, m \in M^{\mathcal{G}}, \text{and} \quad m \in M^{coH}\}.$$

In other words, $M^{\mathcal{G}coH} = M^{\mathcal{G}} \cap M^{coH}$.

Lemma 2. *If H is commutative, then A^{coH} is an H-subcomodule Lie algebra of A: the H-action is trivial.*

Proof. It is well known that A^{coH} is an H-subcomodule algebra of A. For $b, b' \in A^{coH}$, we have

$$[b, b']_0 \otimes [b, b']_1 = [b_0, b'_0] \otimes (b_1 b'_1) = [b, b'] \otimes 1_H,$$

so $[b, b'] \in A^{coH}$, that is, A^{coH} is a Lie subalgebra of A. □

For M and N two objects of ${}_{A,\mathcal{G}}\mathcal{M}^H$, we denote by ${}_{A,\mathcal{G}}Hom^H(M,N)$ the vector space of $(A,\mathcal{G},H)$-comodule homomorphisms from M to N.

For every object M of ${}_{A,\mathcal{G}}\mathcal{M}^H$, we have

$$ {}_{A,\mathcal{G}}Hom^H(A,M) = M^{AcoH} \quad \forall M \in {}_{A,\mathcal{G}}\mathcal{M}^H. $$

Lemma 3. *Assume the antipode of H is bijective. Let M be an $(A,\mathcal{G},H)$-comodule. Then*

(i) $M^{\mathcal{G}}$ is an H-subcomodule of M;

(ii) $A^{\mathcal{G}}$ is a $(\mathcal{G},H)$-subcomodule algebra of A: the $\mathcal{G}$-action is trivial;

(iii) $A^{\mathcal{G}coH}$ is a $(\mathcal{G},H)$-subcomodule algebra of $A^{\mathcal{G}}$: the $\mathcal{G}$-action and the H-coaction are trivial;

(iv) $M^{\mathcal{G}coH}$ is an $A^{\mathcal{G}coH}$-submodule of M: the $\mathcal{G}$-action and the H-coaction are trivial.

Proof. (i) Let $m \in M^{\mathcal{G}}$, $X \in \mathcal{G}$, $\rho(m) = m_0 \otimes m_1$ and $\rho(X) = X_0 \otimes X_1$. Using the relation (7), we have

$$ (X \diamond m_0) \otimes m_1 = (1_A \otimes \mathcal{S}_H^{-1}(X_1)).[(X_0 \diamond m)_0 \otimes (X_0 \diamond m)_1]. $$

Each $X_0 \diamond m$ is equal to 0 since $m \in M^{\mathcal{G}}$. It follows that the right term is equal to 0. We deduce that $(X \diamond m_0) \otimes m_1 = 0$. Now taking the summands $\{m_1\}$ to be linearly independant, we have $X \diamond m_0 = 0$ for each summand m_0, that is, each summand m_0 belongs to $M^{\mathcal{G}}$. So $M^{\mathcal{G}}$ is an H-subcomodule of M.

(ii) By (i), $A^{\mathcal{G}}$ is an H-subcomodule of A. The relation (4) implies that $A^{\mathcal{G}}$ is a subalgebra of A. Clearly, $A^{\mathcal{G}}$ is a $\mathcal{G}$-submodule of A.

(iii) It is clear that $A^{\mathcal{G}coH}$ is an H-subcomodule and a $\mathcal{G}$-submodule of A. The relation (4) implies that $A^{\mathcal{G}coH}$ is a subalgebra of A. □

If H has a bijective antipode, we have $M^{\mathcal{G}coH} = (M^{\mathcal{G}})^{coH}$: this makes sense since by Lemma 3 (i), $M^{\mathcal{G}}$ is an H-comodule.

2. The main results

In this section, H is a Hopf algebra with a bijective antipode $\mathcal{S}_H$, $\mathcal{G}$ is an H-comodule Lie algebra and A is a $(\mathcal{G},H)$-comodule algebra.

We denote by ${}_{\mathcal{G}}\mathcal{M}$ the category of $\mathcal{G}$-modules with $\mathcal{G}$-linear maps. ${}_{\mathcal{G}}\mathcal{M}^H$ is a subcategory of ${}_{\mathcal{G}}\mathcal{M}$.

Lemma 4. *(i) Let N be a $\mathcal{G}$-module. Then $N\otimes H$ is a $(\mathcal{G},H)$-comodule: the H-coaction is $id_N\otimes\Delta_H$, while the $\mathcal{G}$-action is given by*

$$X\diamond(n\otimes h)=(X_0\diamond n)\otimes(X_1h);\quad X\in\mathcal{G},n\in N,h\in H.$$

(ii) If furthermore, N is an $A\#U(\mathcal{G})$-module and H is commutative, then $N\otimes H$ is an $(A,\mathcal{G},H)$-comodule : the A-action is given by

$$a(n\otimes h)=(a_0n)\otimes(a_1h),\quad a\in A,n\in N,h\in H.$$

Proof. (i) We have

$$\begin{aligned}
[X,X']\diamond(n\otimes h)&=([X,X']_0\diamond n)\otimes([X,X']_1h)\\
&=([X_0,X'_0]\diamond n)\otimes((X_1X'_1)h)\\
&=[(X_0\diamond(X'_0\diamond n))-(X'_0\diamond(X_0\diamond n))]\otimes((X_1X'_1)h)\\
&=[(X_0\diamond(X'_0\diamond n))\otimes((X_1X'_1)h)]\\
&\qquad\qquad-[(X'_0\diamond(X_0\diamond n))\otimes((X'_1X_1)h)]\\
&=[(X_0\diamond(X'_0\diamond n))\otimes X_1(X'_1h)]\\
&\qquad\qquad-[(X'_0\diamond(X_0\diamond n))\otimes X'_1(X_1h)]\\
&=[X\diamond(X'\diamond(n\otimes h))]-[X'\diamond(X\diamond(n\otimes h))],
\end{aligned}$$

and the relation (3) is satisfied. We have

$$\begin{aligned}
(X\diamond(n\otimes h))_0\otimes(X\diamond(n\otimes h))_1&=[(X_0\diamond n)\otimes(X_1h)]_0\\
&\qquad\otimes[(X_0\diamond n)\otimes(X_1h)]_1\\
&=(X_0\diamond n)\otimes(X_1h)_1\otimes(X_1h)_2\\
&=(X_0\diamond n)\otimes(X_{11}h_1)\otimes(X_{12}h_2)\\
&=(X_{00}\diamond n)\otimes(X_{01}h_1)\otimes(X_1h_2)\\
&=(X_0\diamond(n\otimes h_1))\otimes(X_1h_2)\\
&=(X_0\diamond(n\otimes h)_0)\otimes(X_1(n\otimes h)_1),
\end{aligned}$$

and the relation (7) is satisfied.

(ii) By (i), the relations (3) and (7) are satisfied. $N\otimes H$ is an A-module for the given A-action. The relation (2) is satisfied: the proof is similar to that given for the relation (7) in part (i). We have

$$\begin{aligned}
X\diamond(a'(n\otimes h))&=X\diamond[(a'_0n)\otimes(a'_1h)]\\
&=[X_0\diamond(a'_0n)]\otimes[X_1(a'_1h)]\\
&=[(X_0\diamond a'_0)n]\otimes(X_1(a'_1h))+[(a'_0(X_0\diamond n))\otimes(X_1(a'_1h))]\\
&=[(X_0\diamond a'_0)n]\otimes((X_1a'_1)h)+[(a'_0(X_0\diamond n))\otimes(a'_1(X_1h))]\\
&=[(X_0\diamond a'_0)n+a'_0(X_0\diamond n)]\otimes(X_1(a'_1h))
\end{aligned}$$

$$= [(X\diamond a')_0 n]\otimes((X\diamond a')_1 h) + a'[(X_0\diamond n)\otimes(X_1 h)]$$
$$= [(X\diamond a')(n\otimes h) + a'[X\diamond(n\otimes h)]$$

and the relation (5) is satisfied. □

From Lemma 4 (ii), if H is commutative, then $A\otimes H$ is an $(A,\mathcal{G},H)$-comodule: the H-coaction is $id_A\otimes\Delta_H$, the A-action and the $\mathcal{G}$-action are given by

$$a(a'\otimes h) = (a_0 a')\otimes(a_1 h),\quad X\diamond(a\otimes h) = (X_0\diamond a)\otimes(X_1 h),$$
$$X\in\mathcal{G}, a, a'\in A, h\in H.$$

For M and N in ${}_{A\#U(\mathcal{G})}\mathcal{M}$ (resp., ${}_{\mathcal{G}}\mathcal{M}$), we denote by ${}_{A\#U(\mathcal{G})}Hom(M,N)$ (resp., ${}_{\mathcal{G}}Hom(M,N)$) the vector space of $A\#U(\mathcal{G})$-linear (resp., $\mathcal{G}$-linear) maps from M to N.

Lemma 5. *(i) Let M be a $(\mathcal{G},H)$-comodule and N a $\mathcal{G}$-module. We have a k-linear isomorphism*

$$\gamma : {}_{\mathcal{G}}Hom^H(M, N\otimes H)\to {}_{\mathcal{G}}Hom(M,N)$$

for all $M\in {}_{\mathcal{G}}\mathcal{M}^H$ and $N\in {}_{\mathcal{G}}\mathcal{M}$. This map γ is defined by $\gamma(f) = (id_N\otimes\epsilon_H)\circ f$. The inverse γ' of γ is given by $\gamma'(g) = (g\otimes id_H)\circ\rho_M$.

(ii) Let H be commutative, M an $(A,\mathcal{G},H)$-comodule and N an $A\#U(\mathcal{G})$-module. We have a k-linear isomorphism

$$\gamma : {}_{A,\mathcal{G}}Hom^H(M, N\otimes H)\to {}_{A,\mathcal{G}}Hom(M,N)$$

for all $M\in {}_{A,\mathcal{G}}\mathcal{M}^H$ and $N\in {}_{A,\mathcal{G}}\mathcal{M}$. This map γ and its inverse γ' are defined as in (i).

Proof. (i) By Lemma 4 (i), $N\otimes H$ is a $(\mathcal{G},H)$-comodule. Choose $f\in {}_{\mathcal{G}}Hom^H(M,N\otimes H)$ and $m\in M$. Set $f(m) = \sum\limits_{i\in I}(n_i\otimes h_i)$ for some $n_i\in N$ and $h_i\in H$, where I a family set of indexes. Then $\gamma(f)(m) = \sum\limits_{i\in I}(n_i\epsilon_H(h_i))$. For $a\in A$, $m\in M$, we have

$$f(X\diamond m) = X\diamond f(m) = \sum_{i\in I}(X_0\diamond n_i)\otimes(X_1 h_i).$$

and

$$\gamma(f)(X\diamond m) = \sum_{i\in I}((X_0\diamond n_i)\epsilon_H(X_1 h_i))$$
$$= \sum_{i\in I}((X_0\diamond n_i)\epsilon_H(X_1)\epsilon_H(h_i))$$

$$
\begin{aligned}
&= \sum_{i\in I}(((X_0\epsilon_H(X_1))\diamond n_i)\epsilon_H(h_i))\\
&= \sum_{i\in I}((X\diamond n_i)\epsilon_H(h_i))\\
&= X\diamond(\sum_{i\in I}(n_i\epsilon_H(h_i)))\\
&= X\diamond(\gamma(f)(m)).
\end{aligned}
$$

Thus $\gamma(f)$ is $\mathcal{G}$-linear. Let $g \in {}_{\mathcal{G}}Hom(M,N)$. We have

$$
\begin{aligned}
\gamma'(g)(m)_0\otimes\gamma'(g)(m)_1 &= (g(m_0)\otimes m_1)_0\otimes(g(m_0)\otimes m_1)_1\\
&= g(m_0)\otimes m_{11}\otimes m_{12}\\
&= g(m_{00})\otimes m_{01}\otimes m_1\\
&= \gamma'(g)(m_0)\otimes m_1.
\end{aligned}
$$

So $\gamma'(g)$ is H-colinear. We also have

$$
\begin{aligned}
\gamma'(g)(X\diamond m) &= g((X\diamond m)_0)\otimes(X\diamond m)_1\\
&= g(X_0\diamond m_0)\otimes(X_1m_1)\\
&= (X_0\diamond g(m_0))\otimes(X_1m_1)\\
&= X\diamond(g(m_0)\otimes m_1)\\
&= X\diamond(\gamma'(g)(m)).
\end{aligned}
$$

So $\gamma'(g)$ is $\mathcal{G}$-linear. We have

$$
\begin{aligned}
[(\gamma\circ\gamma')(g)](m) &= [\gamma(\gamma'(g))](m)\\
&= (id_N\otimes\epsilon_H)[\gamma'(g)(m)]\\
&= (id_N\otimes\epsilon_H)[(g\otimes id_H)(m_0\otimes m_1)]\\
&= (id_N\otimes\epsilon_H)[g(m_0)\otimes m_1]\\
&= g(m_0)\otimes\epsilon_H(m_1) = g(m).
\end{aligned}
$$

So $\gamma\circ\gamma' = id_{{}_{\mathcal{G}}Hom(M,N)}$. We remark that

$$f(m)_0\otimes f(m)_1 = \sum_{i\in I}(n_i\otimes h_{i1}\otimes h_{i2}).$$

Applying this formula, we obtain

$$
\begin{aligned}
[(\gamma'\circ\gamma)(f)](m) &= [\gamma'(\gamma(f))](m)\\
&= [\gamma(f)\otimes id_H](m_0\otimes m_1)\\
&= [\gamma(f)(m_0)]\otimes m_1\\
&= (id_N\otimes\epsilon_H)(f(m_0))\otimes m_1
\end{aligned}
$$

$$\begin{aligned}
&= (id_N \otimes \epsilon_H)(f(m)_0) \otimes f(m)_1 \\
&= \sum_{i\in I}(id_N \otimes \epsilon_H)(n_i \otimes h_{i1}) \otimes h_{i2} \\
&= \sum_{i\in I}(n_i\epsilon_H(h_{i1})) \otimes h_{i2} \\
&= \sum_{i\in I}(n_i \otimes h_i) = f(m).
\end{aligned}$$

The fifth equality is true since f is H-colinear. So $\gamma'\circ\gamma = id_{\mathcal{G}Hom^H(M,N)}$. The proof of (i) is finished.

(ii) By Lemma 4 (ii), $N\otimes H$ is an $(A,\mathcal{G},H)$-comodule. One easily check that that $\gamma(f)$ and $\gamma'(g)$ are A-linear. □

The following result is an immediate consequence of Lemma 5.

Corollary 1. *(i) If N is an injective $\mathcal{G}$-module, then $N\otimes H$ is an injective $(\mathcal{G},H)$-comodule.*

(ii) Let H be commutative. If N is an injective $A\#U(\mathcal{G})$-module, then $N\otimes H$ is an injective $(A,\mathcal{G},H)$-comodule.

We arrive to the first main result of the paper.

Theorem 1. *Assume that there is an H-colinear map $\phi: H\to A^{\mathcal{G}}$ such that $\phi(1_H)=1_A$.*

(i) If H is commutative or if A is commutative and ϕ is an algebra map, then every $(A,\mathcal{G},H)$-comodule which is injective as a $\mathcal{G}$-module is an injective $(\mathcal{G},H)$-comodule;

(ii) If H and A are commutative, then every $(A,\mathcal{G},H)$-comodule which is injective as an $A\#U(\mathcal{G})$-module is an injective $(A,\mathcal{G},H)$-comodule.

Proof. (i) Let M be an $(A,\mathcal{G},H)$-comodule. Let us consider the k-linear map $\lambda: M\otimes H\to M$ defined by

$$\lambda(m\otimes h) = \phi(hS_H^{-1}(m_1))m_0 \quad \forall m\in M,\ \ h\in H.$$

Clearly, λ is a k-linear map. We have

$$\begin{aligned}
(\lambda\circ\rho_M)(m) &= \lambda(\rho_M(m)) = \lambda(m_0\otimes m_1) \\
&= \phi(m_1S_H^{-1}(m_{01}))m_{00} \\
&= \phi(m_{12}S_H^{-1}(m_{11}))m_{(0)} \\
&= \phi(\epsilon_H(m_1)1_H)m_{(0)}
\end{aligned}$$

$$
\begin{aligned}
&= \phi(1_H)(\epsilon_H(m_{(-1)})m_{(0)}) \\
&= \phi(1_H)m = 1_A m = m \quad \text{since} \quad \phi(1_H) = 1_A.
\end{aligned}
$$

Therefore $\lambda \circ \rho_M = id_M$. We have

$$
\begin{aligned}
\lambda(m\otimes h)_0 \otimes \lambda(m\otimes h)_1 &= [\phi(h\mathcal{S}_H^{-1}(m_1))m_0]_0 \otimes [\phi(h\mathcal{S}_H^{-1}(m_1))m_0]_1 \\
&= [\phi(h\mathcal{S}_H^{-1}(m_1))_0 m_{00}] \otimes [\phi(h\mathcal{S}_H^{-1}(m_1))_1 m_{01}] \\
&= [\phi((h\mathcal{S}_H^{-1}(m_1))_0)m_{00}] \otimes [(h\mathcal{S}_H^{-1}(m_1))_1 m_{01}] \\
&= [\phi(h_1\mathcal{S}_H^{-1}(m_1)_1)m_{00}] \otimes [h_2\mathcal{S}_H^{-1}(m_1)_2 m_{01}] \\
&= [\phi(h_1\mathcal{S}_H^{-1}(m_{12}))m_{00}] \otimes [h_2\mathcal{S}_H^{-1}(m_{11})m_{01}] \\
&= [\phi(h_1\mathcal{S}_H^{-1}(m_3))m_0] \otimes [h_2\mathcal{S}_H^{-1}(m_2)m_1] \\
&= [\phi(h_1\mathcal{S}_H^{-1}(m_2))m_0] \otimes [h_2\epsilon_H(m_1)] \\
&= [\phi(h_1\mathcal{S}_H^{-1}(m_1))m_0] \otimes h_2 \\
&= \lambda(m\otimes h_1)\otimes h_2 \\
&= \lambda((m\otimes h)_0)\otimes (m\otimes h)_1
\end{aligned}
$$

and λ is H-colinear. We have

$$
\begin{aligned}
\lambda(X\diamond(m\otimes h)) &= \lambda((X_0\diamond m)\otimes(X_1 h)) \\
&= \phi[(X_1 h)\mathcal{S}_H^{-1}((X_0\diamond m)_1)](X_0\diamond m)_0 \\
&= \phi[X_1 h\mathcal{S}_H^{-1}(X_{01}m_1)](X_{00}\diamond m_0) \\
&= \phi[X_2 h\mathcal{S}_H^{-1}(X_1 m_1)](X_0\diamond m_0) \\
&= \phi[X_2 h\mathcal{S}_H^{-1}(m_1)\mathcal{S}_H^{-1}(X_1)](X_0\diamond m_0).
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
X\diamond[\lambda(m\otimes h)] &= X\diamond[\phi(h\mathcal{S}_H^{-1}(m_1))m_0] \\
&= [X\diamond(\phi(h\mathcal{S}_H^{-1}(m_1)))]m_0 + [\phi(h\mathcal{S}_H^{-1}(m_1))](X\diamond m_0) \\
&= \phi(h\mathcal{S}_H^{-1}(m_1))(X\diamond m_0),
\end{aligned}
$$

since $[X\diamond\phi(h\mathcal{S}_H^{-1}(m_1))]m_0 = 0$. From the computations above, if H is commutative, we have

$$\lambda(X\diamond(m\otimes h)) = X\diamond\lambda(m\otimes h).$$

It follows that λ is $\mathcal{G}$-linear. Thus λ is a homomorphism of $(\mathcal{G}, H)$-comodules. Again, from the computations above, if ϕ is an algebra map and A is commutative, we have

$$\lambda(X\diamond(m\otimes h)) = X\diamond\lambda(m\otimes h).$$

It follows that λ is $\mathcal{G}$-linear. Thus λ is a homomorphism of $(\mathcal{G},H)$-comodules. By Corollary 1 (i), $M\otimes H$ is an injective $(\mathcal{G},H)$-comodule. It is easy to see that the comodule structure map ρ_M is H-colinear and $\mathcal{G}$-linear for the given H-coaction and $\mathcal{G}$-action on $M\otimes H$; that is, ρ_M is a homomorphism of $(\mathcal{G},H)$-comodules. It is well known that $\rho_M : M\to M\otimes H$ is an injective map. So M is a direct summand of $M\otimes H$ as a $(\mathcal{G},H)$-comodule. This implies that M is an injective $(\mathcal{G},H)$-comodule, being a direct summand of the injective $(\mathcal{G},H)$-comodule $M\otimes H$.

(ii) A similar computation as above shows that

$$\lambda(a(m\otimes h)) = \phi[a_2 h\mathcal{S}_H^{-1}(m_1)\mathcal{S}_H^{-1}(a_1)](a_0 m_0).$$

We have

$$a\lambda(m\otimes h) = a[\phi(h\mathcal{S}_H^{-1}(m_1))m_0] = \phi(h\mathcal{S}_H^{-1}(m_1))(am_0),$$

since A is commutative. From these two relations, and since H is commutative, we get

$$\lambda(a(m\otimes h)) = a\lambda(m\otimes h).$$

It follows that λ is A-linear. Thus λ is a homomorphism of $(A,\mathcal{G},H)$-comodules. By Corollary 1 (ii), $M\otimes H$ is an injective $(A,\mathcal{G},H)$-comodule. It is easy to see that the comodule structure map ρ_M is A-linear for the given A-action on $M\otimes H$. Thus, ρ_M is a homomorphism of $(A,\mathcal{G},H)$-comodules. We also know that $\rho_M : M\to M\otimes H$ is an injective map. So M is a direct summand of $M\otimes H$ as an $(A,\mathcal{G},H)$-comodule. It follows that M is an injective $(A,\mathcal{G},H)$-comodule, being a direct summand of the injective $(A,\mathcal{G},H)$-comodule $M\otimes H$. $\square$

Note that in the first part of Theorem 1, we have assumed that H is commutative or A is commutative and ϕ is an algebra map to ensure that the map λ is $\mathcal{G}$-linear. In the second part of Theorem 1, we have assumed that H and A are commutative to ensure that λ is A-linear. By Lemma 4 (ii), the commutativity of H is needed for $M\otimes H$ to be an $(A,\mathcal{G},H)$-comodule.

From Theorem 1, we get the following corollary whose part (i) is a left-right hand version of a result of Doi [3, Theorem 1].

Corollary 2. *Let A be an H-comodule algebra, and M an (A,H)-Hopf module. Assume that there is an H-colinear map $\phi : H\to A$ such that $\phi(1_H)=1_A$.*

(i) Then every (A,H)-Hopf module is an injective H-comodule.

(ii) Let H be commutative. Then every (A,H)-Hopf module which is injective as an A-module is an injective (A,H)-Hopf module.

Proof. Take $\mathcal{G}=k$ with a trivial Lie algebra structure.

(i) Any associative algebra A is a $\mathcal{G}$-module. Thus, $A=A^{\mathcal{G}}$. It is easy to see that any H-comodule algebra A is a $(\mathcal{G},H)$-comodule algebra. Any H-comodule is a $(\mathcal{G},H)$-comodule. Any (A,H)-Hopf module is an $(A,\mathcal{G},H)$-comodule. Any homomorphism of H-comodules is a homomorphism of $(\mathcal{G},H)$-comodules. The map λ in the proof of Theorem 1 (i) is always $\mathcal{G}$-linear (we don't need H to be commutative or ϕ to be an algebra map). The result follows from Theorem 1 (i).

(ii) Any homomorphism of (A,H)-Hopf modules is a homomorphism of $(A,\mathcal{G},H)$-comodules. □

For our second main result we need some preparatory results.

Lemma 6. *Set $B=A^{\mathcal{G}coH}$. Let M be a B-module. Then $A\otimes_B M$ is an $(A,\mathcal{G},H)$-comodule: The associative A-action is the natural one, the $\mathcal{G}$-action is given by*

$$X\diamond(a\otimes_B m)=(X\diamond a)\otimes_B m\quad \forall X\in\mathcal{G},a\in A,m\in M,$$

the coaction is given by

$$\rho_{A\otimes_B M}(a\otimes_B m)=a_0\otimes_B m\otimes a_1,\quad \forall a\in A,m\in M.$$

Proof. It is well known that the coaction is well defined. Let $a,a'\in A$ and $m\in M$. We have

$$\begin{aligned}\rho_{A\otimes_B M}(a'(a\otimes_B m))&=\rho_{A\otimes_B M}((a'a)\otimes_B m)\\&=(a'a)_0\otimes_B m\otimes(a'a)_1\\&=(a'_0a_0)\otimes_B m\otimes(a'_1a_1)\\&=a'_0(a_0\otimes_B m)\otimes(a'_1a_1)\\&=a'_0(a\otimes_B m)_0\otimes(a'_1(a\otimes_B m)_1);\end{aligned}$$

so the relation (2) is satisfied, i.e. $A\otimes_B M$ is an (A,H)-Hopf module.

Let $X\in\mathcal{G}$, $a\in A,b\in B$ and $m\in M$. We get

$$\begin{aligned}X\diamond((ab)\otimes_B m)&=(X\diamond(ab))\otimes_B m\\&=[(X\diamond a)b+a(X\diamond b)]\otimes_B m\\&=[(X\diamond a)b+0]\otimes_B m\\&=(X\diamond a)b\otimes_B m\\&=(X\diamond a)\otimes_B(bm)\\&=X\diamond(a\otimes_B(bm)).\end{aligned}$$

So the $\mathcal{G}$ action is well defined.

We have

$$\begin{aligned} X \diamond [a'(a \otimes_B m)] &= X \diamond (a'a \otimes_B m) \\ &= (X \diamond (a'a)) \otimes_B m \\ &= [(X \diamond a')a + a'(X \diamond a)] \otimes_B m \\ &= [(X \diamond a')a] \otimes_B m + [a'(X \diamond a)] \otimes_B m \\ &= (X \diamond a')(a \otimes_B m) + a'[(X \diamond a) \otimes_B m] \\ &= (X \diamond a')(a \otimes_B m) + a'[X \diamond (a \otimes_B m]; \end{aligned}$$

so the relation (5) is satisfied, i.e. $A \otimes_B M$ is an $A\#U(\mathcal{G})$-module.

Now we have

$$\begin{aligned} \rho_{A\otimes_B M}(X \diamond (a \otimes_B m)) &= \rho_{A\otimes_B M}[(X \diamond a) \otimes_B m] \\ &= (X \diamond a)_0 \otimes_B m \otimes (X \diamond a)_1 \\ &= (X_0 \diamond a_0) \otimes_B m \otimes (X_1 a_1) \\ &= (X_0 \diamond (a_0 \otimes_B m)) \otimes (X_1 a_1) \\ &= [X_0 \diamond (a \otimes_B m)_0] \otimes [X_1 (a \otimes_B m)_1]; \end{aligned}$$

so the relation (7) is satisfied. □

We deduce from Lemma 6 that $A \otimes_B M^{\mathcal{G}coH}$ is an $(A,\mathcal{G},H)$-comodule for every $(A,\mathcal{G},H)$-comodule M.

Lemma 7. *Set $B = A^{\mathcal{G}coH}$. Let M be an $(A,\mathcal{G},H)$-comodule. Then the k-linear map $\alpha : A \otimes_B M^{\mathcal{G}coH} \to M$; $a \otimes_B m \mapsto am$ is a homomorphism of $(A,\mathcal{G},H)$-comodules.*

Proof. Clearly, α is well defined and left A-linear. Let $X \in \mathcal{G}$, $a \in A$ and $m \in M^{\mathcal{G}coH}$. We have

$$\begin{aligned} \alpha(X \diamond (a \otimes_B m)) &= \alpha((X \diamond a) \otimes_B m) = (X \diamond a)m \\ &= (X \diamond a)m + a(X \diamond m) = X \diamond (am) \\ &= X \diamond \alpha(a \otimes_B m). \end{aligned}$$

The third equality is true since $X \diamond m = 0$; m being an element of $M^{\mathcal{G}coH}$. To get the fourth equality, we use the relation (5). Thus α is a $\mathcal{G}$-module map. Finally, we have

$$\begin{aligned} \alpha(a \otimes_B m)_0 \otimes \alpha(a \otimes_B m)_1 &= (am)_0 \otimes (am)_1 \\ &= (a_0 m_0) \otimes (a_1 m_1) = (a_0 m) \otimes a_1 \\ &= \alpha((a \otimes_B m)_0) \otimes (a \otimes_B m)_1. \end{aligned}$$

So α is an H-colinear map. □

Assume that there is a right H-colinear map $\phi : H \to A$ such that $\phi(1_H) = 1_A$. For any $M \in {}_{A,\mathcal{G}}\mathcal{M}^H$, let us consider the k-linear map

$$p_M : M \to M; m \mapsto \phi(\mathcal{S}_H^{-1}(m_1))m_0.$$

Thus for any $M \in {}_{A,\mathcal{G}}\mathcal{M}^H$, we have $p_M(m) = m$ for every $m \in M^{coH}$.

Lemma 8. *Let $M \in {}_{A,\mathcal{G}}\mathcal{M}^H$. Assume that there is a right H-colinear map $\phi : H \to A$ such that $\phi(1_H) = 1_A$. We have*

(i) $p_M(M) = M^{coH}$;

(ii) $p_M \circ p_M = p_M$, and p_M is a projection map.

Proof. (i) Let $m' \in p_M(M)$. Then there is $m \in M$ such that $m' = \phi(\mathcal{S}_H^{-1}(m_1))m_0$. We have

$$\begin{aligned}
[\phi(\mathcal{S}_H^{-1}(m_1))m_0]_0 \otimes [\phi(\mathcal{S}_H^{-1}(m_1))m_0]_1 \\
&= [\phi(\mathcal{S}_H^{-1}(m_1))_0 m_{00}] \otimes [\phi(\mathcal{S}_H^{-1}(m_1))_1 m_{01}] \\
&= [\phi(\mathcal{S}_H^{-1}(m_1))_0 m_{00}] \otimes (\mathcal{S}_H^{-1}(m_1)_1 m_{01}) \\
&= \phi(\mathcal{S}_H^{-1}(m_1)_1) m_{00} \otimes \mathcal{S}_H^{-1}(m_1)_2 m_{01} \\
&= \phi(\mathcal{S}_H^{-1}(m_{12})) m_{00} \otimes \mathcal{S}_H^{-1}(m_{11}) m_{01} \\
&= \phi(\mathcal{S}_H^{-1}(m_3)) m_0 \otimes \mathcal{S}_H^{-1}(m_2) m_1 \\
&= \phi(\mathcal{S}_H^{-1}(m_1)) m_0 \otimes 1_H.
\end{aligned}$$

So $m' \in M^{coH}$, and $p_M(M) \subseteq M^{coH}$. It is clear that $M^{coH} \subseteq p_M(M)$ since $p_M(m) = m$ for every $m \in M^{coH}$.

(ii) Let $m \in M$. We have

$$\begin{aligned}
p_M(p_M(m)) &= p_M(\phi(\mathcal{S}_H^{-1}(m_1))m_0) \\
&= \phi(\mathcal{S}_H^{-1}((\phi(\mathcal{S}_H^{-1}(m_1))m_0)_1))(\phi(\mathcal{S}_H^{-1}(m_1))m_0)_0 \\
&= \phi(\mathcal{S}_H^{-1}(\phi(\mathcal{S}_H^{-1}(m_1))_1 m_{01}))[\phi(\mathcal{S}_H^{-1}(m_1))_0 m_{00}] \\
&= \phi(\mathcal{S}_H^{-1}(\mathcal{S}_H^{-1}(m_1)_2 m_{01}))[\phi(\mathcal{S}_H^{-1}(m_1)_1) m_{00}] \\
&= \phi(\mathcal{S}_H^{-1}(\mathcal{S}_H^{-1}(m_{11}) m_{01}))[\phi(\mathcal{S}_H^{-1}(m_{12})) m_{00}] \\
&= \phi(\mathcal{S}_H^{-1}(\mathcal{S}_H^{-1}(m_2) m_1))[\phi(\mathcal{S}_H^{-1}(m_3)) m_0] \\
&= \phi(\mathcal{S}_H^{-1}(\epsilon_H(m_1)1_H))[\phi(\mathcal{S}_H^{-1}(m_2)) m_0] \\
&= \phi(1_H)[\phi(\mathcal{S}_H^{-1}(m_1)) m_0] \\
&= 1_A[\phi(\mathcal{S}_H^{-1}(m_1)) m_0] \\
&= \phi(\mathcal{S}_H^{-1}(m_1)) m_0 = p_M(m).
\end{aligned}$$

The result also follows from (i) since $p_M(m) \in M^{coH}$ for every $m \in M$ and $p_M(m') = m'$ for every $m' \in M^{coH}$. $\square$

Let $M \in {}_{A,\mathcal{G}}\mathcal{M}^H$. For every $X \in \mathcal{G}$ and $m \in M$, put

$$X \diamond' m = p_M(X \diamond m).$$

Let us record in the following lemma some properties of the projector p_M and the Lie action $\diamond'$.

Lemma 9. *Let A be a commutative $(\mathcal{G}, H)$-comodule algebra. Let $M \in {}_{A,\mathcal{G}}\mathcal{M}^H$ Assume that there is a right H-colinear algebra map $\phi \colon H \to A^{\mathcal{G}}$. Let $X, Y \in \mathcal{G}$, $a \in A$ and $m \in M$. Then*

(i) $p_M(am) = p_A(a)p_M(m)$;

(ii) $X \diamond' p_M(m) = p_M(X \diamond m) = X \diamond' m$;

(iii) $[X, Y] \diamond' m = X \diamond' (Y \diamond' m) - Y \diamond' (X \diamond' m)$;

(iv) $X \diamond p_M(m) = \phi(X_1)(X_0 \diamond' p_M(m))$;

(v) $\phi(m_1)p_M(m_0) = m$.

Proof. (i) Let $a \in A$ and $m \in M$. Then we have

$$\begin{aligned}
p_M(am) &= \phi(\mathcal{S}_H^{-1}((am)_1))(am)_0 \\
&= \phi(\mathcal{S}_H^{-1}(a_1 m_1))(a_0 m_0) \\
&= \phi(\mathcal{S}_H^{-1}(m_1)\mathcal{S}_H^{-1}(a_1))(a_0 m_0) \\
&= \phi(\mathcal{S}_H^{-1}(m_1))\phi(\mathcal{S}_H^{-1}(a_1))(a_0 m_0) \\
&= [\phi(\mathcal{S}_H^{-1}(a_1))a_0][\phi(\mathcal{S}_H^{-1}(m_1))m_0] \\
&= p_A(a)p_M(m).
\end{aligned}$$

The fifth equality is true since A is commutative.

(ii) Let $X \in \mathcal{G}$ and $m \in M$. Then we have

$$\begin{aligned}
X \diamond' p_M(m) &= p_M(X \diamond p_M(m)) \\
&= p_M[X \diamond (\phi(\mathcal{S}_H^{-1}(m_1))m_0)] \\
&= p_M[(X \diamond \phi(\mathcal{S}_H^{-1}(m_1)))m_0 + \phi(\mathcal{S}_H^{-1}(m_1))(X \diamond m_0)] \\
&= p_M[0 + \phi(\mathcal{S}_H^{-1}(m_1))(X \diamond m_0)] \\
&= p_M[\phi(\mathcal{S}_H^{-1}(m_1))(X \diamond m_0)] \\
&= \phi[\mathcal{S}_H^{-1}[\phi(\mathcal{S}_H^{-1}(m_1))(X \diamond m_0)_1]][\phi(\mathcal{S}_H^{-1}(m_1))(X \diamond m_0)]_0 \\
&= \phi[\mathcal{S}_H^{-1}[\phi(\mathcal{S}_H^{-1}(m_1))_1 X_1 m_{01}]][\phi(\mathcal{S}_H^{-1}(m_1))_0(X_0 \diamond m_{00})] \\
&= \phi[\mathcal{S}_H^{-1}[\mathcal{S}_H^{-1}(m_1)_2 X_1 m_{01}]][\phi(\mathcal{S}_H^{-1}(m_1)_1)(X_0 \diamond m_{00})]
\end{aligned}$$

$$\begin{aligned}
&= \phi[\mathcal{S}_H^{-1}[\mathcal{S}_H^{-1}(m_{11})X_1m_{01}]][\phi(\mathcal{S}_H^{-1}(m_{12}))(X_0 \diamond m_{00})] \\
&= \phi[\mathcal{S}_H^{-1}[\mathcal{S}_H^{-1}(m_2)X_1m_1]][\phi(\mathcal{S}_H^{-1}(m_3))(X_0 \diamond m_0)] \\
&= \phi[\mathcal{S}_H^{-1}(X_1m_1)\mathcal{S}_H^{-1}(\mathcal{S}_H^{-1}(m_2))]\phi[\mathcal{S}_H^{-1}(m_3)](X_0 \diamond m_0) \\
&= \phi[\mathcal{S}_H^{-1}(X_1m_1)]\phi[\mathcal{S}_H^{-1}(\mathcal{S}_H^{-1}(m_2))]\phi[\mathcal{S}_H^{-1}(m_3)](X_0 \diamond m_0) \\
&= \phi[\mathcal{S}_H^{-1}(X_1m_1)]\phi[\mathcal{S}_H^{-1}(\mathcal{S}_H^{-1}(m_2))\mathcal{S}_H^{-1}(m_3)](X_0 \diamond m_0)] \\
&= \phi[\mathcal{S}_H^{-1}(X_1m_1)]\phi[\mathcal{S}_H^{-1}(m_3\mathcal{S}_H^{-1}(m_2))](X_0 \diamond m_0)] \\
&= \phi[\mathcal{S}_H^{-1}(X_1m_1)]\phi[\mathcal{S}_H^{-1}(\epsilon_H(m_2)1_H)](X_0 \diamond m_0) \\
&= \phi[\mathcal{S}_H^{-1}(X_1m_1\epsilon_H(m_2))]\phi[\mathcal{S}_H^{-1}(1_H)](X_0 \diamond m_0) \\
&= \phi[\mathcal{S}_H^{-1}(X_1m_1\epsilon_H(m_2))]\phi(1_H)(X_0 \diamond m_0) \\
&= \phi[\mathcal{S}_H^{-1}(X_1m_1)]1_A(X_0 \diamond m_0) \\
&= \phi[\mathcal{S}_H^{-1}(X_1m_1)](X_0 \diamond m_0) \\
&= \phi[\mathcal{S}_H^{-1}((X \diamond m)_1)](X \diamond m)_0 \\
&= p_M(X \diamond m) = X \diamond' m.
\end{aligned}$$

To get the third equality, we use the relation (5). The fourth equality is true since $Im\phi \subseteq A^{\mathcal{G}}$. The sixth equality follows from the definition of p_M. To get the eighth equality, we use the colinearity of ϕ. The twelfth and thirteenth equalities follow from the algebra map structure of ϕ. The fourteenth equality is true since the antipode is an anti-algebra map.

(iii) Let $X, Y \in \mathcal{G}$ and $m \in M$. Then we have

$$\begin{aligned}
[X,Y] \diamond' m &= [X,Y] \diamond' p_M(m) \\
&= p_M([X,Y] \diamond p_M(m)) \\
&= p_M[X \diamond (Y \diamond p_M(m)) - Y \diamond (X \diamond p_M(m))] \\
&= p_M[X \diamond (Y \diamond p_M(m))] - p_M[Y \diamond (X \diamond p_M(m))] \\
&= X \diamond' (Y \diamond p_M(m)) - Y \diamond' (X \diamond p_M(m)) \\
&= X \diamond' p_M(Y \diamond p_M(m)) - Y \diamond' p_M(X \diamond p_M(m)) \\
&= X \diamond' (Y \diamond' p_M(m)) - Y \diamond' (X \diamond' p_M(m)) \\
&= X \diamond' (Y \diamond' m) - Y \diamond' (X \diamond' m).
\end{aligned}$$

The first equality follows from (ii). We have used the definition of $\diamond'$ in the second, the fifth and the seventh equalities. To get the sixth and eighth equalities, we have used (ii). The third equality is true since M is a $\mathcal{G}$-module under $\diamond$.

(iv) Let $X \in \mathcal{G}$ and $m \in M$. Then we have

$$\begin{aligned}
\phi(X_1)(X_0 \diamond' p_M(m)) &= \phi(X_1)(X_0 \diamond' m)\\
&= \phi(X_1)p_M(X_0 \diamond m)\\
&= \phi(X_1)\phi(\mathcal{S}_H^{-1}((X_0 \diamond m)_1))(X_0 \diamond m)_0\\
&= \phi(X_1)\phi(\mathcal{S}_H^{-1}(X_{01}m_1))(X_{00} \diamond m_0)\\
&= \phi(X_1\mathcal{S}_H^{-1}(X_{01}m_1))(X_{00} \diamond m_0)\\
&= \phi(X_2\mathcal{S}_H^{-1}(X_1m_1))(X_0 \diamond m_0)\\
&= \phi(X_2\mathcal{S}_H^{-1}(m_1)\mathcal{S}_H^{-1}(X_1))(X_0 \diamond m_0)\\
&= \phi(X_2)\phi(\mathcal{S}_H^{-1}(m_1))\phi(\mathcal{S}_H^{-1}(X_1))(X_0 \diamond m_0)\\
&= \phi(X_2)\phi(\mathcal{S}_H^{-1}(X_1))\phi(\mathcal{S}_H^{-1}(m_1))(X_0 \diamond m_0)\\
&= \phi(X_2\mathcal{S}_H^{-1}(X_1))\phi(\mathcal{S}_H^{-1}(m_1))(X_0 \diamond m_0)\\
&= \phi(\epsilon_H(X_1)1_H)\phi(\mathcal{S}_H^{-1}(m_1))(X_0 \diamond m_0)\\
&= \phi(1_H)\phi(\mathcal{S}_H^{-1}(m_1))(X \diamond m_0)\\
&= 1_A\phi(\mathcal{S}_H^{-1}(m_1))(X \diamond m_0)\\
&= \phi(\mathcal{S}_H^{-1}(m_1))(X \diamond m_0)\\
&= X \diamond (\phi(\mathcal{S}_H^{-1}(m_1))m_0)\\
&= X \diamond p_M(m).
\end{aligned}$$

The first equality follows from (ii). The second equality uses the definition of $\diamond'$. To get the third equality, we have used the definition of p_M. The eighth and tenth equalities follow from the algebra map structure of ϕ. The ninth equality uses the commutativity of A. The fifteenth equality uses the relation (5) and the fact that each $X \diamond (\phi(\mathcal{S}_H^{-1}(m_1))$ is equal to 0.

(v) Let $m \in M$. Then we have

$$\begin{aligned}
\phi(m_1)p_M(m_0) &= \phi(m_1)\phi(\mathcal{S}_H^{-1}(m_{01}))m_{00}\\
&= \phi(m_1\mathcal{S}_H^{-1}(m_{01}))m_{00}\\
&= \phi(m_2\mathcal{S}_H^{-1}(m_1))m_0\\
&= \phi(\epsilon_H(m_1)1_H)m_0\\
&= \phi(\epsilon_H(m_1)1_H)m_0\\
&= \phi(1_H)m = 1_Am = m.
\end{aligned}$$

The second equality is true since ϕ is an algebra homomorphism. □

If A is commutative, it follows from Lemma 9 (iii) that every $(A, \mathcal{G}, H)$-comodule M is a $\mathcal{G}$-module under the new $\mathcal{G}$-action defined by

$$X \diamond' m = p_M(X \diamond m); a \in A, m \in M$$

with M^{coH} as a $\mathcal{G}$-submodule. Since the $(\mathcal{G}, H)$-comodule algebra A itself is an $(A, \mathcal{G}, H)$-comodule, A is a $\mathcal{G}$-module under the $\mathcal{G}$-action $\diamond'$ defined by

$$X \diamond' c = p_A(X \diamond c), \quad \forall X \in \mathcal{G}, \quad c \in A,$$

and A^{coH} is a $\mathcal{G}$-submodule of A under $\diamond'$. Note that M^{coH} and A^{coH} are not $\mathcal{G}$-modules under $\diamond$.

Lemma 10. *Let A be a commutative $(\mathcal{G}, H)$-comodule algebra. Assume that there is a right H-colinear algebra map $\phi : H \to A^{\mathcal{G}}$. Then*

(i) A^{coH} considered as a $\mathcal{G}$-module via $\diamond'$ is a left $U(\mathcal{G})$-module algebra, and we can form the algebra smash product $A^{coH} \# U(\mathcal{G})$;

(ii) for any $(A, \mathcal{G}, H)$-comodule M, M^{coH} is a left $A^{coH} \# U(\mathcal{G})$-module: the $\mathcal{G}$-module action is given by $\diamond'$.

Proof. (i) We know that A^{coH} is a subalgebra of A. We have mentioned above that A^{coH} is a $\mathcal{G}$-module via $\diamond'$. Let $a \in A$, and $b, b' \in A^{coH}$. We have

$$\begin{aligned} X \diamond' (bb') &= p_A(X \diamond (bb')) \\ &= p_A((X \diamond b)b' + b(X \diamond b')) \\ &= p_A((X \diamond b)b') + p_A(b(X \diamond b')) \\ &= p_A(X \diamond b)p_A(b') + p_A(b)p_A(X \diamond b')) \\ &= p_A(X \diamond b)b' + bp_A(X \diamond b')) \\ &= (X \diamond' b)b' + b(X \diamond' b'). \end{aligned}$$

The fourth equality uses Lemma 9 (i). The fifth equality is true since the elements of A^{coH} are invariant under p_A. It is well known that $X \diamond' 1_A = 0_A$ for all $a \in A$.

(ii) Set $B = A^{coH}$. It is well known that M^{coH} is a B-module and we have mentioned that it is a $\mathcal{G}$-module with the given $\mathcal{G}$- action. Let $X \in \mathcal{G}$, $b \in B$ and $m \in M^{coH}$. We have

$$\begin{aligned} X \diamond' (bm) &= p_M(X \diamond (bm)) \\ &= p_M((X \diamond b)m + b(X \diamond m)) \\ &= p_M((X \diamond b)m) + p_M(b(X \diamond m)) \\ &= p_A(X \diamond b)p_M(m) + p_A(b)p_M(X \diamond m) \end{aligned}$$

$$= p_A(X \diamond b)m + bp_M(X \diamond m)$$
$$= (X \diamond' b)m + b(X \diamond' m).$$

So the relation (5) is satisfied. The second equality uses the relation (5) for M under the $\mathcal{G}$-action $\diamond$. The fourth equality follows from Lemma 9 (i). The fifth equality is true since the elements of M^{coH} and A^{coH} are respectively invariant under p_M and p_A. The last equality uses the definition of $\diamond'$. $\square$

Let A be a commutative $(\mathcal{G}, H)$-comodule algebra and M an $(A, \mathcal{G}, H)$-comodule. Then M^{coH} is a trivial $\mathcal{G}$-module under $\diamond'$ if and only if $p_M(X \diamond m) = 0$ for all $X \in \mathcal{G}$ and $m \in M^{coH}$.

Lemma 11. *Let A be a commutative $(\mathcal{G}, H)$-comodule algebra. Assume that there is a right H-colinear algebra map $\phi : H \to A^{\mathcal{G}}$.*

(i) Let M be an object of ${}_{A,\mathcal{G}}\mathcal{M}^H$. If the $\mathcal{G}$-action $\diamond'$ on M^{coH} is trivial, then $M^{\mathcal{G}coH} = M^{coH}$.

(ii) If the $\mathcal{G}$-action $\diamond'$ on A^{coH} is trivial, then $A^{\mathcal{G}coH} = A^{coH}$.

Proof. (i) We know that $M^{\mathcal{G}coH} \subseteq M^{coH}$. Let $X \in \mathcal{G}$, and $m \in M^{coH}$. We have

$$X \diamond m = X \diamond p_M(m) = \phi(X_1)(X_0 \diamond' p_M(m)) = 0.$$

The second equality follows from Lemma 9 (iv). So m is an element of $M^{\mathcal{G}coH}$, that is, $M^{coH} \subseteq M^{\mathcal{G}coH}$. $\square$

We are now in the position to provide the Fundamental Theorem for $(A, \mathcal{G}, H)$-comodules.

Theorem 2. *Let A be a commutative $(\mathcal{G}, H)$-comodule algebra. Set $B = A^{\mathcal{G}coH}$. Let M be an $(A, \mathcal{G}, H)$-comodule. Assume that there is a right H-colinear algebra map $\phi : H \to A^{\mathcal{G}}$. Suppose M^{coH} and A^{coH} are trivial $\mathcal{G}$-modules under $\diamond'$. Then the k-linear map $\alpha : A \otimes_B M^{\mathcal{G}coH} \to M$; $a \otimes_B m \mapsto am$ is an isomorphism of $(A, \mathcal{G}, H)$-comodules.*

Proof. By Lemma 7, α is a homomorphism of $(A, \mathcal{G}, H)$-comodules. Let $X \in \mathcal{G}$ and $m \in M$. By Lemma 9 (iv), we have $X \diamond p_M(m) = \phi(X_1)(X_0 \diamond' p_M(m)) = 0$, since the $\mathcal{G}$-action $\diamond'$ on M^{coH} is trivial and $p_M(m) \in M^{coH}$. It follows that $p_M(m) \in M^{\mathcal{G}}$. Thus, $p_M(m) \in M^{\mathcal{G}coH}$. So we get a well-defined k-linear map

$$\beta : M \to A \otimes_B M^{\mathcal{G}coH}; m \mapsto \phi(m_1) \otimes_B p_M(m_0).$$

Let $m \in M$. We have

$$\begin{aligned}(\alpha \circ \beta)(m) &= \alpha(\phi(m_1) \otimes_B p_M(m_0)) \\ &= \phi(m_1)p_M(m_0) \\ &= \phi(m_1)\phi(\mathcal{S}_H^{-1}(m_{01}))m_{00} \\ &= \phi(m_1\mathcal{S}_H^{-1}(m_{01}))m_{00} \\ &= \phi(m_2\mathcal{S}_H^{-1}(m_1))m_0 \\ &= \phi(1_H\epsilon_H(m_1))m_0 \\ &= \phi(1_H)\epsilon_H(m_1)m_0 = 1_A m = m.\end{aligned}$$

So $\alpha \circ \beta = id_M$. Let $a \in A$ and $m \in M^{\mathcal{G}coH}$. We have

$$\begin{aligned}(\beta \circ \alpha)(a \otimes_B m) &= \beta(am) \\ &= \phi((am)_1) \otimes_B p_M((am)_0) \\ &= \phi((am)_1) \otimes_B \phi(\mathcal{S}_H^{-1}((am)_{01}))(am)_{00} \\ &= \phi((am)_2) \otimes_B \phi(\mathcal{S}_H^{-1}((am)_1))(am)_0 \\ &= \phi(a_2m_2) \otimes_B \phi(\mathcal{S}_H^{-1}(a_1m_1))(a_0m_0) \\ &= \phi(a_2 1_H) \otimes_B \phi(\mathcal{S}_H^{-1}(a_1 1_H))(a_0m) \\ &= \phi(a_2) \otimes_B \phi(\mathcal{S}_H^{-1}(a_1))(a_0m) \\ &= \phi(a_1) \otimes_B \phi(\mathcal{S}_H^{-1}(a_{01}))(a_{00}m) \\ &= \phi(a_1)[\phi(\mathcal{S}_H^{-1}(a_{01}))a_{00}] \otimes_B m \\ &= \phi(a_2)[\phi(\mathcal{S}_H^{-1}(a_1))a_0] \otimes_B m \\ &= \phi(a_2\mathcal{S}_H^{-1}(a_1))a_0 \otimes_B m \\ &= \phi(1_H\epsilon_H(a_1))a_0 \otimes_B m \\ &= \phi(1_H)\epsilon_H(a_1)a_0 \otimes_B m \\ &= 1_A a \otimes_B m = a \otimes_B m.\end{aligned}$$

So $\beta \circ \alpha = id_{A\otimes_B M^{\mathcal{G}coH}}$. Thus, α is a k-isomorphism with inverse β. □

Theorem 2 shows that there is a one-to-one correspondence (up to equivalence) between B-modules and $(A, \mathcal{G}, H)$-comodules. From Theorem 2, we get the following left-right hand version of a result of Doi [3, Theorem 3].

Corollary 3. *Let A be an H-comodule algebra, and M an (A, H)-Hopf module. Assume that there is a right H-colinear algebra map $\phi : H \to A$. Set $B = A^{coH}$. Then the k-linear map $\alpha : A \otimes_B M^{coH} \to M$; $a \otimes_B m \mapsto am$ is an isomorphism of (A, H)-Hopf modules.*

Proof. Take $\mathcal{G} = k$ with a trivial Lie algebra structure. Any vector space is considered as a $\mathcal{G}$-module via the trivial action. It follows that $A^{\mathcal{G}} = A$. Any H-comodule (algebra) A is a $(\mathcal{G}, H)$-comodule (algebra). Any (A, H)-Hopf module is an $(A, \mathcal{G}, H)$-comodule. Any homomorphism of (A, H)-Hopf modules is a homomorphism of $(A, \mathcal{G}, H)$-comodules. Since the $\mathcal{G}$-action on M is trivial, the $\mathcal{G}$-action $\diamond'$ on M is trivial. Thus Lemma 9 (ii), (iii), (iv) and Lemma 11 are true if A is commutative or not. We don't use Lemma 10 and Lemma 9 (i) in the proof of the theorem. $\square$

By Lemma 1, $Z(\mathcal{G})$ is an H-comodule Lie algebra. It is easy to see that A is a $(Z(\mathcal{G}), H)$-comodule algebra: the $Z(\mathcal{G})$-action is induced by that of $\mathcal{G}$. By Lemma 3, $A^{Z(\mathcal{G})}$ is a $(Z(\mathcal{G}), H)$-comodule algebra.

Corollary 4. *Let A be a commutative $(\mathcal{G}, H)$-comodule algebra. Let M be an $(A^{Z(\mathcal{G})}, Z(\mathcal{G}), H)$-comodule. Set $B = A^{Z(\mathcal{G})coH}$. Assume that there is a right H-colinear algebra map $\phi : H \to A^{Z(\mathcal{G})}$. Suppose M^{coH} is a trivial $Z(\mathcal{G})$-module under $\diamond'$. Then the k-linear map $\alpha : A^{Z(\mathcal{G})} \otimes_B M^{Z(\mathcal{G})coH} \to M$; $a \otimes_B m \mapsto am$ is an isomorphism of $(A^{Z(\mathcal{G})}, Z(\mathcal{G}), H)$-comodules.*

Proof. We want to show that $(A^{Z(\mathcal{G})})^{Z(\mathcal{G})coH} = A^{Z(\mathcal{G})coH}$. Let $a \in (A^{Z(\mathcal{G})})^{Z(\mathcal{G})coH}$. Then $a \in A^{Z(\mathcal{G})}$, is $Z(\mathcal{G})$-invariant and H-coinvariant. It follows that $a \in A$ is $Z(\mathcal{G})$-invariant, and H-coinvariant. Then $a \in A^{Z(\mathcal{G})coH}$. Let $a \in A^{Z(\mathcal{G})coH}$. Then $a \in A^{Z(\mathcal{G})}$ and $a \in A^{coH}$. So $a \in A$ and a is $Z(\mathcal{G})$-invariant. So $a \in A^{Z(\mathcal{G})}$ and a is $Z(\mathcal{G})$-invariant. We deduce that a is $Z(\mathcal{G})$-invariant in $A^{Z(\mathcal{G})}$. Thus $a \in (A^{Z(\mathcal{G})})^{Z(\mathcal{G})coH}$. So we get $(A^{Z(\mathcal{G})})^{Z(\mathcal{G})coH} = A^{Z(\mathcal{G})coH}$. In the same way, we have $(A^{Z(\mathcal{G})})^{Z(\mathcal{G})} = A^{Z(\mathcal{G})}$. We also have $X \diamond c = 0$ for all $X \in Z(\mathcal{G})$ and $c \in (A^{Z(\mathcal{G})})^{coH}$. So $X \diamond' c = 0$ for all $X \in Z(\mathcal{G})$ and $c \in (A^{Z(\mathcal{G})})^{coH}$. The result follows from Theorem 2. $\square$

Let A be a $(\mathcal{G}, H)$-comodule algebra. Let I be a vector subspace of A. We say that I is

(i) an H-ideal of A if I is an ideal of A and an H-subcomodule of A;

(ii) a $\mathcal{G}$-ideal of A if I is an ideal of A and a $\mathcal{G}$-submodule of A;

(iii) a $(\mathcal{G}, H)$-ideal of A if I is an H-ideal of A and a $\mathcal{G}$-ideal of A, that is, an ideal of A, an H-subcomodule of A and a $\mathcal{G}$-submodule of A.

We say that a $(\mathcal{G}, H)$-comodule algebra A is $(\mathcal{G}, H)$-simple if the only $(\mathcal{G}, H)$-ideals of A are 0 and A.

Lemma 12. *Let A be a commutative $(\mathcal{G}, H)$-simple $(\mathcal{G}, H)$-comodule algebra. Then $A^{\mathcal{G}coH}$ is a field.*

Proof. Let a be a nonzero element of $A^{\mathcal{G}coH}$. Then Aa is an ideal of A. For all $X \in \mathcal{G}$ and $u \in A$, we have

$$X \diamond (ua) = (X \diamond u)a + u(X \diamond a) = (X \diamond u)a \in Aa;$$

this means that Aa is a $\mathcal{G}$-submodule of A. Finally, we have

$$\rho_A(ua) = (ua)_0 \otimes (ua)_1 = u_0 a_0 \otimes u_1 a_1 = u_0 a \otimes u_1 \in (Aa) \otimes H;$$

this means that Aa is an H-subcomodule of A, that is, Aa is an H-ideal of A. Therefore, Aa is a $(\mathcal{G}, H)$-ideal of A containing the nonzero element a. So Aa is nonzero. Thus $Aa = A$ since A is $(\mathcal{G}, H)$-simple. So $1_A \in Aa$ and there is an element $a' \in A$ such that $a'a = 1_A$, that is, a is invertible. $\square$

From Theorem 2, we get the following corollary.

Corollary 5. *Let A be a commutative $(\mathcal{G}, H)$-simple $(\mathcal{G}, H)$-comodule algebra and M an $(A, \mathcal{G}, H)$-comodule. Set $B = A^{\mathcal{G}coH}$. Assume that there is a right H-colinear algebra map $\phi : H \to A^{\mathcal{G}}$. Suppose M^{coH} and A^{coH} are trivial $\mathcal{G}$-modules under $\diamond'$. Then M is free as an A-module with rank the dimension of the vector space $M^{\mathcal{G}coH}$ over $A^{\mathcal{G}coH}$.*

Let A be a $(\mathcal{G}, H)$-comodule algebra. Set $B = A^{\mathcal{G}coH}$. For any morphism $f : M \to N$ in ${}_{A,\mathcal{G}}\mathcal{M}^H$, it is not hard to show that $f(M^{\mathcal{G}coH}) \subseteq N^{\mathcal{G}coH}$. This gives rise to a functor

$$G = (-)^{\mathcal{G}coH} : {}_{A,\mathcal{G}}\mathcal{M}^H \to {}_B\mathcal{M}; M \mapsto M^{\mathcal{G}coH}.$$

Using Lemma 6, we also have a functor

$$F = A \otimes_B - : {}_B\mathcal{M} \to {}_{A,\mathcal{G}}\mathcal{M}^H; M \mapsto A \otimes_B M.$$

Proposition 1. *Set $B = A^{\mathcal{G}coH}$. Let $M \in {}_{A,\mathcal{G}}\mathcal{M}^H$ and $N \in {}_B\mathcal{M}$. There is a functorial isomorphism of $(A, \mathcal{G}, H)$-comodules*

$$\psi : {}_{A,\mathcal{G}}Hom^H(A \otimes_B N, M) \to {}_BHom(N, M^{\mathcal{G}coH}); f \mapsto [n \mapsto f(1_A \otimes_B n)],$$

with inverse map ψ' given by $g \mapsto [a \otimes_B n \mapsto ag(n)]$.

Thus, the functors F and G form an adjoint pair with unit and counit

$$\eta_N : N \to (A \otimes_B N)^{\mathcal{G}coH}, n \mapsto 1_A \otimes_B n$$

and

$$\epsilon_M : A \otimes_B M^{\mathcal{G}coH} \to M; a \otimes_B m \mapsto am.$$

Proof. We show that $f(1_A \otimes_B n) \in M^{coH}$: we have

$$f(1_A \otimes_B n)_0 \otimes f(1_A \otimes_B n)_1 = f((1_A \otimes_B n)_0) \otimes (1_A \otimes_B n)_1 = f(1_A \otimes_B n) \otimes 1_H,$$

since f is H-colinear. Next, we show that $f(1_A \otimes_B n)$ is $\mathcal{G}$-invariant:

$$X \diamond f(1_A \otimes_B n) = f(X \diamond (1_A \otimes_B n)) = f((X \diamond 1_A) \otimes_B n) = 0,$$

since f is $\mathcal{G}$-linear and $X \diamond 1_A = 0_A$. It follows that $f(1_A \otimes_B n) \in M^{\mathcal{G}coH}$. Now let us show that $\psi(f)$ is B-linear: we have

$$\begin{aligned}\psi(f)(bn) &= f(1_A \otimes_B (bn)) = f((1_A b) \otimes_B n) \\ &= f((b1_A) \otimes_B n) = f(b(1_A \otimes_B n)) \\ &= bf(1_A \otimes_B n) = b(\psi(f)(n)),\end{aligned}$$

since f is A-linear, à fortiori B-linear. Thus, the map ψ is well defined. Clearly, $\psi'(g)$ is A-linear. We show that $\psi'(g)$ is $\mathcal{G}$-linear:

$$\begin{aligned}\psi'(g)(X \diamond (a \otimes_B n)) &= \psi'(g)((X \diamond a) \otimes_B n) = (X \diamond a)g(n) \\ &= X \diamond (ag(n)) = X \diamond (\psi'(g)(a \otimes_B n)),\end{aligned}$$

since $g(n) \in M^{\mathcal{G}coH}$. Next, we show that $\psi'(g)$ is H-colinear:

$$\begin{aligned}\psi'(g)(a \otimes_B n)_0 \otimes \psi'(g)(a \otimes_B n)_1 &= (ag(n))_0 \otimes (ag(n))_1 \\ &= a_0 g(n)_0 \otimes a_1 g(n)_1 \\ &= a_0 g(n) \otimes a_1 1_H \\ &= \psi'(g)(a_0 \otimes_B n) \otimes a_1 \\ &= \psi'(g)((a \otimes_B n)_0) \otimes (a \otimes_B n)_1.\end{aligned}$$

Thus the map ψ' is well defined. It is easy to check that $\psi \circ \psi'$ and $\psi' \circ \psi$ are respectively the identity of ${}_B Hom(N, M^{\mathcal{G}coH})$ and ${}_{A,\mathcal{G}}Hom^H(A \otimes_B N, M)$. This means that ψ is bijective with inverse ψ'. □

Corollary 6. *Let A be a commutative $(\mathcal{G}, H)$-comodule algebra. Let M be an $(A, \mathcal{G}, H)$-comodule. Assume that there is a right H-colinear algebra map $\phi : H \to A^{\mathcal{G}}$. Suppose M^{coH} and A^{coH} are trivial $\mathcal{G}$-modules under $\diamond'$. Then the functor $G = (-)^{coH} : {}_{A,\mathcal{G}}\mathcal{M}^H \to {}_B\mathcal{M}$ is dual Maschke, that is, every object of ${}_{A,\mathcal{G}}\mathcal{M}^H$ is G-relative projective.*

Proof. The result follows from Theorem 2, Proposition 1 and [2, Theorem 3.4]. □

Let us give four examples of $(\mathcal{G}, H)$-comodule algebras.

Example 1. Let G be an abelian group. A G-graded vector space M is a vector space M with a fixed decomposition $M = \bigoplus_{g\in G} M^{(g)}$, where each $M^{(g)}$ is a vector subspace of M. A G-graded algebra A is an algebra A which is a G-graded vector space $A = \bigoplus_{g\in G} A^{(g)}$ such that $a^{(g)}a^{(h)} \in A^{(gh)}$ for all $a^{(g)} \in A^{(g)}$ and $a^{(h)} \in A^{(h)}$. It is well known that a G-graded vector space is a kG-comodule, and a G-graded algebra is a kG-comodule algebra.

A G-graded Lie algebra $\mathcal{G}$ is a Lie algebra $\mathcal{G}$ which is a G-graded vector space $\mathcal{G} = \bigoplus_{g\in G} \mathcal{G}^{(g)}$ such that $[X^{(g)}, X^{(h)}] \in \mathcal{G}^{(gh)}$ for all $X^{(g)} \in \mathcal{G}^{(g)}$ and $X^{(h)} \in \mathcal{G}^{(h)}$.

Let $\mathcal{G}$ be a G-graded Lie algebra. By [8, Example 6] $\mathcal{G}$ is a kG-comodule Lie algebra, where $\rho(X^{(g)}) = X^{(g)} \otimes g$ for all $g \in G$ and $X^{(g)} \in \mathcal{G}^{(g)}$.

A G-graded $\mathcal{G}$-module M is a G-graded vector space $M = \bigoplus_{g\in G} M^{(g)}$ which is a $\mathcal{G}$-module such that $X^{(g)} \diamond m^{(h)} \in M^{(gh)}$ for all $X^{(g)} \in \mathcal{G}^{(g)}$ and $m^{(h)} \in M^{(h)}$.

A G-graded $\mathcal{G}$-module algebra A is a G-graded $\mathcal{G}$-module $A = \bigoplus_{g\in G} A^{(g)}$ which is a G-graded algebra such that

$$X^{(g)} \diamond (a^{(h)}a^{(h')}) = (X^{(g)} \diamond (a^{(h)}))a^{(h')} + a^{(h)}(X^{(g)} \diamond (a^{(h')}))$$

for all $X^{(g)} \in \mathcal{G}^{(g)}$, $a^{(h)} \in A^{(h)}$, and $a^{(h')} \in A^{(h')}$.

Let G be an abelian group and A a G-graded $\mathcal{G}$-module algebra. Then A is a $(\mathcal{G}, kG)$-comodule algebra: $\rho(a^{(g)}) = a^{(g)} \otimes g$ for all $g \in G$ and $a^{(g)} \in A^{(g)}$.

Example 2. We refer to [4, 5, 12, 13, 16, 17] for further information on rational actions of an algebraic group.

Let G be an affine algebraic group. By [17], a k-vector space M is a rational G-module if it is a G-module, and for every $m \in M$, the translates of m span a finite dimensional subspace N of M and the induced map $G \to Aut_k(N)$ is a morphism of algebraic groups. A rational G-module algebra is a commutative algebra A which is a rational G-module such that

$$g.(aa') = (g.a)(g.a') \quad \forall a, a' \in A, g \in G.$$

For a rational G-module algebra A, a rational (A, G)-module M is an A-module M which is a rational G-module such that

$$g(am) = (g.a)(gm) \quad \forall g \in G, a \in A, m \in M.$$

Let $k[G]$ be the affine coordinate ring of G (it is a commutative Hopf algebra). It is well known that rational G-modules are $k[G]$-comodules with (right) coaction $\rho : M \to M \otimes k[G]$ characterized by the condition

$$\rho(m) = m_0 \otimes m_1 \in M \otimes k[G] \Leftrightarrow g.m = m_0 m_1(g) \quad \forall g \in G.$$

A rational G-module algebra is a commutative $k[G]$-comodule algebra. According to these definitions, we will say that a rational G-module Lie algebra is a Lie algebra $\mathcal{G}$ which is a rational G-module such that

$$g.[X, X'] = [g.X, g.X'] \quad \forall\, X, X' \in \mathcal{G}, g \in G.$$

Let $\mathcal{G}$ be a rational G-module Lie algebra. Then $\mathcal{G}$ is a right $k[G]$-comodule. We have

$$\rho([X, X']) = [X, X']_0 \otimes [X, X']_1$$

if and only if

$$g.[X, X'] = [X, X']_0 [X, X']_1(g)$$

for all $g \in G$. Since $g.[X, X'] = [g.X, g.X']$, we have

$$[X, X']_0 [X, X']_1(g) = [X_0, X'_0](X_1 X'_1)(g) \quad \forall g \in G.$$

It follows that

$$[X, X']_0 \otimes [X, X']_1 = [X_0, X'_0] \otimes (X_1 X'_1),$$

that is, $\mathcal{G}$ is a right $k[G]$-comodule Lie algebra.

A k-vector space M is a rational $(\mathcal{G}, G)$-module if it is a $\mathcal{G}$-module, a rational G-module, and

$$g(X \diamond m) = (g.X) \diamond (gm), \quad X \in \mathcal{G}, g \in G, m \in M.$$

A vector space A is a rational $(\mathcal{G}, G)$-module algebra if A is a rational $(\mathcal{G}, G)$-module which is also a $\mathcal{G}$-module algebra and a rational G-module algebra. We can show that a rational $(\mathcal{G}, G)$-module algebra is a $(\mathcal{G}, k[G])$-comodule algebra.

Let A be a rational $(\mathcal{G}, G)$-module algebra. A vector space M is a rational $(A, \mathcal{G}, G)$-module if M is an $A\#U(\mathcal{G})$-module, a rational (A, G)-module and a rational $(\mathcal{G}, G)$-module. Clearly, A is a rational $(A, \mathcal{G}, G)$-module. The rational $(A, \mathcal{G}, G)$-modules are exactly the $(A, \mathcal{G}, k[G])$-comodules.

Example 3. Let $\mathcal{G}$ a Lie algebra, A a commutative $\mathcal{G}$-module algebra and T a multiplicative subset of A. Let us consider the localization $T^{-1}A = AT^{-1}$ of A with product

$$(at^{-1})(a't'^{-1}) = (aa')(tt')^{-1} \quad \forall a, a' \in A, t, t' \in T.$$

Define a $\mathcal{G}$-action on $T^{-1}A$ by

$$X \diamond (at^{-1}) = [(X \diamond a)t - a(X \diamond t)]t^{-2} \quad \forall a \in A, t \in T.$$

Then $T^{-1}A$ is an algebra and a $\mathcal{G}$-module. We have

$$\begin{aligned} X \diamond [(at^{-1})(a't'^{-1})] &= X \diamond [(aa')(tt')^{-1}] \\ &= [(X \diamond (aa'))(tt') - (aa')(X \diamond (tt'))](tt')^{-2} \\ &= [((X \diamond a)a' + a(X \diamond a'))(tt') \\ &- (aa')((X \diamond t)t' + t(X \diamond t'))](tt')^{-2} \\ &= [((X \diamond a)a' + a(X \diamond a'))](tt')^{-1} \\ &- (aa')[((X \diamond t)t' - t(X \diamond t'))](tt')^{-2} \\ &= (X \diamond a)(a't^{-1}t'^{-1}) + (X \diamond a')(at^{-1}t'^{-1}) \\ &- (X \diamond t)(aa't^{-2}t'^{-1}) - (X \diamond t')](aa't^{-1}t'^{-2}). \end{aligned}$$

We also have

$$\begin{aligned} (X \diamond (at^{-1}))(a't'^{-1}) &+ (at^{-1})(X \diamond (a't'^{-1})) \\ &= [(X \diamond a)t - a(X \diamond t)]t^{-2}(a't'^{-1}) \\ &+ (at^{-1})[(X \diamond a')t' - a'(X \diamond t')]t'^{-2} \\ &= [(X \diamond a)t]t^{-2}(a't'^{-1}) + (at^{-1})[(X \diamond a')t't'^{-2} \\ &- a(X \diamond t)]t^{-2}(a't'^{-1}) - (at^{-1})a'(X \diamond t')]t'^{-2} \\ &= (X \diamond a)(a't^{-1}t'^{-1}) + (X \diamond a')(at^{-1}t'^{-1}) \\ &- (X \diamond t)(aa't^{-2}t'^{-1}) - (X \diamond t')(aa't^{-1}t'^{-2}). \end{aligned}$$

It follows from the above computations that

$$X \diamond [(at^{-1})(a't'^{-1})] = (X \diamond (at^{-1}))(a't'^{-1}) + (at^{-1})(X \diamond (a't'^{-1})) \quad \text{and}$$

the relation (4) is satisfied. Thus $T^{-1}A$ is a commutative $\mathcal{G}$-module algebra.

Let H be a Hopf algebra with a bijective antipode, $\mathcal{G}$ an H-comodule Lie algebra, and A a commutative $(\mathcal{G}, H)$-comodule algebra which is an integral domain. Let T be the set of the nonzero H-coinvariant elements of A. Then T is a multiplicative subset of A, and $T^{-1}A$ becomes an H-comodule with the structure map given by

$$(at^{-1})_0 \otimes (at^{-1})_1 = a_0 t^{-1} \otimes a_1, \quad \forall a \in A, t \in T.$$

Thus (setting $1_A = 1$), we get

$$(a1^{-1})_0 \otimes (a1^{-1})_1 = a_0 1^{-1} \otimes a_1,$$

and the comodule structure map ρ_A is compatible with the localization map $A \to AT^{-1}$. We have

$$\begin{aligned}
&[(at^{-1})(a't'^{-1})]_0 \otimes [(at^{-1})(a't'^{-1})]_1 \\
&\qquad = [(aa')(tt')^{-1}]_0 \otimes [(aa')(tt')^{-1}]_1 \\
&\qquad = [(aa')_0(tt')^{-1}] \otimes (aa')_1 \\
&\qquad = [(a_0a'_0)(t^{-1}t'^{-1})] \otimes (a_1a'_1) \\
&\qquad = [(a_0t^{-1})(a'_0t'^{-1})] \otimes (a_1a'_1) \\
&\qquad = [(at^{-1})_0(a't'^{-1})_0] \otimes [(at^{-1})_1(a't'^{-1})_1],
\end{aligned}$$

and the relation (1) is satisfied, that is, $T^{-1}A$ is an H-comodule algebra. We have

$$\begin{aligned}
&(X \diamond (at^{-1}))_0 \otimes (X \diamond (at^{-1}))_1 \\
&\quad = ([(X \diamond a)t - a(X \diamond t)]t^{-2})_0 \otimes ([(X \diamond a)t - a(X \diamond t)]t^{-2})_1 \\
&\quad = ([(X \diamond a)t - a(X \diamond t)]_0t^{-2}) \otimes ([(X \diamond a)t - a(X \diamond t)]_1) \\
&\quad = ([(X \diamond a)t]_0t^{-2} \otimes [(X \diamond a)t]_1) - (([a(X \diamond t)]_0)t^{-2} \otimes [a(X \diamond t)]_1) \\
&\quad = ([(X \diamond a)_0t_0]t^{-2} \otimes [(X \diamond a)_1t_1]) - ([a_0(X \diamond t)_0]t^{-2} \otimes [a_1(X \diamond t)_1]) \\
&\quad = ([(X \diamond a)_0t]t^{-2} \otimes [(X \diamond a)_1]) - ([a_0(X \diamond t)_0]t^{-2} \otimes [a_1(X \diamond t)_1]) \\
&\quad = [[(X_0 \diamond a_0)t]t^{-2} \otimes (a_1X_1)] - [[a_0(X_0 \diamond t_0)]t^{-2} \otimes [a_1(X_1t_1)]] \\
&\quad = [[(X_0 \diamond a_0)t]t^{-2} \otimes (a_1X_1)] - [[a_0(X_0 \diamond t)]t^{-2}] \otimes (a_1X_1) \\
&\quad = ([(X_0 \diamond a_0)tt^{-2}] - [a_0(X_0 \diamond t)t^{-2}]) \otimes (a_1X_1) \\
&\quad = ([(X_0 \diamond a_0)t^{-1}] - [a_0(X_0 \diamond t)t^{-2}]) \otimes (a_1X_1) \\
&\quad = [(X_0 \diamond (at^{-1})_0] \otimes [X_1(at^{-1})_1],
\end{aligned}$$

and the relation (7) is satisfied. We deduce that if H is a Hopf algebra with a bijective antipode, $\mathcal{G}$ an H-comodule Lie algebra, A a commutative

$(\mathcal{G}, H)$-comodule algebra which is an integral domain and if T is the set of the nonzero H-coinvariant elements of A, then $T^{-1}A$ is a commutative $(\mathcal{G}, H)$-comodule algebra.

We refer to [6] and [7] for interesting results on Poisson algebras and Poisson modules.

Example 4. Let A be a Poisson algebra with Poisson bracket $\{,\}$. We say that A is an H-comodule Poisson algebra (see [10]) if A is an H-comodule algebra such that the H-coaction is compatible with the Poisson bracket; that is, A is an H-comodule algebra satisfying the relation

$$\{a, a'\}_0 \otimes \{a, a'\}_1 = \{a_0, a'_0\} \otimes a_1 a'_1 \quad \forall a, a' \in A.$$

Clearly, an H-comodule Poisson algebra is a $(\mathcal{G}, H)$-comodule algebra, where $\mathcal{G} = A$ as a Lie algebra.

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Received by the editors: 07.10.2024
and in final form 04.11.2024.