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Automorphisms of the endomorphism semigroup of a free commutative *g*-dimonoid

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Abstract. We determine all isomorphisms between the endomorphism semigroups of free commutative g-dimonoids and prove that all automorphisms of the endomorphism semigroup of a free commutative g-dimonoid are quasi-inner.

1. Introduction

A *dimonoid* is an algebra $(D, \dashv, \vdash)$ with two binary associative operations $\dashv$ and $\vdash$ such that for all $x, y, z \in D$ the following conditions hold:

$$(D_1) \qquad (x \dashv y) \dashv z = x \dashv (y \vdash z),$$
$$(D_2) \qquad (x \vdash y) \dashv z = x \vdash (y \dashv z),$$
$$(D_3) \qquad (x \dashv y) \vdash z = x \vdash (y \vdash z).$$

This notion was introduced by Jean-Louis Loday in [1] and now it plays a prominent role in problems from the theory of Leibniz algebras. A vector space equipped with the structure of a dimonoid is called a *dialgebra*. Thus, a dialgebra is a linear analog of a dimonoid. It is known that Leibniz algebras are a non-commutative variation of Lie algebras and dialgebras are a variation of associative algebras.

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There exist some generalizations of dimonoids, for example, 0-dialgebras and duplexes (see, e.g., [2], [3]), g-dimonoids etc. Omitting the axiom (D_2) of an inner associativity in the definition of a dimonoid, we obtain the notion of a g-dimonoid. An associative 0-dialgebra, that is, a vector space equipped with two binary associative operations $\dashv$ and $\vdash$ satisfying the axioms (D_1) and (D_3), is a linear analog of a g-dimonoid. Free g-dimonoids and free n-nilpotent g-dimonoids were constructed in [4], [5] and [5], respectively. The construction of a free commutative g-dimonoid and the least commutative congruence on a free g-dimonoid were described in [6]. Defining identities of a g-dimonoid appear also in axioms of trialgebras and of trioids [7–9].

Endomorphism semigroups of algebraic systems have been studied by numerous authors. The problem of studying the endomorphism semigroup for free algebras in a certain variety was raised by B.I. Plotkin in his papers on universal algebraic geometry (see, e.g., [10], [11]). In this direction there are many papers devoted to describing automorphisms of endomorphism semigroups of free finitely generated universal algebras of some varieties: groups [12], semigroups [13], associative algebras [14], inverse semigroups [15], modules and semimodules [16], Lie algebras [17] and other algebras (see also [18]). In this paper we solve the similar problem for the variety of commutative g-dimonoids.

The paper is organized in the following way. In Section 2, we give necessary definitions and statements. In Section 3, we define the notion of a crossed isomorphism of g-dimonoids and prove auxiliary lemmas. In Section 4, we describe all isomorphisms between the endomorphism monoids of free commutative g-dimonoids of rank 1. In Section 5, we prove that automorphisms of the endomorphism semigroup of a free commutative g-dimonoid of a non-unity rank are inner or "mirror inner". We show also that the automorphism group of the endomorphism semigroup of a free commutative g-dimonoid is isomorphic to the direct product of a symmetric group and a 2-element group.

2. Preliminaries

Let $\mathfrak{D}_1 = (D_1, \dashv_1, \vdash_1)$ and $\mathfrak{D}_2 = (D_2, \dashv_2, \vdash_2)$ be arbitrary g-dimonoids. A mapping $\varphi : D_1 \to D_2$ is called a *homomorphism* of $\mathfrak{D}_1$ into $\mathfrak{D}_2$ if

$$(x \dashv_1 y)\varphi = x\varphi \dashv_2 y\varphi, \ \ (x \vdash_1 y)\varphi = x\varphi \vdash_2 y\varphi$$

for all $x, y \in D_1$.

A bijective homomorphism $\varphi : D_1 \to D_2$ is called an *isomorphism* of $\mathfrak{D}_1$ onto $\mathfrak{D}_2$. In this case g-dimonoids $\mathfrak{D}_1$ and $\mathfrak{D}_2$ are called *isomorphic.*

A g-dimonoid $(D, \dashv, \vdash)$ is called *commutative* if for all $x, y \in D$,

$$x \dashv y = y \dashv x, \qquad x \vdash y = y \vdash x.$$

Firstly we give an example of a g-dimonoid which is not a dimonoid.

Let A be an arbitrary nonempty set and $\overline{A} = \{\overline{x} \mid x \in A\}$. For every $x \in A$ assume $\widetilde{\overline{x}} = x$ and introduce a mapping $\alpha = \alpha_A : A \cup \overline{A} \to A$ by the following rule:

$$y\alpha = \begin{cases} y, & y \in A, \\ \widetilde{y}, & y \in \overline{A}. \end{cases}$$

Give an arbitrary semigroup S and define operations $\prec$ and $\succ$ on $S \cup \overline{S}$ as follows:

$$a \prec b = (a\alpha_S)(b\alpha_S), \qquad a \succ b = \overline{(a\alpha_S)(b\alpha_S)}$$

for all $a, b \in S \cup \overline{S}$. The algebra $(S \cup \overline{S}, \prec, \succ)$ is denoted by $S^{(\alpha)}$.

Proposition 1 ([6]). *$S^{(\alpha)}$ is a g-dimonoid but not a dimonoid.*

We note that if X is a generating set of a semigroup S, then $S^{(\alpha)} \setminus \overline{X}$ is a g-subdimonoid of $S^{(\alpha)}$ generated by X.

For an arbitrary commutative semigroup S, obviously, $S^{(\alpha)}$ is a commutative g-dimonoid.

Recall the construction of a free commutative g-dimonoid. Let $F[A]$ be the free commutative semigroup generated by a set A.

Theorem 1 ([6]). *$F[A]^{(\alpha)} \setminus \overline{A}$ is the free commutative g-dimonoid.*

Observe that A is a generating set of $F[A]^{(\alpha)} \setminus \overline{A}$, the cardinality of A is the *rank* of $F[A]^{(\alpha)} \setminus \overline{A}$ and this g-dimonoid is uniquely determined up to an isomorphism by $|A|$.

Further the free commutative g-dimonoid generated by A will be denoted by $\mathfrak{FCD}^g_A$.

In particular, we consider the free commutative g-dimonoid of rank 1. Let $\mathbb{N}$ be the set of all natural numbers and $\mathbb{N}^* = (\mathbb{N} \cup \overline{\mathbb{N}}) \setminus \{\overline{1}\}$. Define operations $\prec$ and $\succ$ on $\mathbb{N}^*$ by

$$\begin{gathered} m \prec n = m + n, \qquad \overline{q} \prec \overline{r} = q + r, \\ m \prec \overline{r} = m + r, \qquad \overline{q} \prec n = q + n, \\ a \succ b = \overline{a \prec b}, \end{gathered}$$

for all $m, n \in \mathbb{N}$, $\overline{q}, \overline{r} \in \overline{\mathbb{N}} \setminus \{\overline{1}\}$ and $a, b \in \mathbb{N}^*$.

Proposition 2 ([6]). *The free commutative g-dimonoid $\mathfrak{FCD}_A^g$ of rank 1 is isomorphic to* $(\mathbb{N}^*, \prec, \succ)$.

Recall that the content of $\omega = x_1x_2\dots x_n \in F[A]$ is the set $c(\omega) = \{x_1, x_2, \dots, x_n\}$ and the length of ω is the number $l(\omega) = n$.

For every $\omega \in \mathfrak{FCD}_A^g$, the set $c(\omega\alpha)$ and the number $l(\omega\alpha)$ we call the *content* and the *length* of ω, respectively, and denote it by $c(\omega)$ and $l(\omega)$. For example, for $w = \overline{bacda}$ we have $c(w) = \{a, b, c, d\}$ and $l(w) = 5$.

3. Auxiliary statements

We start this section with the following lemma.

Lemma 1. *Let $\mathfrak{FCD}_X^g$ and $\mathfrak{FCD}_Y^g$ be free commutative g-dimonoids generated by X and Y, respectively. Every bijection $\varphi : X \to Y$ induces an isomorphism $\varepsilon_\varphi : \mathfrak{FCD}_X^g \to \mathfrak{FCD}_Y^g$ such that*

$$\omega\varepsilon_\varphi = \begin{cases} x_1\varphi \prec x_2\varphi \prec \dots \prec x_m\varphi, & \omega = x_1x_2\dots x_m,\ m \geqslant 1, \\ x_1\varphi \succ x_2\varphi \succ \dots \succ x_m\varphi, & \omega = \overline{x_1x_2\dots x_m},\ m > 1 \end{cases}$$

for all $\omega \in \mathfrak{FCD}_X^g$.

Proof. The proof of this statement is obvious. □

Now we introduce the notion of a crossed isomorphism of g-dimonoids. A mapping $\varphi : D_1 \to D_2$ we call a *crossed homomorphism* of a g-dimonoid $\mathfrak{D}_1 = (D_1, \dashv_1, \vdash_1)$ into a g-dimonoid $\mathfrak{D}_2 = (D_2, \dashv_2, \vdash_2)$ if for all $x, y \in D_1$,

$$(x \dashv_1 y)\varphi = x\varphi \vdash_2 y\varphi, \qquad (x \vdash_1 y)\varphi = x\varphi \dashv_2 y\varphi.$$

A bijective crossed homomorphism $\varphi : D_1 \to D_2$ will be called a *crossed isomorphism* of $\mathfrak{D}_1$ onto $\mathfrak{D}_2$. In such case g-dimonoids $\mathfrak{D}_1$ and $\mathfrak{D}_2$ we call *crossed isomorphic*.

An example of crossed isomorphic g-dimonoids gives the next lemma.

Lemma 2. *Let $\mathfrak{FCD}_X^g$ and $\mathfrak{FCD}_Y^g$ be free commutative g-dimonoids generated by X and Y, respectively. Every bijection $\varphi : X \to Y$ induces a crossed isomorphism $\varepsilon_\varphi^* : \mathfrak{FCD}_X^g \to \mathfrak{FCD}_Y^g$ such that*

$$\omega\varepsilon_\varphi^* = \begin{cases} x_1\varphi \succ x_2\varphi \succ \dots \succ x_m\varphi, & \omega = x_1x_2\dots x_m,\ m \geqslant 1, \\ x_1\varphi \prec x_2\varphi \prec \dots \prec x_m\varphi, & \omega = \overline{x_1x_2\dots x_m},\ m > 1 \end{cases}$$

for all $\omega \in \mathfrak{FCD}_X^g$.

Proof. It is clear that ε_φ^* is a bijection. Take arbitrary $u, v \in \mathfrak{FCD}_X^g$ and consider the following cases.

Case 1. $u = u_1u_2\dots u_m, v = v_1v_2\dots v_n \in F[X]$, then

$$\begin{aligned}(u \prec v)\varepsilon_\varphi^* &= (u\alpha v\alpha)\varepsilon_\varphi^* = (uv)\varepsilon_\varphi^* \\ &= u_1\varphi \succ \dots \succ u_m\varphi \succ v_1\varphi \succ \dots \succ v_n\varphi = u\varepsilon_\varphi^* \succ v\varepsilon_\varphi^*,\end{aligned}$$

$$\begin{aligned}(u \succ v)\varepsilon_\varphi^* &= (\overline{u\alpha v\alpha})\varepsilon_\varphi^* = (\overline{uv})\varepsilon_\varphi^* \\ &= u_1\varphi \prec \dots \prec u_m\varphi \prec v_1\varphi \prec \dots \prec v_n\varphi \\ &= \overline{u_1\varphi\dots u_m\varphi} \prec \overline{v_1\varphi\dots v_n\varphi} = u\varepsilon_\varphi^* \prec v\varepsilon_\varphi^*.\end{aligned}$$

Case 2. $u = u_1u_2\dots u_m \in F[X], \overline{v} = \overline{v_1v_2\dots v_n} \in \overline{F[X]} \setminus \overline{X}$, then

$$\begin{aligned}(u \prec \overline{v})\varepsilon_\varphi^* &= (uv)\varepsilon_\varphi^* = \overline{u_1\varphi\dots u_m\varphi v_1\varphi\dots v_n\varphi} \\ &= \overline{u_1\varphi\dots u_m\varphi} \succ (v_1\varphi\dots v_n\varphi) = u\varepsilon_\varphi^* \succ \overline{v}\varepsilon_\varphi^*,\end{aligned}$$

$$\begin{aligned}(u \succ \overline{v})\varepsilon_\varphi^* &= (\overline{uv})\varepsilon_\varphi^* = u_1\varphi\dots u_m\varphi v_1\varphi\dots v_n\varphi \\ &= \overline{u_1\varphi\dots u_m\varphi} \prec (v_1\varphi\dots v_n\varphi) = u\varepsilon_\varphi^* \prec \overline{v}\varepsilon_\varphi^*.\end{aligned}$$

Case 3, where $\overline{u} = \overline{u_1u_2\dots u_m} \in \overline{F[X]} \setminus \overline{X}, v = v_1v_2\dots v_n \in F[X]$, can be omited since operations $\prec$ and $\succ$ are commutative.

Case 4, where $\overline{u} = \overline{u_1u_2\dots u_m}, \overline{v} = \overline{v_1v_2\dots v_n} \in \overline{F[X]} \setminus \overline{X}$, is analogous to the case 1.

From cases 1–4 it follows that ε_φ^* is a crossed homomorphism which completes the proof of this statement. $\square$

For an arbitrary algebra A, we denote the endomorphism semigroup and the automorphism group of A by $\mathrm{End}(A)$ and $\mathrm{Aut}(A)$, respectively.

Anywhere the composition of mappings is defined from left to right.

Lemma 3. *Let $\mathfrak{D}_1 = (D_1, \dashv_1, \vdash_1)$ and $\mathfrak{D}_2 = (D_2, \dashv_2, \vdash_2)$ be arbitrary g-dimonoids, and φ be any isomorphism or a crossed isomorphism of $\mathfrak{D}_1$ onto $\mathfrak{D}_2$. The mapping*

$$\Phi : f \mapsto f\Phi = \varphi^{-1}f\varphi, \quad f \in \mathrm{End}(\mathfrak{D}_1),$$

is an isomorphism of $\mathrm{End}(\mathfrak{D}_1)$ *onto* $\mathrm{End}(\mathfrak{D}_2)$.

Proof. Let φ be a crossed isomorphism of $\mathfrak{D}_1$ onto $\mathfrak{D}_2$. Clearly, φ^{-1} is a crossed isomorphism of $\mathfrak{D}_2$ onto $\mathfrak{D}_1$. For all $u, v \in D_2$ and $f \in \mathrm{End}(\mathfrak{D}_1)$,

$$\begin{aligned}(u \dashv_2 v)\varphi^{-1}f\varphi &= (u\varphi^{-1} \vdash_1 v\varphi^{-1})f\varphi \\ &= (u\varphi^{-1}f \vdash_1 v\varphi^{-1}f)\varphi = u(\varphi^{-1}f\varphi) \dashv_2 v(\varphi^{-1}f\varphi).\end{aligned}$$

In similar way, $\varphi^{-1}f\varphi \in \mathrm{End}(D_2, \vdash_2)$ and so $f\Phi \in \mathrm{End}(\mathfrak{D}_2)$ for all $f \in \mathrm{End}(\mathfrak{D}_1)$. The remaining part of the proof is trivial. $\square$

We call Φ from Lemma 3 as the isomorphism induced by the isomorphism or the crossed isomorphism φ.

For an arbitrary nonempty set X the identity transformation of X is denoted by id_X. By Lemma 2, $\varepsilon^*_{id_X}$ is a crossed automorphism of the free commutative g-dimonoid $\mathfrak{FCD}^g_X$.

By Lemma 3, a transformation Φ_1 of the endomorphism monoid $\mathrm{End}(\mathfrak{FCD}^g_X)$ defined by $\eta\Phi_1 = (\varepsilon^*_{id_X})^{-1}\eta\varepsilon^*_{id_X}$ for all $\eta \in \mathrm{End}(\mathfrak{FCD}^g_X)$, is an automorphism. Obviously, $(\varepsilon^*_{id_X})^{-1} = \varepsilon^*_{id_X}$.

The automorphism Φ_1 we will call the *mirror automorphism* of the endomorphism monoid $\mathrm{End}(\mathfrak{FCD}^g_X)$. By Φ_0 we denote the identity automorphism of $\mathrm{End}(\mathfrak{FCD}^g_X)$. It is clear that $\{\Phi_0, \Phi_1\}$ is a group with respect to the composition of permutations.

Let $\mathfrak{FCD}^g_X$ be the free commutative g-dimonoid generated by X. Each endomorphism ξ of $\mathfrak{FCD}^g_X$ is uniquely determined by a mapping $\varphi : X \to \mathfrak{FCD}^g_X$. Really, to define ξ, it suffices to put

$$\omega\xi = \begin{cases} x_1\varphi \prec x_2\varphi \prec \ldots \prec x_m\varphi, & \omega = x_1x_2\ldots x_m,\ m \geqslant 1, \\ x_1\varphi \succ x_2\varphi \succ \ldots \succ x_m\varphi, & \omega = \overline{x_1x_2\ldots x_m},\ m > 1 \end{cases}$$

for all $\omega \in \mathfrak{FCD}^g_X$.

In particular, an endomorphism ξ of $\mathfrak{FCD}^g_X$ is an automorphism if and only if a restriction ξ on X belong to the symmetric group $S(X)$. Therefore, the group $\mathrm{Aut}(\mathfrak{FCD}^g_X)$ is isomorphic to $S(X)$ (see [6]).

Let $u \in \mathfrak{FCD}^g_X$. An endomorphism $\theta_u \in \mathrm{End}(\mathfrak{FCD}^g_X)$ is called *constant* if $x\theta_u = u$ for all $x \in X$.

Lemma 4. (i) *Let $u \in \mathfrak{FCD}^g_X$, $\xi \in \mathrm{End}(\mathfrak{FCD}^g_X)$. Then $\theta_u\xi = \theta_{u\xi}$.*

(ii) *An endomorphism ξ of $\mathfrak{FCD}^g_X$ is constant if and only if $\psi\xi = \xi$ for all $\psi \in \mathrm{Aut}(\mathfrak{FCD}^g_X)$.*

(iii) *An endomorphism ξ of $\mathfrak{FCD}^g_X$ is constant idempotent if and only if $\xi = \theta_x$ for some $x \in X$.*

Proof. (i) It is obvious.

(ii) Take a constant $\theta_u \in \mathrm{End}(\mathfrak{FCD}_X^g)$ for some $u \in \mathfrak{FCD}_X^g$, and let $\psi \in \mathrm{Aut}(\mathfrak{FCD}_X^g)$. Then $x(\psi\theta_u) = (x\psi)\,\theta_u = u = x\theta_u$ for all $x \in X$.

Now let $\psi\xi = \xi$ for all $\psi \in \mathrm{Aut}(\mathfrak{FCD}_X^g)$ and some $\xi \in \mathrm{End}(\mathfrak{FCD}_X^g)$. Fixing $x \in X$, we obtain $x\xi = x\,(\psi\xi) = (x\psi)\,\xi = y\xi$, where $y = x\psi$. Since $\{x\psi \mid \psi \in \mathrm{Aut}(\mathfrak{FCD}_X^g)\} = X$, then $x\xi = y\xi$ for all $y \in X$. Consequently, $\xi = \theta_u$ for $u = x\xi$.

(iii) Let $\xi \in \mathrm{End}(\mathfrak{FCD}_X^g)$ be a constant idempotent. Then $\xi = \theta_u, u \in \mathfrak{FCD}_X^g$, and by (i) of this lemma, $\theta_u = \theta_u\theta_u = \theta_{u\theta_u}$. This implies $u = u\theta_u$ and, therefore, $l(u) = 1$ and $u \in X$. Converse is obvious. □

4. The automorphism group of $\mathrm{End}(\mathfrak{FCD}_X^g)$, $|X| = 1$

The free commutative g-dimonoid $\mathfrak{FCD}_X^g$ on an n-element set X we denote by $\mathfrak{FCD}_n^g$. Recall that the g-dimonoid $\mathfrak{FCD}_1^g$ is isomorphic to $(\mathbb{N}^*, \prec, \succ)$ (see Proposition 2). Therefore, we will identify elements of $\mathfrak{FCD}_1^g$ with corresponding elements of $(\mathbb{N}^*, \prec, \succ)$.

Define a binary operation $\odot$ on $\mathbb{N}^* = (\mathbb{N} \cup \overline{\mathbb{N}}) \setminus \{\overline{1}\}$ by

$$m \odot n = m \odot \overline{n} = m \cdot n, \qquad \overline{m} \odot n = \overline{m} \odot \overline{n} = \overline{m \cdot n},$$
$$1 \odot x = x \odot 1 = x$$

for all $m, n \in \mathbb{N} \setminus \{1\}$, $\overline{m}, \overline{n} \in \overline{\mathbb{N}} \setminus \{\overline{1}\}$ and $x \in \mathbb{N}^*$.

Lemma 5. (i) *The operation $\odot$ is associative.*

(ii) *The operation $\odot$ is distributive with respect to $\prec$ and $\succ$.*

Proof. It can be verified directly. □

From Lemma 5 (i) it follows that $(\mathbb{N}^*, \odot)$ is a semigroup.

Lemma 6. *The semigroups* $\mathrm{End}(\mathfrak{FCD}_1^g)$ *and* $(\mathbb{N}^*, \odot)$ *are isomorphic.*

Proof. Let φ be an arbitrary endomorphism of $(\mathbb{N}^*, \prec, \succ)$ and $1\varphi = k$ for some $k \in \mathbb{N}^*$. For all $a \in \mathbb{N}$ and $\overline{b} \in \overline{N} \setminus \{\overline{1}\}$ we obtain

$$a\varphi = (\underbrace{1 \prec 1 \prec \ldots \prec 1}_{a})\varphi = a \odot k, \quad \overline{b}\varphi = (\underbrace{1 \succ 1 \succ \ldots \succ 1}_{b})\varphi = \overline{b \odot k}.$$

Converse, any transformation $\varphi_k : \mathbb{N}^* \to \mathbb{N}^*, k \in \mathbb{N}^*$, defined by

$$a\varphi_k = a \odot k$$

is an endomorphism of $(\mathbb{N}^*, \prec, \succ)$. Indeed, using the condition (ii) of Lemma 5, for all $a, b \in \mathbb{N}^*$ and $\star \in \{\prec, \succ\}$ we obtain

$$(a \star b)\varphi_k = (a \star b) \odot k = (a \odot k) \star (b \odot k) = a\varphi_k \star b\varphi_k.$$

Consequently,

$$\mathrm{End}(\mathbb{N}^*, \prec, \succ) = \{\varphi_k \mid k \in \mathbb{N}^*\}.$$

Define a mapping Θ of $\mathrm{End}(\mathbb{N}^*, \prec, \succ)$ into $(\mathbb{N}^*, \odot)$ by $\varphi_k\Theta = k$ for all $\varphi_k \in \mathrm{End}(\mathbb{N}^*, \prec, \succ)$. An immediate verification shows that Θ is an isomorphism. $\square$

Remark 1. Note that all endomorphisms of a g-dimonoid $(\mathbb{N}^*, \prec, \succ)$ are injective but they are not surjective (except an identity automorphism). So that the automorphism group of $(\mathbb{N}^*, \prec, \succ)$ is singleton.

Let $\mathbb{P}$ be the set of all prime numbers, $\overline{\mathbb{P}} = \{\overline{x} \mid x \in \mathbb{P}\}$ and $\mathbb{P}^* = \mathbb{P} \cup \overline{\mathbb{P}}$. For any mapping $f : A \to B$ and a nonempty subset $C \subseteq A$, we denote the restriction of f to C by $f|_C$.

Further let $A, B \subseteq N \setminus \{1\}$, $C \subseteq \overline{N} \setminus \{\overline{1}\}$ be nonempty subsets and $\varphi : A \to B$, $\psi : B \to C$ be arbitrary mappings. Denote by $\overline{\varphi}$ and $\overrightarrow{\psi}$ the mappings $\overline{A} \to \overline{B}$ and, respectively, $\overline{B} \to C\alpha$ (the mapping α was defined in Section 2) such that

$$\overline{a}\ \overline{\varphi} = \overline{b} \ \ if \ \ a\varphi = b \ \ and \ \ \overline{b}\ \overrightarrow{\psi} = c \ \ if \ \ b\psi = \overline{c}.$$

Proposition 3. *Let* $\mathrm{End}(\mathfrak{FCD}_X^g) \cong \mathrm{End}(\mathfrak{FCD}_Y^g)$, *where* X *is a singleton set,* Y *is an arbitrary set. Then* $|Y| = 1$ *and the isomorphisms of* $\mathrm{End}(\mathfrak{FCD}_X^g)$ *onto* $\mathrm{End}(\mathfrak{FCD}_Y^g)$ *are in a natural one-to-one correspondence with permutations* $f : \mathbb{P}^* \to \mathbb{P}^*$ *such that*

$$\mathbb{P}f = \mathbb{P}, f|_{\overline{\mathbb{P}}} = \overline{f|_{\mathbb{P}}} \ \ or \ \ \mathbb{P}f = \overline{\mathbb{P}}, f|_{\overline{\mathbb{P}}} = \overrightarrow{f|_{\mathbb{P}}}.$$

Proof. According to Lemma 6, $\mathrm{End}(\mathfrak{FCD}_1^g) \cong (\mathbb{N}^*, \odot)$. Let $|Y| \geqslant 2$ and $a, b \in Y, a \neq b$. Define a binary relation ρ on $\mathbb{N}^*$ by

$$(a; b) \in \rho \Leftrightarrow a = b = 1 \ \ or \ \ a \neq 1 \neq b, a \odot b = b \odot a.$$

Obviously, ρ is an equivalence and $\mathbb{N}^*/\rho = \{\mathbb{N} \setminus \{1\}, \overline{\mathbb{N}} \setminus \{\overline{1}\}, \{1\}\}$. Since $\mathrm{End}(\mathfrak{FCD}_Y^g) \cong (\mathbb{N}^*, \odot)$, we will use the relation ρ for $\mathrm{End}(\mathfrak{FCD}_Y^g)$ too. For constants $\theta_{\overline{ab}}, \theta_a, \theta_{ab} \in \mathrm{End}(\mathfrak{FCD}_Y^g)$ and some $y \in Y$ we have

$$y(\theta_{\overline{ab}}\theta_a) = \overline{ab}\theta_a = \overline{aa} \neq \overline{ab} = a\theta_{\overline{ab}} = y(\theta_a\theta_{\overline{ab}}),$$
$$y(\theta_{\overline{ab}}\theta_{ab}) = \overline{ab}\theta_{ab} = \overline{abab} \neq abab = ab\theta_{\overline{ab}} = y(\theta_{ab}\theta_{\overline{ab}}),$$

therefore $(\theta_{\overline{ab}}, \theta_a) \notin \rho$ and $(\theta_{\overline{ab}}, \theta_{ab}) \notin \rho$. From here it follows that $(\theta_{ab}, \theta_a) \in \rho$ which contradicts the fact that $\theta_{ab}\theta_a \neq \theta_a\theta_{ab}$. Then $|Y| = 1$.

It is clear that the semigroup $(\mathbb{N}^* \setminus \{1\}, \odot)$ is generated by $\mathbb{P}^*$ and $\mathbb{P}^* f = \mathbb{P}^*$ for all $f \in \mathrm{Aut}(\mathbb{N}^*, \odot)$. Assume that there exist $p, q \in \mathbb{P}$ such that $pf = p' \in \mathbb{P}$ and $qf = \overline{q'} \in \overline{\mathbb{P}}$ for some $f \in \mathrm{Aut}(\mathbb{N}^*, \odot)$. Then

$$p' \cdot q' = p' \odot \overline{q'} = (p \cdot q)f = (q \cdot p)f = \overline{q'} \odot p' = \overline{p' \cdot q'}.$$

It means that $\mathbb{P}f = \mathbb{P}$ and so $\overline{\mathbb{P}}f = \overline{\mathbb{P}}$, or $\mathbb{P}f = \overline{\mathbb{P}}$ and then $\overline{\mathbb{P}}f = \mathbb{P}$.

If $\mathbb{P}f = \mathbb{P}$, then for all $p \in \mathbb{P}$ we have $(pf)^2 = p^2 f = (p \odot \overline{p})f = pf \odot \overline{p}f$, whence $\overline{p}f = \overline{pf}$. Thus, $f|_{\overline{\mathbb{P}}} = \overline{f|_{\mathbb{P}}}$. In a similar way it can be shown that in the case $\mathbb{P}f = \overline{\mathbb{P}}$ we obtain $f|_{\overline{\mathbb{P}}} = \overrightarrow{f|_{\mathbb{P}}}$.

On the other hand, as it is not hard to check, every permutation $f : \mathbb{P}^* \to \mathbb{P}^*$ such that $\mathbb{P}f = \mathbb{P}, f|_{\overline{\mathbb{P}}} = \overline{f|_{\mathbb{P}}}$, or $\mathbb{P}f = \overline{\mathbb{P}}, f|_{\overline{\mathbb{P}}} = \overrightarrow{f|_{\mathbb{P}}}$, uniquely determines an automorphism of $(\mathbb{N}^*, \odot)$. These permutations and hence the isomorphisms $\mathrm{End}(\mathfrak{FCD}^g_X) \to \mathrm{End}(\mathfrak{FCD}^g_Y)$, are in a natural one-to-one correspondence. $\square$

An automorphism $\Phi : \mathrm{End}(\mathfrak{FCD}^g_X) \to \mathrm{End}(\mathfrak{FCD}^g_X)$ is called *quasi-inner* if there exists a permutation α of $\mathfrak{FCD}^g_X$ such that $\beta\Phi = \alpha^{-1}\beta\alpha$ for all $\beta \in \mathrm{End}(\mathfrak{FCD}^g_X)$. If α turns out to be an automorphism of $\mathfrak{FCD}^g_X$, Φ is an inner automorphism of $\mathrm{End}(\mathfrak{FCD}^g_X)$.

We denote the symmetric group on a set X by $S(X)$. A 2-element group with identity e is denoted by $C_2 = \{e, a\}$.

Proposition 4. *Automorphisms of the monoid* $\mathrm{End}(\mathfrak{FCD}^g_1)$ *are quasi-inner. In addition, the automorphism group of* $\mathrm{End}(\mathfrak{FCD}^g_1)$ *is isomorphic to the direct product* $S(\mathbb{P}) \times C_2$.

Proof. Let $\Psi : \mathrm{End}(\mathfrak{FCD}^g_1) \to \mathrm{End}(\mathfrak{FCD}^g_1)$ be an arbitrary automorphism. Define a bijection $\psi : N^* \to N^*$ putting $x\psi = y$ if $\varphi_x\Psi = \varphi_y$. It is clear that $\psi \in \mathrm{Aut}(\mathbb{N}^*, \odot)$, however $\psi \notin \mathrm{Aut}(\mathbb{N}^*, \prec, \succ)$ except the identity permutation (see Remark 1). Then for all $x \in N^*$ and some $\varphi_i \in \mathrm{End}(\mathfrak{FCD}^g_1)$, $i \in N^*$, we have

$$\begin{aligned} x(\psi^{-1}\varphi_i\psi) &= (x\psi^{-1})\varphi_i\psi = ((x\psi^{-1}) \odot i)\psi \\ &= (x\psi^{-1})\psi \odot i\psi = x \odot i\psi = x\varphi_{i\psi}. \end{aligned}$$

Thus, $\psi^{-1}\varphi_i\psi = \varphi_i\Psi$ and Ψ is a quasi-inner automorphism.

The immediate check shows that a mapping Θ of $\mathrm{Aut}(\mathbb{N}^*, \odot)$ onto $S(\mathbb{P}) \times C_2$ defined as follows:

$$\xi\Theta = \begin{cases} (\xi|_P, e), & P\xi = P, \\ (\xi|_P, a), & P\xi = \overline{P} \end{cases}$$

for all $\xi \in \mathrm{Aut}(\mathbb{N}^*, \odot)$, is an isomorphism.

By Lemma 6, $\mathrm{End}(\mathfrak{FCD}_1^g) \cong (\mathbb{N}^*, \odot)$, therefore $\mathrm{Aut}(\mathrm{End}(\mathfrak{FCD}_1^g))$ and $S(\mathbb{P}) \times C_2$ are isomorphic. □

5. The automorphism group of $\mathrm{End}(\mathfrak{FCD}_X^g)$, $|X| \geqslant 2$

An automorphism Ψ of the endomorphism monoid $\mathrm{End}(\mathfrak{FCD}_X^g)$ of the free commutative g-dimonoid $\mathfrak{FCD}_X^g$ is called *stable* if Ψ induces the identity permutation of X, that is, $\theta_x\Psi = \theta_x$ for all $x \in X$.

Lemma 7. *For all $u, v \in F[X] \setminus X$ the following equalities hold:*

$$\theta_u\theta_v = \theta_u\theta_{\overline{v}} \text{ and } \theta_{\overline{u}}\theta_{\overline{v}} = \theta_{\overline{u}}\theta_v.$$

Proof. It is obvious. □

Lemma 8. *Let Ψ be a stable automorphism of $\mathrm{End}(\mathfrak{FCD}_X^g)$, $u, v \in F[X] \setminus X$, $x \in X$ and $\xi \in \mathrm{End}(\mathfrak{FCD}_X^g)$. Then*

(i) $\theta_{x\xi}\Psi = \theta_{x(\xi\Psi)}$;

(ii) $\theta_u\Psi = \theta_v$ *implies* $\theta_{\overline{u}}\Psi = \theta_{\overline{v}}$;

(iii) $\theta_u\Psi = \theta_{\overline{v}}$ *implies* $\theta_{\overline{u}}\Psi = \theta_v$.

Proof. (i) By Lemma 4 (i), $\theta_{x\xi}\Psi = (\theta_x\xi)\Psi = \theta_x(\xi\Psi) = \theta_{x(\xi\Psi)}$.

(ii) Let $\theta_u\Psi = \theta_v$. By (i) of this lemma, $\theta_{\overline{u}}\Psi = \theta_w$ for some $w \in \mathfrak{FCD}_X^g$. Using Lemma 7, we obtain

$$\begin{aligned} \theta_{v^{l(v)}} &= \theta_v^2 = (\theta_u\Psi)^2 = (\theta_u^2)\Psi \\ &= (\theta_u\theta_{\overline{u}})\Psi = \theta_u\Psi\theta_{\overline{u}}\Psi = \theta_v\theta_w = \theta_{w^{l(v)}}, \end{aligned}$$

where $w^{l(v)} = \underbrace{w \prec w \prec \ldots \prec w}_{l(v)}$. From here $w = v$ or $w = \overline{v}$. In the first case we have $\theta_{\overline{u}}\Psi = \theta_v$ which contradicts to injectivity of Ψ, therefore $\theta_{\overline{u}}\Psi = \theta_{\overline{v}}$.

(iii) This statement is anologous to the case (ii). □

An endomorphism θ of the free commutative g-dimonoid $\mathfrak{FCD}_X^g$ is called *linear* if $x\theta \in X$ for all $x \in X$.

Lemma 9. *Let Ψ be a stable automorphism of* $\mathrm{End}(\mathfrak{FCD}_X^g)$*,* $u, v\in\mathfrak{FCD}_X^g$*,* $x \in X$ *and* $\xi \in \mathrm{End}(\mathfrak{FCD}_X^g)$*. The following conditions hold:*

(i) $\xi\Psi = \xi$*, if* ξ *is linear;*

(ii) $c(u) = c(v)$*, if* $\theta_u\Psi = \theta_v$*;*

(iii) $l(x\xi) = l(x(\xi\Psi))$*.*

Proof. (i) If ξ is linear, then $x\xi \in X$ for all $x \in X$. Hence by stability of Ψ, $\theta_{x(\xi\Psi)} = \theta_{x\xi}\Psi = \theta_{x\xi}$. From here, $\xi\Psi = \xi$.

(ii) Let $\theta_u\Psi = \theta_v$ and $c(u) \setminus c(v) \neq \varnothing$. We take $z \in c(u) \setminus c(v)$, and $x \in X, x \neq z$, and $\xi \in \mathrm{End}(\mathfrak{FCD}_X^g)$ such that $z\xi = x$ and $y\xi = y$ for all $y \in X, y \neq z$. Then ξ is linear, $v\xi = v$ and

$$\theta_u\Psi = \theta_v = \theta_{v\xi} = \theta_v\xi = (\theta_u\Psi)(\xi\Psi) = (\theta_u\xi)\Psi = \theta_{u\xi}\Psi.$$

From here $\theta_u = \theta_{u\xi}$ and then $u = u\xi$ which contradicts to the definition of ξ, so $c(u)\setminus c(v) = \varnothing$. If $z \in c(v)\setminus c(u) \neq \varnothing,\ z \neq x$ and $\xi \in \mathrm{End}(\mathfrak{FCD}_X^g)$ the same as above, then

$$\theta_v = \theta_u\Psi = \theta_{u\xi}\Psi = (\theta_u\xi)\Psi = (\theta_u\Psi)(\xi\Psi) = \theta_v\xi = \theta_{v\xi},$$

whence $v=v\xi$ which contradicts to the definition of ξ. Thus, $c(v) \setminus c(u)=\varnothing$ and therefore, $c(u) = c(v)$.

(iii) Let $\xi_1, \xi_2 \in \mathrm{End}(\mathfrak{FCD}_X^g)$ such that $l(x\xi_1) = l(x\xi_2) = m$ and $l(x(\xi_1\Psi)) = r,\ \ l(x(\xi_2\Psi)) = s$. For all $t \in X$ we obtain

$$t(\theta_x\xi_1\theta_x) = (x\xi_1)\theta_x = \begin{cases} x^m = t\theta_{x^m}, & x\xi_1 \in F[X], \\ \overline{x^m} = t\theta_{\overline{x^m}}, & x\xi_1 \in \overline{F[X]} \setminus \overline{X}. \end{cases}$$

Analogously it is proved that $\theta_x\xi_2\theta_x = \begin{cases} \theta_{x^m}, & x\xi_2 \in F[X], \\ \theta_{\overline{x^m}}, & x\xi_2 \in \overline{F[X]} \setminus \overline{X}. \end{cases}$

Consider following four cases.

Case 1. $x\xi_1, x\xi_2 \in F[X]$. Using that Ψ is stable, we have

$$\theta_{x^m}\Psi = (\theta_x\xi_1\theta_x)\Psi = \theta_x(\xi_1\Psi)\theta_x = \begin{cases} \theta_{x^r}, & x(\xi_1\Psi) \in F[X], \\ \theta_{\overline{x^r}}, & x(\xi_1\Psi) \in \overline{F[X]} \setminus \overline{X}, \end{cases}$$

$$\theta_{x^m}\Psi = (\theta_x\xi_2\theta_x)\Psi = \theta_x(\xi_2\Psi)\theta_x = \begin{cases} \theta_{x^s}, & x(\xi_2\Psi) \in F[X], \\ \theta_{\overline{x^s}}, & x(\xi_2\Psi) \in \overline{F[X]} \setminus \overline{X}. \end{cases}$$

If $x(\xi_1\Psi) \in F[X]$, $x(\xi_2\Psi) \in \overline{F[X]}\setminus\overline{X}$ or $x(\xi_1\Psi) \in \overline{F[X]}\setminus\overline{X}$, $x(\xi_2\Psi) \in F[X]$, then we obtain $\theta_{x^r} = \theta_{\overline{x^s}}$ or $\theta_{\overline{x^r}} = \theta_{x^s}$ which is false. Otherwise, we have $r = s$.

Case 2. $x\xi_1, x\xi_2 \in \overline{F[X]} \setminus \overline{X}$. It is similar to the case 1.

Case 3. $x\xi_1 \in F[X]$, $x\xi_2 \in \overline{F[X]} \setminus \overline{X}$. Assume that $\theta_{x^m}\Psi = \theta_{x^r}$, then by (ii) of Lemma 8 we have $\theta_{\overline{x^m}}\Psi = \theta_{\overline{x^r}}$. On the other hand,

$$\theta_{\overline{x^m}}\Psi = (\theta_x\xi_2\theta_x)\Psi = \theta_x(\xi_2\Psi)\theta_x = \begin{cases} \theta_{x^s}, & x(\xi_2\Psi) \in F[X], \\ \theta_{\overline{x^s}}, & x(\xi_2\Psi) \in \overline{F[X]} \setminus \overline{X}. \end{cases}$$

For $x(\xi_2\Psi) \in F[X]$ we obtain $\overline{x^r} = x^s$ which is false. If $x(\xi_2\Psi) \in \overline{F[X]} \setminus \overline{X}$, then $\theta_{\overline{x^r}} = \theta_{\overline{x^s}}$, whence $r = s$.

In similar way we can show that $r = s$ if $\theta_{x^m}\Psi = \theta_{\overline{x^r}}$.

Case 4. $x\xi_1 \in \overline{F[X]} \setminus \overline{X}$, $x\xi_2 \in F[X]$. It is analogous to the case 3.

Thus, cases 1–4 imply that r and s coincides.

Further, let A be a nonempty finite subset of X and

$$\mathrm{End}_A^m(x) = \{\xi \in \mathrm{End}(\mathfrak{FCD}_X^g) \mid l(x\xi) = m, c(x\xi) = A\}.$$

For $\theta_{x\xi} \in \mathrm{End}_A^m(x)$ by (i) of Lemma 8 we have $\theta_{x\xi}\Psi = \theta_{x(\xi\Psi)}$. By (ii) of given lemma, $c(x\xi) = c(x(\xi\Psi))$. Taking into account the previous arguments, there exists k such that $\mathrm{End}_A^m(x)\Psi \subseteq \mathrm{End}_A^k(x)$. Since Ψ is bijective, $k = m$. Thus, $l(x\xi) = l(x(\xi\Psi))$ for all $\xi \in \mathrm{End}(\mathfrak{FCD}_X^g)$ and $x \in X$. $\square$

Corollary 1. *Let Ψ be a stable automorphism of $\mathrm{End}(\mathfrak{FCD}_X^g)$ and $x_1, x_2 \in X$ are distinct. Then*

$$\theta_{x_1x_2}\Psi = \theta_{x_1x_2} \ \textit{or}\ \theta_{x_1x_2}\Psi = \theta_{\overline{x_1x_2}}.$$

Proof. By Lemma 8 (i), $\theta_{x_1x_2}\Psi = \theta_u$ for some $u \in \mathfrak{FCD}_X^g$, and by (ii) of Lemma 9, $c(u) = \{x_1, x_2\}$. By (iii) of Lemma 9, $l(u) = 2$. Thus, $u = x_1x_2$ or $u = \overline{x_1x_2}$. $\square$

Lemma 10. *Let Ψ be a stable automorphism of $\mathrm{End}(\mathfrak{FCD}_X^g)$ and $x_1, x_2 \in X$ are distinct. Then*

(i) $\theta_{x_1x_2}\Psi = \theta_{x_1x_2}$ *implies* $\Psi = \Phi_0$*;*

(ii) $\theta_{x_1x_2}\Psi = \theta_{\overline{x_1x_2}}$ *implies* $\Psi = \Phi_1$*.*

Proof. (i) By induction on the length of u we show that $\theta_u\Psi = \theta_u$ for all $u \in F[X]$. By stability of Ψ, $\theta_x\Psi = \theta_x$ for all $x \in X$. Assume that $\theta_v\Psi = \theta_v$ for all $v \in F[X]$ with $l(v) < n$, and let $u = u_1 \dots u_n \in F[X]$, where $n \geqslant 2$. Let $v_1 = u_1 \dots u_{n-1}$, $v_2 = u_n$ and $f \in \mathrm{End}(\mathfrak{FCD}_X^g)$ is such that $x_1f = v_1$, $x_2f = v_2$ and $yf = y$ for all $y \in X \setminus \{x_1, x_2\}$. Then $x(\theta_{x_1x_2}f) = (x_1x_2)f = x_1fx_2f = u = x\theta_u$ for all $x \in X$.

By Lemma 8 (i) and the induction hypothesis, we have

$$\theta_{x_i(f\Psi)} = \theta_{x_i f}\Psi = \theta_{v_i}\Psi = \theta_{v_i} = \theta_{x_i f}, \quad i \in \{1,2\},$$
$$\theta_{x(f\Psi)} = \theta_{xf}\Psi = \theta_x\Psi = \theta_x = \theta_{xf}, \quad x \in X \setminus \{x_1, x_2\}.$$

So, $f\Psi = f$ and then for all $u \in F[X]$ with $l(u) \geqslant 2$,

$$\theta_u\Psi = (\theta_{x_1x_2}f)\Psi = (\theta_{x_1x_2}\Psi)(f\Psi) = \theta_{x_1x_2}f = \theta_u.$$

By (ii) of Lemma 8, $\theta_{\overline{u}}\Psi = \theta_{\overline{u}}$ for all $\overline{u} \in \overline{F[X]} \setminus \overline{X}$, so that $\theta_u\Psi = \theta_u$ for all $u \in \mathfrak{FCD}^g_X$. Now for all $x \in X$ and $\varphi \in \mathrm{End}(\mathfrak{FCD}^g_X)$,

$$\theta_{x(\varphi\Psi)} = \theta_{x\varphi}\Psi = \theta_{x\varphi}.$$

This implies $\varphi\Psi = \varphi$ for all $\varphi \in \mathrm{End}(\mathfrak{FCD}^g_X)$, that is, $\Psi = \Phi_0$.

(ii) Take the crossed automorphism $\varepsilon^*_{id_X}$ of $\mathfrak{FCD}^g_X$ (see Lemma 2). For all $u \in \mathfrak{FCD}^g_X$ and $f \in \mathrm{End}(\mathfrak{FCD}^g_X)$ we use denotations $u^* = u\varepsilon^*_{id_X}$ and $f^* = (\varepsilon^*_{id_X})^{-1} f \varepsilon^*_{id_X}$.

By induction on $l(u)$ we show that $\theta_u\Psi = \theta_{u^*}$ for all $u \in F[X]$. The induction base follows from the fact that Ψ is stable.

Let us suppose that $\theta_v\Psi = \theta_{v^*}$ for all $v \in F[X]$ such that $l(v) < n$, and let $u = u_1 \dots u_n \in F[X], n \geqslant 2$. We put $v_1 = u_1, v_2 = u_2 \dots u_n$, and take the endomorphism f of $\mathfrak{FCD}^g_X$ such that $x_1 f = v_1$, $x_2 f = v_2$, and $yf = y$ for all $y \in X \setminus \{x_1, x_2\}$.

Similarly as in (i) of this lemma, we can show that $\theta_{x_1x_2}f = \theta_u$. By Lemma 8 (i) and the induction hypothesis,

$$\theta_{x_i(f\Psi)} = \theta_{x_i f}\Psi = \theta_{v_i}\Psi = \theta_{v_i^*} = \theta_{x_i f^*}, \quad i \in \{1,2\},$$
$$\theta_{x(f\Psi)} = \theta_{xf}\Psi = \theta_x\Psi = \theta_{x^*} = \theta_{xf^*}, \quad x \in X \setminus \{x_1, x_2\}.$$

From here, $f\Psi = f^*$. Then for all $u \in F[X]$ with $l(u) \geqslant 2$,

$$\theta_u\Psi = (\theta_{x_1x_2}f)\Psi = (\theta_{x_1x_2}\Psi)(f\Psi) = \theta_{\overline{x_1x_2}}f^* = \theta_{\overline{u}} = \theta_{u^*}.$$

Taking into account Lemma 8 (iii), $\theta_{\overline{u}}\Psi = \theta_u$ for all $\overline{u} \in \overline{F[X]} \setminus \overline{X}$. It means that $\theta_u\Psi = \theta_{u^*}$ for all $u \in \mathfrak{FCD}^g_X$.

Finally, for all $x \in X$ and $\varphi \in \mathrm{End}(\mathfrak{FCD}^g_X)$ we have

$$\theta_{x(\varphi\Psi)} = \theta_{x\varphi}\Psi = \theta_{(x\varphi)^*} = \theta_{x\varphi^*}.$$

Hence, $\varphi\Psi = \varphi^*$ for all $\varphi \in \mathrm{End}(\mathfrak{FCD}^g_X)$, that is, $\Psi = \Phi_1$. $\square$

Theorem 2. *Let X be an arbitrary set with $|X| \geqslant 2$. Every isomorphism $\Phi : \mathrm{End}(\mathfrak{FCD}_X^g) \to \mathrm{End}(\mathfrak{FCD}_Y^g)$ is induced either by the isomorphism ε_f or by the crossed isomorphism ε_f^* of $\mathfrak{FCD}_X^g$ onto $\mathfrak{FCD}_Y^g$ for a uniquely determined bijection $f : X \to Y$.*

Proof. Let $\Phi : \mathrm{End}(\mathfrak{FCD}_X^g) \to \mathrm{End}(\mathfrak{FCD}_Y^g)$ be an arbitrary isomorphism. In similar way as in the case of free abelian dimonoids (see [19, Theorem 3]), using Lemma 4 can be shown that for every $x \in X$ there exists $y \in Y$ such that $\theta_x\Phi = \theta_y$. Define a bijection $f : X \to Y$ putting $xf = y$ if $\theta_x\Phi = \theta_y$. In this case we say that f is induced by Φ.

By Lemma 1, f induces the isomorphism $\varepsilon_f : \mathfrak{FCD}_X^g \to \mathfrak{FCD}_Y^g$. According to Lemma 3, $E_f : \eta \mapsto \varepsilon_f^{-1}\eta\varepsilon_f$ is an isomorphism of $\mathrm{End}(\mathfrak{FCD}_X^g)$ onto $\mathrm{End}(\mathfrak{FCD}_Y^g)$. From this it follows that the composition ΦE_f^{-1} is an automorphism of $\mathrm{End}(\mathfrak{FCD}_X^g)$.

Further for all $x \in X$ we have

$$\theta_x(\Phi E_f^{-1}) = (\theta_x\Phi)E_f^{-1} = \theta_{xf}E_f^{-1} = \theta_{(xf)f^{-1}} = \theta_x,$$

which implies stability of ΦE_f^{-1}.

Using Corollary 1 and Lemma 10, we obtain ΦE_f^{-1} is either the identity automorphism Φ_0 or the mirror automorphism Φ_1. Assume, $\Phi E_f^{-1} = \Phi_0$, then $\Phi = E_f$ which means that Φ is an isomorphism induced by ε_f. If $\Phi E_f^{-1} = \Phi_1$, then $\Phi = \Phi_1 E_f$ which means that Φ is an isomorphism induced by ε_f^*. □

The following statement gives the positive solution of the definability problem of free commutative g-dimonoids by its endomorphism semigroups.

Corollary 2. *Let $\mathfrak{FCD}_X^g$ and $\mathfrak{FCD}_Y^g$ be free commutative g-dimonoids such that $\mathrm{End}(\mathfrak{FCD}_X^g) \cong \mathrm{End}(\mathfrak{FCD}_Y^g)$. Then $\mathfrak{FCD}_X^g$ and $\mathfrak{FCD}_Y^g$ are isomorphic.*

Proof. As shown in the proof of Theorem 2, every isomorphism $\Phi : \mathrm{End}(\mathfrak{FCD}_X^g) \to \mathrm{End}(\mathfrak{FCD}_Y^g)$ induces a bijection $X \to Y$, therefore obviously we obtain $\mathfrak{FCD}_X^g \cong \mathfrak{FCD}_Y^g$. □

We recall that an automorphism $\Phi : \mathrm{End}(\mathfrak{FCD}_X^g) \to \mathrm{End}(\mathfrak{FCD}_X^g)$ is quasi-inner if there exists $\alpha \in S(\mathfrak{FCD}_X^g)$ such that $\beta\Phi = \alpha^{-1}\beta\alpha$ for all $\beta \in \mathrm{End}(\mathfrak{FCD}_X^g)$.

At the end we consider the automorphism group of $\mathrm{End}(\mathfrak{FCD}_X^g)$.

Theorem 3. *Let X be an arbitrary set with $|X| \geqslant 2$. Then*

(i) *all automorphisms of* $\mathrm{End}(\mathfrak{FCD}_X^g)$ *are quasi-inner;*

(ii) *the automorphism group* $\mathrm{Aut}(\mathrm{End}(\mathfrak{FCD}_X^g))$ *is isomorphic to the direct product* $S(X) \times C_2$.

Proof. (i) Let $X = Y$ in Theorem 2, then it will be the part of Theorem 3. It is not hard to see that every automorphism Φ of $\mathrm{End}(\mathfrak{FCD}_X^g)$ is either an inner automorphism or the product of a mirror automorphism and an inner automorphism. Namely, we have $\Phi = E_\varphi$ or $\Phi = \Phi_1 E_\varphi$ for a suitable bijection $\varphi : X \to X$. It means that all automorphisms of $\mathrm{End}(\mathfrak{FCD}_X^g)$ are quasi-inner.

(ii) It is clear that the automorphism group $\{\Phi_0, \Phi_1\}$ of $\mathrm{End}(\mathfrak{FCD}_X^g)$ is isomorphic to C_2. Define a mapping $\zeta : \mathrm{Aut}(\mathrm{End}(\mathfrak{FCD}_X^g)) \to S(X) \times C_2$ as follows:

$$\Phi\zeta = \begin{cases} (\varphi, \Phi_0), & \Phi = E_\varphi, \\ (\varphi, \Phi_1), & \Phi = \Phi_1 E_\varphi \end{cases}$$

for all $\Phi \in \mathrm{Aut}(\mathrm{End}(\mathfrak{FCD}_X^g))$.

It is easy to see that ζ is a bijection. Since for all $\varphi, \psi \in S(X)$ and $f \in \mathrm{End}(\mathfrak{FCD}_X^g)$,

$$\begin{aligned} f(E_\varphi E_\psi) &= (\varepsilon_\varphi^{-1} f \varepsilon_\varphi) E_\psi = (\varepsilon_\varphi \varepsilon_\psi)^{-1} f (\varepsilon_\varphi \varepsilon_\psi) \\ &= \varepsilon_{\varphi\psi}^{-1} f \varepsilon_{\varphi\psi} = f E_{\varphi\psi} \end{aligned}$$

and

$$\begin{aligned} f(E_\varphi \Phi_1) &= (\varepsilon_\varphi^{-1} f \varepsilon_\varphi)\Phi_1 = (\varepsilon_{id_X}^* \varepsilon_\varphi^{-1}) f (\varepsilon_\varphi \varepsilon_{id_X}^*) \\ &= (\varepsilon_\varphi^{-1} \varepsilon_{id_X}^*) f (\varepsilon_{id_X}^* \varepsilon_\varphi) = (\varepsilon_{id_X}^* f \varepsilon_{id_X}^*) E_\varphi = f(\Phi_1 E_\varphi), \end{aligned}$$

we obtain $E_\varphi E_\psi = E_{\varphi\psi}$ and $E_\varphi \Phi_1 = \Phi_1 E_\varphi$.

The immediate check shows that ζ is a homomorphism. □

References

[1] Loday J.-L., Dialgebras, in: Dialgebras and related operads, *Lect. Notes Math.* **1763**, Springer-Verlag, Berlin, 2001, 7–66.

[2] Pozhidaev A. P., 0-dialgebras with bar-unity and nonassociative Rota- Baxter algebras, *Sib. Math. J.* **50** (2009), no. 6, 1070–1080.

[3] Pirashvili T., Sets with two associative operations, *Cent. Eur. J. Math.* **2** (2003), 169–183.

[4] Movsisyan Y., Davidov S., Safaryan Mh., Construction of free g-dimonoids, *Algebra and Discrete Math.* **18** (2014), no. 1, 138–148.

[5] Zhuchok Yul. V. On one class of algebras, *Algebra and Discrete Math.* **18**:2 (2014), no. 2, 306-–320.

[6] Zhuchok A. V., Zhuchok Yu. V., Free commutative g-dimonoids, *Chebyshevskii Sb.* **16** (2015), no. 3, 276–284.

[7] Loday J.-L., Ronco M. O., Trialgebras and families of polytopes, *Contemp. Math.* **346** (2004), 369–398.

[8] Casas J. M., Trialgebras and Leibniz 3-algebras, *Bolet??n de la Sociedad Matematica Mexicana* **12** (2006), no. 2, 165–178.

[9] Zhuchok Yu.V., The endomorphism monoid of a free troid of rank 1, *Algebra Universalis*, 2016. (to appear)

[10] Plotkin B.I., Seven Lectures on the Universal Algebraic Geometry, *Preprint, Institute of Mathematics*, Hebrew University, 2000.

[11] Plotkin B.I., Algebras with the same (algebraic) geometry, *Proc. of the Steklov Institute of Mathematics* **242** (2003), 176–207.

[12] Formanek E., A question of B. Plotkin about the semigroup of endomorphisms of a free group, *Proc. American Math. Soc.* **130** (2001), 935–937.

[13] Mashevitsky G., Schein B.M., Automorphisms of the endomorphism semigroup of a free monoid or a free semigroup. *Proc. American Math. Soc.* **131** (2003), no. 6, 1655–1660.

[14] Kanel-Belov A., Berzins A., Lipyanski R., Automorphisms of the semigroup of endomorphisms of free associative algebras, arXiv:math/0512273v3 [math.RA], 2005.

[15] Mashevitsky G., Schein B.M., Zhitomirski G.I., Automorphisms of the endomorphism semigroup of a free inverse semigroup, *Communic. in Algebra* **34** (2006), no. 10, 3569–3584.

[16] Katsov Y., Lipyanski R., Plotkin B.I., Automorphisms of categories of free modules, free semimodules, and free Lie modules, *Communic. in Algebra* **35** (2007), no. 3, 931–952.

[17] Mashevitzky G., Plotkin B., Plotkin E., Automorphisms of the category of free Lie algebras, *J. of Algebra* **282** (2004), 490–512.

[18] Mashevitzky G., Plotkin B., Plotkin E., Automorphisms of categories of free algebras of varieties, *Electronic research announvements of American Math. Soc.* **8** (2002), 1–10.

[19] Zhuchok Yu.V., Free abelian dimonoids, *Algebra and Discrete Math.* **20** (2015), no. 2, 330–342.

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