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RESEARCH ARTICLE

On the semigroup of injective transformations with restricted range that equal gap and defect

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Abstract. Let X be an infinite set and $I(X)$ the symmetric inverse semigroup on X. Let $A(X) = \{\alpha \in I(X) : |X \setminus \operatorname{dom} \alpha| = |X \setminus X\alpha|\}$, it is known that $A(X)$ is the largest factorizable subsemigroup of $I(X)$. In this article, for any nonempty subset Y of X, we consider the subsemigroup $A(X, Y)$ of $A(X)$ of all transformations with range contained in Y. We give a complete description of Green's relations on $A(X, Y)$. With respect to the natural partial order on a semigroup, we determine when two elements in $A(X, Y)$ are related and find all the maximum, minimum, maximal, minimal, lower cover and upper cover elements. We also describe elements which are compatible and we investigate the greatest lower bound and the least upper bound of two elements in $A(X, Y)$.

1. Introduction

Suppose that X is a nonempty set and let $P(X)$ denote the set of all partial transformations of X, i.e., all transformations α whose domain, $\operatorname{dom} \alpha$, and range, $X\alpha$ are subsets of X. Also, let $I(X)$ denote the

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symmetric inverse semigroup on X: that is, the set of all injective mappings in $P(X)$. As usual, $|X|$ denotes the cardinality of X and we write $X\backslash Y = \{x \in X : x \notin Y\}$, where Y is a set. We also write

$$g(\alpha) = |X\backslash \mathrm{dom}\ \alpha|, \ \ d(\alpha) = |X\backslash X\alpha| \ \text{ and } r(\alpha) = |X\alpha|,$$

and refer to these cardinals as the *gap*, the *defect* and the *rank* of α, respectively. In [1], when X is an infinite set, the authors studied the semigroup

$$A(X) = \{\alpha \in I(X) : g(\alpha) = d(\alpha)\}.$$

They showed that any inverse semigroup can be embedded in some $A(X)$ and $A(X)$ is the largest factorizable subsemigroup of $I(X)$.

In the paper [6], Sanwong and Sullivan gave a complete proof that $A(X)$ is an inverse subsemigroup of $I(X)$. They also described Green's relations and ideals in $A(X)$ and determined all maximal subsemigroups of $A(X)$ when X is uncountable. Recently, in 2022, Singha [7] described all prime ideals of $A(X)$ and applied these results to characterize all maximal subsemigroups of $A(X)$ for an arbitrary infinite set X. In [4], Mendes-Gonçalves and Mendes Araujo studied a linear version of this semigroup by defining the semigroup $A(V)$ consisting of all linear transformation α on a vector space V with equal gap and defect by changing the meaning of the gap and the defect of α to the co-dimension of dom α and the co-dimension of $X\alpha$ respectively.

In this paper, we generalize $A(X)$ by letting a nonempty subset Y of X and defining

$$A(X, Y) = \{\alpha \in I(X) : g(\alpha) = d(\alpha) \text{ and } X\alpha \subseteq Y\}.$$

Clearly $A(X, Y)$ is a subsemigroup of $A(X)$ and when $X = Y$, we obtain that $A(X, Y) = A(X)$. Thus, we may regard $A(X, Y)$ as a generalization of $A(X)$.

In 1986, Mitsch [5] introduced the definition of the natural partial order on an arbitrary semigroup S by

$$a \le b \ \text{ if and only if } \ a = xb = by \text{ and } \ a = ay \ \text{ for some } \ x, y \in S^1,$$

where the notation S^1 means S itself if S contains the identity element, otherwise S^1 denotes the semigroup obtained from S by adjoining an extra identity element 1. A wide range of properties of the natural partial order have been studied on various transformation semigroups. In 2003, Marques-Smith and Sullivan [3] studied various properties of the natural partial order $\leq$ and the another partial order $\subseteq$ on $P(X)$, namely the containment order defined by

$$\alpha \subseteq \beta \text{ if and only if } \operatorname{dom} \alpha \subseteq \operatorname{dom} \beta \text{ and } x\alpha = x\beta \text{ for all } x \in \operatorname{dom} \alpha.$$

They showed that these two partial orders are different and each of them is not contained in the other. Unlike for $I(X)$, the result was shown in [2, p. 153], that $\leq$ and $\subseteq$ are equal in $I(X)$. For the description of the natural partial order on $A(X)$, it was not characterized before.

The main objective of this paper is to study the semigroup $A(X,Y)$. In Section 2, we give a brief introduction to the notations used and we investigate some elementary results of $A(X,Y)$. In Section 3, we determine Green's relations on $A(X,Y)$. The results in this section generalize the corresponding results for $A(X)$ obtained in [6]. In Section 4, we use the results from Section 3 to describe the natural partial order on this semigroup.

2. Preliminary

Throughout this paper, we suppose that X is an infinite set with $|X| = n \geq \aleph_0$ and Y is a nonempty subset of X. For each mapping $\alpha \in A(X,Y)$, we write

$$\alpha = \begin{pmatrix} a_i \\ y_i \end{pmatrix},$$

where the subscript i belongs to some unmentioned index set I, the abbreviation $\{y_i\}$ denotes $\{y_i : i \in I\}$, $X\alpha = \{y_i\} \subseteq Y$, $\operatorname{dom} \alpha = \{a_i\}$ and $a_i\alpha = y_i$. For a subset A of X, we denote by $\alpha|_A$ the restriction of α to A and we write $A\alpha$ instead of $(A \cap \operatorname{dom} \alpha)\alpha$ for convenience. Also, denote by id_A the identity function on A and we write $A = B \mathbin{\dot\cup} C$ to denote A is a disjoint union of B and C. As usual, $\emptyset$ denotes the emptyset, but in some contexts, $\emptyset$ is used to refer to the empty (one-to-one) transformation which is the zero element in $I(X)$ and $A(X,Y)$.

We refer to [6, Lemma 1] that, if $\alpha \in I(X)$ and $Y, Z \subseteq X$, then $(Y \setminus Z)\alpha = Y\alpha \setminus Z\alpha$. This fact has been used in [6, Lemma 2] to show that $A(X)$ is closed under the composition of mapping and we will use this idea as well for $A(X,Y)$. We begin with some basic results on $A(X,Y,)$ which will be used in the following.

Proposition 1. *$A(X,Y)$ is a semigroup containing zero element and it contains the identity element which is* id_X *precisely when $X = Y$.*

Proof. Let $\alpha, \beta \in A(X,Y)$, then $g(\alpha) = d(\alpha)$ and $g(\beta) = d(\beta)$, whence

$$\begin{aligned}
d(\alpha\beta) &= |X \setminus X\beta| + |X\beta \setminus X\alpha\beta| \\
&= d(\beta) + |((X \setminus X\alpha) \cap \operatorname{dom}\beta)\beta| \\
&= g(\beta) + |(X \setminus X\alpha) \cap \operatorname{dom}\beta| \\
&= |(X \setminus \operatorname{dom}\beta) \cap X\alpha| + |(X \setminus \operatorname{dom}\beta) \cap (X \setminus X\alpha)| \\
&\quad + |(X \setminus X\alpha) \cap \operatorname{dom}\beta| \\
&= |(X \setminus \operatorname{dom}\beta)\alpha^{-1}| + d(\alpha) \\
&= |(X \setminus \operatorname{dom}\beta)\alpha^{-1}| + g(\alpha) \\
&= |X\alpha^{-1} \setminus (X\alpha \cap \operatorname{dom}\beta)\alpha^{-1}| + |X \setminus X\alpha^{-1}| \\
&= |X \setminus (X\alpha \cap \operatorname{dom}\beta)\alpha^{-1}| \\
&= g(\alpha\beta).
\end{aligned}$$

Moreover, as $X\alpha\beta \subseteq X\beta \subseteq Y$, whence $\alpha\beta \in A(X,Y)$ as required. Next, as X is an infinite set, we can choose $x \in X, y \in Y$ for which $x \neq y$ and define

$$\alpha = \begin{pmatrix} y \\ y \end{pmatrix}, \beta = \begin{pmatrix} x \\ y \end{pmatrix}.$$

Then $\alpha, \beta \in A(X,Y)$ and thus $\emptyset = \alpha\beta \in A(X,Y)$. Finally, it is clear that when $X = Y$, we have $A(X,Y) = A(X)$ and in this case id_X is the identity of $A(X,Y)$. Suppose that $Y \subsetneq X$. We can choose $x \in X \setminus Y$, $y \in Y$ and define

$$\gamma = \begin{pmatrix} x \\ y \end{pmatrix} \in A(X,Y).$$

If $\mu \in A(X,Y)$ is the identity element, then $\mu\gamma = \gamma$. Therefore $x\mu\gamma = x\gamma$. As γ is injective, we have that $x\mu = x \notin Y$, this contradicts to that $\mu \in A(X,Y)$. Hence, $A(X,Y)$ has no identity when $X \neq Y$. □

To close this section, we discuss the regularity of $A(X,Y)$. As mentioned in the introduction that $A(X)$ is an inverse semigroup, therefore it is regular. Here, we observe that, in general $A(X,Y)$ is not a regular semigroup. For example, let $X = \mathbb{N}$ be the set of all positive integers, let Y be the set of all positive even integers. Let $\alpha \in A(X,Y)$ be defined by

$$\alpha = \begin{pmatrix} 3 & 5 \\ 2 & 4 \end{pmatrix}.$$

Suppose that $\alpha = \alpha\beta\alpha$ for some $\beta \in A(X,Y)$. Then, $3\alpha = 3\alpha\beta\alpha$. As α is injective, we have that $3\alpha\beta = 3 \in X\beta \setminus Y$, this contradicts to that $\beta \in A(X,Y)$. Hence, $A(X,Y)$ is not a regular semigroup.

3. Green's Relations

In this section, we characterize the Green's relations on $A(X,Y)$ by using some ideas of the proof for $A(X)$ in [6] with the idea of restricted range concerned. For the definition of Green's relations $\mathcal{L}$, $\mathcal{R}$, $\mathcal{H}$, $\mathcal{D}$, and $\mathcal{J}$ on a semigroup, see Chapter 2 of [2]. We also recall from Proposition 1 that, in general $A(X,Y)$ has no the identity element, so the notation $A(X,Y)^1$ denotes a monoid obtained from $A(X,Y)$ by adjoining an identity 1.

For comparison with what follows, we recall the following descriptions of the Green's relations in the semigroups $A(X)$ which was proved in 2009 by Sanwong and Sullivan.

Theorem 1 ([6, Section 3])**.** *Let $\alpha,\beta \in A(X)$. Then the following statements hold.*

(*a*) *$\alpha\mathcal{R}\beta$ if and only if* $\operatorname{dom}\alpha = \operatorname{dom}\beta$.
(*b*) *$\alpha\mathcal{L}\beta$ if and only if $X\alpha = X\beta$.*
(*c*) *$\alpha\mathcal{H}\beta$ if and only if* $\operatorname{dom}\alpha = \operatorname{dom}\beta$ *and $X\alpha = X\beta$.*
(*d*) *$\alpha\mathcal{D}\beta$ if and only if* $r(\alpha) = r(\beta)$ *and* $d(\alpha) = d(\beta)$.
(*e*) *$\mathcal{D} = \mathcal{J}$ on $A(X)$.*

We begin by characterizing the $\mathcal{R}$ and $\mathcal{L}$ relations on $A(X,Y)$. As discussed in Section 2 that $A(X,Y)$ is not a regular subsemigroup of $A(X)$, so Hall's Theorem cannot be used to describe these two relations on $A(X,Y)$ in terms of their characterization on $A(X)$. However, the following result for the $\mathcal{R}$ relation on $A(X,Y)$ appears to coincide with the results in Theorem 1 when $\alpha\mathcal{R}\beta$ in $A(X)$.

Theorem 2. *Let $\alpha,\beta \in A(X,Y)$. Then $\alpha = \beta\mu$ for some $\mu \in A(X,Y)$ if and only if* $\operatorname{dom}\alpha \subseteq \operatorname{dom}\beta$. *In other word, $\alpha\mathcal{R}\beta$ in $A(X,Y)$ if and only if* $\operatorname{dom}\alpha = \operatorname{dom}\beta$.

Proof. It is clear that, if $\alpha = \beta\mu$ for some $\mu \in A(X,Y)$, then $\operatorname{dom}\alpha \subseteq \operatorname{dom}\beta$. For the converse, we suppose that $\operatorname{dom}\alpha \subseteq \operatorname{dom}\beta$, whence $X \setminus \operatorname{dom}\alpha = (X \setminus \operatorname{dom}\beta) \dot\cup (\operatorname{dom}\beta \setminus \operatorname{dom}\alpha)$ and so $g(\alpha) = g(\beta) + |\operatorname{dom}\beta \setminus \operatorname{dom}\alpha|$. We can write

$$\alpha = \begin{pmatrix} a_i \\ b_i \end{pmatrix} \text{ and } \beta = \begin{pmatrix} a_i & x_j \\ c_i & y_j \end{pmatrix},$$

where $\operatorname{dom}\alpha = \{a_i\} \subseteq \operatorname{dom}\beta$. We define $\mu = \begin{pmatrix} c_i \\ b_i \end{pmatrix}$. Then $\alpha = \beta\mu$ and $X\mu = X\alpha \subseteq Y$, $g(\mu) = d(\beta) + |\{y_j\}| = g(\beta) + |\operatorname{dom}\beta \setminus \operatorname{dom}\alpha| = g(\alpha) =$

$d(\alpha) = d(\mu)$, whence $\mu \in A(X, Y)$ and as required. Next, suppose that $\alpha\mathcal{R}\beta$ in $A(X, Y)$. Then $\alpha = \beta\mu$ and $\beta = \alpha\lambda$ for some $\mu, \lambda \in A(X, Y)^1$. If $\mu = 1$ or $\lambda = 1$, then $\alpha = \beta$ and so $\operatorname{dom} \alpha = \operatorname{dom} \beta$. On the other hand, if $\mu \neq 1 \neq \lambda$, then $\mu, \lambda \in A(X, Y)$. Thus, we can deduce from the first part of this proof that $\operatorname{dom} \alpha = \operatorname{dom} \beta$. Conversely, if $\operatorname{dom} \alpha = \operatorname{dom} \beta$, then $\alpha = \beta\mu$ and $\beta = \alpha\lambda$ for some $\mu, \lambda \in A(X, Y)$, whence $\alpha\mathcal{R}\beta$. $\square$

In order to characterize the $\mathcal{L}$ relation on $A(X, Y)$, the following lemma is needed.

Lemma 1. *Let $\alpha, \beta \in A(X, Y)$. Then $\alpha = \lambda\beta$ for some $\lambda \in A(X, Y)$ if and only if the following conditions hold.*

(*a*) *$X\alpha \subseteq X\beta$.*

(*b*) *$(X\alpha)\beta^{-1} \subseteq Y$.*

Proof. Suppose that $\alpha = \lambda\beta$ for some $\lambda \in A(X, Y)$. Then $X\alpha \subseteq X\beta$ and we suppose

$$\alpha = \begin{pmatrix} a_i \\ b_i \end{pmatrix} \text{ and } \beta = \begin{pmatrix} x_i & x_k \\ b_i & b_k \end{pmatrix},$$

where $X\alpha = \{b_i\} \subseteq X\beta$ and $\{x_i\} = (X\alpha)\beta^{-1}$. Thus $a_i\lambda\beta = a_i\alpha = b_i = x_i\beta$. Since β is injective, we have that $x_i = a_i\lambda \in Y$, that is $(X\alpha)\beta^{-1} \subseteq Y$.

Conversely, suppose that the conditions (*a*) and (*b*) hold. Then we can write

$$\alpha = \begin{pmatrix} a_i \\ b_i \end{pmatrix} \text{ and } \beta = \begin{pmatrix} x_i & x_k \\ b_i & b_k \end{pmatrix},$$

where $X\alpha = \{b_i\} \subseteq X\beta$, $\{x_i\} = (X\alpha)\beta^{-1} \subseteq Y$ and $X\beta \setminus X\alpha = \{b_k\}$. As $X\alpha \subseteq X\beta$, we have that $X \setminus X\alpha = (X \setminus X\beta) \dot{\cup} (X\beta \setminus X\alpha)$ and so $d(\alpha) = d(\beta) + |X\beta \setminus X\alpha|$. Now, define $\lambda = \begin{pmatrix} a_i \\ x_i \end{pmatrix}$. Then $\alpha = \lambda\beta$, $X\lambda \subseteq Y$ and $d(\lambda) = g(\beta) + |\{x_k\}| = d(\beta) + |X\beta \setminus X\alpha| = d(\alpha) = g(\alpha) = g(\lambda)$, whence $\lambda \in A(X, Y)$. $\square$

Using Lemma 1 , we have the following theorem.

Theorem 3. *Let $\alpha, \beta \in A(X, Y)$. Then $\alpha\mathcal{L}\beta$ if and only if*

$$\alpha = \beta \;\; or \;\; (X\alpha = X\beta, \; \operatorname{dom} \alpha \subseteq Y \;\; and \;\; \operatorname{dom} \beta \subseteq Y).$$

Proof. Suppose that $\alpha\mathcal{L}\beta$ in $A(X,Y)$. Then $\alpha = \lambda\beta$ and $\beta = \mu\alpha$ for some $\lambda, \mu \in A(X,Y)^1$. If $\alpha \neq \beta$, then $\lambda, \mu \in A(X,Y)$. Thus, Lemma 1 implies that the following conditions hold:

$$(a_1)\ X\alpha \subseteq X\beta,\ (a_2)\ (X\alpha)\beta^{-1} \subseteq Y,\ (b_1)\ X\beta \subseteq X\alpha,\ (b_2)\ (X\beta)\alpha^{-1} \subseteq Y.$$

Then (a_1) and (b_1) imply $X\alpha = X\beta$. Consequently, (a_2) implies $\operatorname{dom}\beta = (X\beta)\beta^{-1} = (X\alpha)\beta^{-1} \subseteq Y$. Similarly, (b_2) implies $\operatorname{dom}\alpha \subseteq Y$ as required.

Conversely, it is clear that if $\alpha = \beta$, then $\alpha\mathcal{L}\beta$. Now we suppose that $X\alpha = X\beta$, $\operatorname{dom}\alpha \subseteq Y$ and $\operatorname{dom}\beta \subseteq Y$. Then $(X\alpha)\beta^{-1} = (X\beta)\beta^{-1} = \operatorname{dom}\beta \subseteq Y$. Thus, by Lemma 1, $\alpha = \lambda\beta$ for some $\lambda \in A(X,Y)$. Similarly, we can show that $\beta = \mu\alpha$ for some $\mu \in A(X,Y)$. Hence, $\alpha\mathcal{L}\beta$ as required. □

According to Theorem 2 and Theorem 3, we have the following conclusion readily for $\mathcal{H}$ relation on $A(X,Y)$.

Corollary 1. *Let $\alpha, \beta \in A(X,Y)$. Then $\alpha\mathcal{H}\beta$ in $A(X,Y)$ if and only if*

$$\alpha = \beta \ \textit{ or } \ (X\alpha = X\beta \ \textit{ and } \ \operatorname{dom}\alpha = \operatorname{dom}\beta \subseteq Y).$$

The following theorem characterizes the relation $\mathcal{D}$ on $A(X,Y)$.

Theorem 4. *Let $\alpha, \beta \in A(X,Y)$. Then $\alpha\mathcal{D}\beta$ if and only if* $\operatorname{dom}\alpha = \operatorname{dom}\beta$ *or* $(r(\alpha) = r(\beta), \operatorname{dom}\alpha \subseteq Y, \operatorname{dom}\beta \subseteq Y$ *and* $g(\alpha) = g(\beta))$.

Proof. Suppose that $\alpha\mathcal{D}\beta$. Then there exists $\gamma \in A(X,Y)$ such that $\alpha\mathcal{L}\gamma$ and $\gamma\mathcal{R}\beta$. Then by Theorem 2 and Theorem 3, we get $\operatorname{dom}\gamma = \operatorname{dom}\beta$ and

$$(a)\ \ \alpha = \gamma \quad \text{or} \quad (b)\ \ X\alpha = X\gamma, \operatorname{dom}\alpha \subseteq Y \ \text{ and } \ \operatorname{dom}\gamma \subseteq Y.$$

If (a) holds, then $\operatorname{dom}\alpha = \operatorname{dom}\gamma = \operatorname{dom}\beta$. Otherwise, if (b) holds, then $\operatorname{dom}\beta = \operatorname{dom}\gamma \subseteq Y$, $\operatorname{dom}\alpha \subseteq Y$, $g(\alpha) = |X \setminus X\alpha| = |X \setminus X\gamma| = g(\gamma) = g(\beta)$ and $|X\alpha| = |X\gamma| = |\operatorname{dom}\gamma| = |\operatorname{dom}\beta| = |X\beta|$, whence $r(\alpha) = r(\beta)$ as required.

For the converse, if $\operatorname{dom}\alpha = \operatorname{dom}\beta$, then $\alpha\mathcal{R}\beta$ in $A(X,Y)$. Consequently, as $\mathcal{D}$ is an equivalence relation containing $\mathcal{R}$, we have that $\alpha\mathcal{D}\beta$ as required. Next, we assume $r(\alpha) = r(\beta)$, $\operatorname{dom}\alpha \subseteq Y$, $\operatorname{dom}\beta \subseteq Y$ and $g(\alpha) = g(\beta)$. As $r(\alpha) = r(\beta)$, we may write

$$\alpha = \begin{pmatrix} a_i \\ b_i \end{pmatrix} \text{ and } \beta = \begin{pmatrix} c_i \\ d_i \end{pmatrix}.$$

We define $\gamma = \begin{pmatrix} c_i \\ b_i \end{pmatrix}$. Then $X\gamma = X\alpha \subseteq Y$ and $g(\gamma) = g(\beta) = g(\alpha) = d(\alpha) = d(\gamma)$, whence $\gamma \in A(X,Y)$. Moreover, $\mathrm{dom}\,\gamma = \mathrm{dom}\,\beta \subseteq Y$. Hence, by Theorem 2 and Theorem 3, we get $\gamma\mathcal{R}\beta$ and $\alpha\mathcal{L}\gamma$ which imply $\alpha\mathcal{D}\beta$ in $A(X,Y)$. □

In order to describe the $\mathcal{J}$ relation on $A(X,Y)$, we need the following lemma.

Lemma 2. *Let $\alpha, \beta \in A(X,Y)$. Then $\beta = \lambda\alpha\mu$ for some $\lambda, \mu \in A(X,Y)$ if and only if $g(\alpha) + |\mathrm{dom}\,\alpha \setminus Y| \leq g(\beta)$ and $|\mathrm{dom}\,\beta| \leq |\mathrm{dom}\,\alpha \cap Y|$.*

Proof. Suppose that $\beta = \lambda\alpha\mu$ for some $\lambda, \mu \in A(X,Y)$. Since $X\lambda \subseteq Y$, we have that $|\mathrm{dom}\,\beta| \leq |X\lambda \cap \mathrm{dom}\,\alpha| \leq |Y \cap \mathrm{dom}\,\alpha|$. Moreover, we get $\mathrm{dom}\,\beta \subseteq \mathrm{dom}\,\lambda$ and, as $X\lambda \subseteq Y$, we have that $(\mathrm{dom}\,\beta)\lambda \subseteq \mathrm{dom}\,\alpha \cap Y$, whence $X\lambda \cap (X \setminus \mathrm{dom}\,\alpha) = (\mathrm{dom}\,\lambda \setminus \mathrm{dom}\,\beta)\lambda \cap (X \setminus \mathrm{dom}\,\alpha)$ which implies

$$\begin{aligned}
g(\beta) &= |X \setminus \mathrm{dom}\,\lambda| + |\mathrm{dom}\,\lambda \setminus \mathrm{dom}\,\beta| \\
&= |X \setminus X\lambda| + |(\mathrm{dom}\,\lambda \setminus \mathrm{dom}\,\beta)\lambda| \\
&= |X \setminus Y| + |Y \setminus X\lambda| + |(\mathrm{dom}\,\lambda \setminus \mathrm{dom}\,\beta)\lambda| \\
&= |(X \setminus Y) \cap \mathrm{dom}\,\alpha| + |(X \setminus Y) \cap (X \setminus \mathrm{dom}\,\alpha)| \\
&\quad + |(Y \setminus X\lambda) \cap \mathrm{dom}\,\alpha| + |(Y \setminus X\lambda) \cap (X \setminus \mathrm{dom}\,\alpha)| \\
&\quad + |(\mathrm{dom}\lambda \setminus \mathrm{dom}\,\beta)\lambda \cap \mathrm{dom}\,\alpha| + |(\mathrm{dom}\,\lambda \setminus \mathrm{dom}\,\beta)\lambda \cap (X \setminus \mathrm{dom}\,\alpha)| \\
&\geq |(X \setminus Y) \cap \mathrm{dom}\,\alpha| + |(X \setminus Y) \cap (X \setminus \mathrm{dom}\,\alpha)| \\
&\quad + |(Y \setminus X\lambda) \cap (X \setminus \mathrm{dom}\,\alpha)| + |(\mathrm{dom}\,\lambda \setminus \mathrm{dom}\,\beta)\lambda \cap (X \setminus \mathrm{dom}\,\alpha)| \\
&= |\mathrm{dom}\,\alpha \setminus Y| + |(X \setminus Y) \cap (X \setminus \mathrm{dom}\,\alpha)| \\
&\quad + |(Y \setminus X\lambda) \cap (X \setminus \mathrm{dom}\,\alpha)| + |X\lambda \cap (X \setminus \mathrm{dom}\,\alpha)| \\
&= |\mathrm{dom}\,\alpha \setminus Y| + g(\alpha).
\end{aligned}$$

Conversely, suppose that the conditions hold and assume $\beta = \begin{pmatrix} b_i \\ y_i \end{pmatrix}$. We will consider two cases.

Case 1. If $|\mathrm{dom}\,\beta| = n$, then $|\mathrm{dom}\,\alpha \cap Y| = n$ since $|\mathrm{dom}\,\beta| \leq |\mathrm{dom}\,\alpha \cap Y|$. Moreover, as $g(\alpha) + |\mathrm{dom}\,\alpha \setminus Y| \leq g(\beta)$, there exists a cardinal t for which $g(\alpha) + |\mathrm{dom}\,\alpha \setminus Y| + t = g(\beta)$. Then we may write $\mathrm{dom}\,\alpha \cap Y = \{a_i\} \,\dot\cup\, T$ for some set T with $|T| = t$ and $|\{a_i\}| = |\{b_i\}| = n$. Then we define

$$\lambda = \begin{pmatrix} b_i \\ a_i \end{pmatrix} \text{ and } \mu = \begin{pmatrix} a_i\alpha \\ y_i \end{pmatrix}.$$

It is clear that $X\lambda \subseteq Y$ and $X\mu = X\beta \subseteq Y$. Moreover, $d(\lambda) = g(\alpha) + |\text{dom } \alpha \setminus Y| + |T| = g(\beta) = g(\lambda)$ and $g(\mu) = d(\alpha) + |(\text{dom } \alpha \setminus Y)\alpha| + |T\alpha| = g(\alpha) + |\text{dom } \alpha \setminus Y| + |T| = g(\beta) = d(\beta) = d(\mu)$. Therefore, $\lambda, \mu \in A(X, Y)$ and $\beta = \lambda\alpha\mu$.

Case 2. If $|\text{dom } \beta| < n$, then $g(\beta) = n$. As $|\text{dom } \beta| \leq |\text{dom } \alpha \cap Y|$, there exists a subset $\{c_i\}$ of $\text{dom } \alpha \cap Y$ where $|\{c_i\}| = |\{b_i\}| < n$. It follows that $|X \setminus \{c_i\}| = n = |X \setminus \{c_i\alpha\}|$. Then we redefine

$$\lambda = \begin{pmatrix} b_i \\ c_i \end{pmatrix} \text{ and } \mu = \begin{pmatrix} c_i\alpha \\ y_i \end{pmatrix}.$$

We see that $X\lambda \subseteq Y$, $X\mu = X\beta \subseteq Y$ and $g(\lambda) = g(\beta) = n = d(\beta) = d(\mu)$. Moreover, $d(\lambda) = |X \setminus \{c_i\}| = n = |X \setminus \{c_i\alpha\}| = g(\mu)$. Therefore, $\lambda, \mu \in A(X, Y)$ and $\beta = \lambda\alpha\mu$ as required. □

The following theorem is a consequence of the above lemma.

Theorem 5. *Let $\alpha, \beta \in A(X, Y)$. Then $\alpha\mathcal{J}\beta$ if and only if*

(a) $\text{dom } \alpha = \text{dom } \beta$ *or*

(b) $|\text{dom } \alpha| = |\text{dom } \alpha \cap Y| = |\text{dom } \beta| = |\text{dom } \beta \cap Y|$ *and*

$$g(\alpha) + |\text{dom } \alpha \setminus Y| = g(\alpha) = g(\beta) = g(\beta) + |\text{dom } \beta \setminus Y|.$$

Proof. Suppose that $\alpha\mathcal{J}\beta$ in $A(X, Y)$. Then there exist $\sigma, \delta, \sigma', \delta' \in A(X, Y)^1$ such that $\alpha = \sigma\beta\delta$ and $\beta = \sigma'\alpha\delta'$. If $\sigma = 1 = \sigma'$, then $\alpha = \beta\delta$ and $\beta = \alpha\delta'$. So $\alpha\mathcal{R}\beta$, whence $\text{dom } \alpha = \text{dom } \beta$ by Theorem 2. If $\delta = 1 = \delta'$, then $\alpha = \sigma\beta$ and $\beta = \sigma'\alpha$ which imply $\alpha\mathcal{L}\beta$. Then, by Theorem 3, we have $\alpha = \beta$ or $X\alpha = X\beta, \text{dom } \alpha \subseteq Y$ and $\text{dom } \beta \subseteq Y$. Here, if $\alpha = \beta$, then we obtain that $\text{dom } \alpha = \text{dom } \beta$. Otherwise, if the latter holds, then $|\text{dom } \alpha \setminus Y| = 0 = |\text{dom } \beta \setminus Y|$, which implies $g(\alpha) + |\text{dom } \alpha \setminus Y| = g(\alpha)$ and $g(\beta) + |\text{dom } \beta \setminus Y| = g(\beta)$. As $X\alpha = X\beta$, we have that $d(\alpha) = d(\beta)$, whence $g(\alpha) = g(\beta)$ and thus $g(\alpha) + |\text{dom } \alpha \setminus Y| = g(\alpha) = g(\beta) = g(\beta) + |\text{dom } \beta \setminus Y|$. Moreover, we obtain that $|\text{dom } \alpha \cap Y| = |\text{dom } \alpha| = |X\alpha| = |X\beta| = |\text{dom } \beta| = |\text{dom } \beta \cap Y|$ as required. In other cases, it is a routine to check that $\alpha = \lambda\beta\mu$ and $\beta = \lambda'\alpha\mu'$ for some $\lambda, \lambda', \mu, \mu' \in A(X, Y)$ (for example, if $\sigma = 1$ and $\delta, \delta', \sigma' \in A(X, Y)$, then $\alpha = \beta\delta$ and $\beta = \sigma'\alpha\delta'$. So $\alpha = \beta\delta = (\sigma'\alpha\delta')\delta = \sigma'(\beta\delta)\delta'\delta = \sigma'\beta(\delta\delta'\delta)$, where $\delta\delta'\delta \in A(X, Y)$). Thus, Lemma 2 implies that $|\text{dom } \alpha| \leq |\text{dom } \beta \cap Y| \leq |\text{dom } \beta| \leq |\text{dom } \alpha \cap Y| \leq |\text{dom } \alpha|$ and $g(\alpha) + |\text{dom } \alpha \setminus Y| \leq g(\beta) \leq g(\beta) + |\text{dom } \beta \setminus Y| \leq g(\alpha) \leq g(\alpha) + |\text{dom } \alpha \setminus Y|$. Hence, $|\text{dom } \alpha| = |\text{dom } \alpha \cap Y| = |\text{dom } \beta| = |\text{dom } \beta \cap Y|$ and $g(\alpha) + |\text{dom } \alpha \setminus Y| = g(\alpha) = g(\beta) = g(\beta) + |\text{dom } \beta \setminus Y|$ as required.

Conversely, if $\mathrm{dom}\,\alpha = \mathrm{dom}\,\beta$, then $\alpha\mathcal{R}\beta$ and so $\alpha\mathcal{J}\beta$. Now, we assume that $|\mathrm{dom}\,\alpha| = |\mathrm{dom}\,\alpha \cap Y| = |\mathrm{dom}\,\beta| = |\mathrm{dom}\,\beta \cap Y|$ and $g(\alpha) + |\mathrm{dom}\,\alpha \setminus Y| = g(\alpha) = g(\beta) = g(\beta) + |\mathrm{dom}\,\beta \setminus Y|$. Then Lemma 2 implies $\alpha = \lambda\beta\mu$ and $\beta = \lambda'\alpha\mu'$ for some $\lambda, \lambda', \mu, \mu' \in A(X,Y)$. Therefore, $\alpha\mathcal{J}\beta$ and the proof is complete. □

Recall that $\mathcal{D} \subseteq \mathcal{J}$ on any semigroups and $\mathcal{D} = \mathcal{J}$ on some well known transformation semigroups, for example, on $P(X), I(X)$ and $A(X)$. However, this is not true for $A(X,Y)$ as shown in the following example.

Example 1. Let $X = \mathbb{N}$ denote the set of all positive integers and let Y be the set of all positive even integers. We define $\alpha = \begin{pmatrix} 5n \\ 2n \end{pmatrix}$, where $n \in \mathbb{N}$, and $\beta = \mathrm{id}_Y$. It can be verified that $\alpha, \beta \in A(X,Y)$. We see that $|\mathrm{dom}\,\alpha \cap Y| = |\{10n : n \in \mathbb{N}\}| = \aleph_0 = |\mathrm{dom}\,\alpha|$ and $|\mathrm{dom}\,\beta \cap Y| = |\mathrm{dom}\,\beta| = |Y| = \aleph_0$. Moreover, $g(\alpha) = \aleph_0 = g(\beta)$, whence $g(\alpha) = g(\alpha) + |\mathrm{dom}\,\alpha \setminus Y| = \aleph_0 = g(\beta) = g(\beta) + |\mathrm{dom}\,\beta \setminus Y|$. So $\alpha\mathcal{J}\beta$ in $A(X,Y)$ by Theorem 5. But α and β are not $\mathcal{D}$-related in $A(X,Y)$ by Theorem 4 since $\mathrm{dom}\,\alpha \neq \mathrm{dom}\,\beta$ and $\mathrm{dom}\,\alpha \nsubseteq Y$.

To close this section, it is worth noticing that, unlike the $\mathcal{R}$ relation, when $X \neq Y$ the relations $\mathcal{L}$, $\mathcal{H}$, $\mathcal{D}$ and $\mathcal{J}$ on $A(X,Y)$ are not the restriction of the corresponding relations from $A(X)$ to $A(X,Y)$. We provide some examples below.

Example 2. Let X and Y be as in Example 1.

(1) Let $\alpha = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix}$ and $\beta = \begin{pmatrix} 3 & 5 & 1 \\ 2 & 4 & 6 \end{pmatrix}$. We can verify that $\alpha, \beta \in A(X,Y) \subseteq A(X)$ and $X\alpha = X\beta$, $\mathrm{dom}\,\alpha = \mathrm{dom}\,\beta$. So $\alpha\mathcal{H}\beta$ in $A(X)$ by Theorem 1, whence $\alpha\mathcal{L}\beta$ in $A(X)$. But α and β are not $\mathcal{L}$-related in $A(X,Y)$ by Theorem 3 since $\alpha \neq \beta$ and $\mathrm{dom}\,\alpha \nsubseteq Y$. Consequently, they are not $\mathcal{H}$-related in $A(X,Y)$.

(2) Let $\gamma = \begin{pmatrix} 1 & 3 & 4 \\ 2 & 4 & 6 \end{pmatrix}$ and $\mu = \begin{pmatrix} 2 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix}$. Then $\gamma, \mu \in A(X,Y) \subseteq A(X)$, $r(\gamma) = 3 = r(\mu)$ and $d(\gamma) = d(\mu) = \aleph_0$. So $\gamma\mathcal{D}\mu$ in $A(X)$ by Theorem 1. Since $\mathcal{D} = \mathcal{J}$ in $A(X)$, we obtain that $\gamma\mathcal{J}\mu$ in $A(X)$. But by Theorem 5, γ and μ are not $\mathcal{J}$-related in $A(X,Y)$ since $\mathrm{dom}\,\gamma \neq \mathrm{dom}\,\mu$ and $|\mathrm{dom}\,\gamma| = 3$ whereas $|\mathrm{dom}\,\gamma \cap Y| = 1$. Consequently, they are not $\mathcal{D}$-related in $A(X,Y)$.

4. Natural Partial Order

In this section, we investigate various properties of the natural partial order $\leq$ on $A(X,Y)$. We first observe that, for $\alpha, \beta \in A(X,Y)$ with $\alpha \subseteq \beta$, since they are injective, it is a routine matter to show that the following results are true.

(1) $|\mathrm{dom}\, \beta \setminus \mathrm{dom}\, \alpha| = |X\beta \setminus X\alpha|$.

(2) $(X\alpha)\alpha^{-1} = (X\alpha)\beta^{-1}$.

(3) If $\mathrm{dom}\, \alpha = \mathrm{dom}\, \beta$ or $X\alpha = X\beta$, then $\alpha = \beta$.

When $\alpha \subseteq \beta$, the above results will be used throughout this section without further mention. We denote by $\alpha \subset \beta$ when $\alpha \subseteq \beta$ and $\alpha \neq \beta$. Similarly, we write $\alpha < \beta$ when $\alpha \leq \beta$ and $\alpha \neq \beta$.

Theorem 6. *Let $\alpha, \beta \in A(X,Y)$. Then $\alpha \leq \beta$ if and only if*

$$\alpha = \beta \ \textit{or} \ (\alpha \subseteq \beta \ \textit{and} \ \mathrm{dom}\, \alpha \subseteq Y).$$

Proof. Suppose that $\alpha \leq \beta$ in $A(X,Y)$. Then $\alpha = \lambda\beta = \beta\mu$ and $\alpha = \alpha\mu$ for some $\lambda, \mu \in A(X,Y)^1$ which imply $X\alpha \subseteq X\beta$ and $\mathrm{dom}\, \alpha \subseteq \mathrm{dom}\, \beta$. If $\alpha \neq \beta$, then $\lambda, \mu \in A(X,Y)$. So, by Lemma 1, we have $(X\alpha)\beta^{-1} \subseteq Y$. Next, as $\alpha = \beta\mu = \alpha\mu$, we have that $x\alpha\mu = x\beta\mu$ for all $x \in \mathrm{dom}\, \alpha$. Thus, $x\alpha = x\beta$ as μ is injective. Therefore, $\alpha \subseteq \beta$ and so $\mathrm{dom}\, \alpha = (X\alpha)\alpha^{-1} = (X\alpha)\beta^{-1} \subseteq Y$ as required.

For the converse, if $\alpha = \beta$, then it is clear that $\alpha \leq \beta$. Now we assume $\alpha \subseteq \beta$ and $\mathrm{dom}\, \alpha \subseteq Y$. We may write

$$\alpha = \begin{pmatrix} a_i \\ x_i \end{pmatrix} \text{ and } \beta = \begin{pmatrix} a_i & a_j \\ x_i & x_j \end{pmatrix}.$$

Let $\mu = \begin{pmatrix} x_i \\ x_i \end{pmatrix}$, clearly $\alpha = \beta\mu = \alpha\mu$, where $X\mu = X\alpha \subseteq Y$ and $g(\mu) = d(\mu)$, whence $\mu \in A(X,Y)$. Moreover, as $\alpha \subseteq \beta$, we also obtain that $X\alpha \subseteq X\beta$ and $(X\alpha)\beta^{-1} = (X\alpha)\alpha^{-1} = \mathrm{dom}\, \alpha \subseteq Y$. Thus, by Lemma 1, $\alpha = \lambda\beta$ for some $\lambda \in A(X,Y)$. Hence, $\alpha \leq \beta$ as required. $\square$

Notice that, from Theorem 6 above, the natural partial order $\leq$ is contained in $\subseteq$ on $A(X,Y)$. Moreover, by applying Theorem 6 to the case $Y = X$, we obtain the following corollary where the natural partial order on $A(X)$ is described.

Corollary 2. *On $A(X)$, the partial order $\leq$ is equal to the partial order $\subseteq$.*

Theorem 7. *The following statements hold for the maximum and minimum elements with respect to $\leq$ in $A(X,Y)$.*

(*a*) *$A(X,Y)$ has no maximum element with respect to $\leq$.*

(*b*) *$\emptyset$ is the minimum element with respect to $\leq$.*

Proof. In order to prove (a), we choose $a, b \in X$ and $c \in Y$ with $a \neq b$ and define

$$\beta = \begin{pmatrix} a \\ c \end{pmatrix} \text{ and } \gamma = \begin{pmatrix} b \\ c \end{pmatrix}.$$

It can be verified that $\beta, \gamma \in A(X,Y)$. If $\alpha \in A(X,Y)$ is the maximum under $\leq$, then $\beta, \gamma \leq \alpha$, whence $\beta, \gamma \subseteq \alpha$. This implies $a\alpha = a\beta = c = b\gamma = b\alpha$, whence $a = b$ (as α is injective), a contradiction. Therefore, $A(X,Y)$ has no the maximum element under $\leq$.

To prove (b), let $\alpha \in A(X,Y)$. It is clear that $\emptyset \subseteq \alpha$ and $\operatorname{dom} \emptyset = \emptyset \subseteq Y$, whence $\emptyset \leq \alpha$. Hence, $\emptyset$ is the the minimum element under $\leq$. □

Theorem 8. *Let $\alpha \in A(X,Y)$. Then α is a maximal element in $A(X,Y)$ with respect to $\leq$ if and only if $X\alpha = Y$ or* $\operatorname{dom} \alpha \nsubseteq Y$.

Proof. Suppose that $\alpha = \begin{pmatrix} a_i \\ x_i \end{pmatrix}$ is a maximal element in $A(X,Y)$. If $X\alpha \subsetneq Y$ and $\operatorname{dom} \alpha \subseteq Y$, then $g(\alpha) = d(\alpha) > 0$. So, we can choose $a \in X \setminus \operatorname{dom} \alpha$, $b \in Y \setminus X\alpha$ and define $\beta = \begin{pmatrix} a_i & a \\ x_i & b \end{pmatrix}$. Clearly, $X\beta \subseteq Y$ and $g(\beta) = |(X \setminus \operatorname{dom} \alpha) \setminus \{a\}| = |(X \setminus X\alpha) \setminus \{b\}| = d(\beta)$, whence $\beta \in A(X,Y)$. In addition, we see that $\alpha \subset \beta$. Therefore, $\alpha < \beta$ by Theorem 6, which contradicts the maximality of α. Hence, $X\alpha = Y$ or $\operatorname{dom} \alpha \nsubseteq Y$.

For the converse, assume that the conditions hold and suppose $\alpha \leq \beta$ where $\beta \in A(X,Y)$. It follows that $\alpha \subseteq \beta$ and so $X\alpha \subseteq X\beta$. If $X\alpha = Y$, then $Y = X\alpha \subseteq X\beta \subseteq Y$, whence $X\alpha = X\beta$. As $\alpha \subseteq \beta$ and β is injective, it follows that $\alpha = \beta$. On the other hand, if $\operatorname{dom} \alpha \nsubseteq Y$, then $\alpha = \beta$ by Theorem 6. In both cases, α is maximal under $\leq$. □

The next result characterizes all the minimal elements under $\leq$ in $A(X,Y)$.

Theorem 9. *Let $\alpha \in A(X,Y)$. Then α is a non-zero minimal element with respect to $\leq$ in $A(X,Y)$ if and only if* $|\operatorname{dom} \alpha| = 1$ *or* $\operatorname{dom} \alpha \cap Y = \emptyset$.

Proof. Let α be a non-zero minimal element under $\leq$. If $|\mathrm{dom}\,\alpha| > 1$ and $\mathrm{dom}\,\alpha \cap Y \neq \emptyset$, then we choose $a \in \mathrm{dom}\,\alpha \cap Y$ and define $\beta = \begin{pmatrix} a \\ a\alpha \end{pmatrix}$. Clearly $\beta \in A(X,Y)$ and $\emptyset \neq \beta \subset \alpha$ since $|\mathrm{dom}\,\alpha| > 1$. Moreover, as $a \in Y$, we get $\emptyset \neq \beta < \alpha$ by Theorem 6, which contradicts the minimality of α. Hence, $|\mathrm{dom}\,\alpha| = 1$ or $\mathrm{dom}\,\alpha \cap Y = \emptyset$.

Conversely, suppose the conditions hold and let $\beta \in A(X,Y)$ be such that $\emptyset \neq \beta \leq \alpha$. Then $\emptyset \neq \beta \subseteq \alpha$ and so $\emptyset \neq \mathrm{dom}\,\beta \subseteq \mathrm{dom}\,\alpha$. Consequently, if $|\mathrm{dom}\,\alpha| = 1$, then $\mathrm{dom}\,\alpha = \mathrm{dom}\,\beta$ and so $\alpha = \beta$. Otherwise, if $\mathrm{dom}\,\alpha \cap Y = \emptyset$, then $\mathrm{dom}\,\beta \cap Y = \emptyset$ and we obtain by Theorem 6 that $\alpha = \beta$. In both cases, we deduce that α is non-zero minimal under $\leq$. □

Next, we examine the compatibility of the natural partial order on $A(X,Y)$. To do this, we first recall from [3, p. 104], that, the containment order $\subseteq$ is both left and right compatible on $P(X)$, in other words, if $\alpha \subseteq \beta$, then $\gamma\alpha \subseteq \gamma\beta$ and $\alpha\gamma \subseteq \beta\gamma$ for all $\alpha, \beta, \gamma \in P(X)$. Therefore, it is also left and right compatible on $A(X,Y)$ since $A(X,Y)$ is contained in $P(X)$.

Theorem 10. *The following statements hold for the compatibility properties with respect to $\leq$ on $A(X,Y)$.*

(a) *The natural partial order is right compatible on $A(X,Y)$.*

(b) *$\alpha \in A(X,Y)$ is left compatible with respect to $\leq$ if and only if $|\mathrm{dom}\,\alpha| = 1$ or $\mathrm{dom}\,\alpha \subseteq Y$.*

Proof. In order to prove (a), let $\alpha, \beta, \gamma \in A(X,Y)$ be such that $\alpha \leq \beta$. Clearly, if $\alpha = \beta$, then $\alpha\gamma = \beta\gamma$, whence $\alpha\gamma \leq \beta\gamma$. Now, we suppose that $\alpha \neq \beta$. Then by Theorem 6, we have $\alpha \subseteq \beta$ and $\mathrm{dom}\,\alpha \subseteq Y$, whence $\alpha\gamma \subseteq \beta\gamma$ and $\mathrm{dom}\,\alpha\gamma \subseteq \mathrm{dom}\,\alpha \subseteq Y$. By Theorem 6 again, we deduce that $\alpha\gamma \leq \beta\gamma$. Therefore γ is right compatible as required.

To prove (b), let α be a left compatible element under $\leq$ and suppose $|\mathrm{dom}\,\alpha| \neq 1$. If $\alpha = \emptyset$, then it is clear that $\mathrm{dom}\,\alpha = \emptyset \subseteq Y$. On the other hand, if $\alpha \neq \emptyset$, then $|\mathrm{dom}\,\alpha| > 1$. Here, for any $x \in \mathrm{dom}\,\alpha$, we let $\lambda_x = \mathrm{id}_{X\alpha \setminus \{x\alpha\}}$ and $\mu = \mathrm{id}_{X\alpha}$. Clearly $\lambda_x, \mu \in A(X,Y)$ and $\alpha\lambda_x \neq \alpha\mu = \alpha$. Moreover, $\lambda_x \subset \mu$ and $\mathrm{dom}\,\lambda_x \subseteq Y$, whence $\lambda_x < \mu$ by Theorem 6. Hence, $\alpha\lambda_x < \alpha\mu$ since α is left compatible. So, by Theorem 6 again, we obtain that $\mathrm{dom}\,\alpha \setminus \{x\} = \mathrm{dom}\,\alpha\lambda_x \subseteq Y$. Since x is an arbitrary element in $\mathrm{dom}\,\alpha$, we get $\mathrm{dom}\,\alpha = \bigcup_{x \in \mathrm{dom}\,\alpha} \mathrm{dom}\,\alpha \setminus \{x\} \subseteq Y$ as required.

Conversely, suppose that the conditions hold. Let $\gamma, \mu \in A(X,Y)$ be such that $\gamma \leq \mu$, then $\gamma \subseteq \mu$. First, assume that $|\mathrm{dom}\,\alpha| = 1$, where $\operatorname{dom}\alpha = \{x\}$. If $x\alpha \notin \operatorname{dom}\gamma$, then $\alpha\gamma = \emptyset \leq \alpha\mu$ since $\emptyset$ is the minimum element. Otherwise, if $x\alpha \in \operatorname{dom}\gamma$, then, as $\gamma \subseteq \mu$, we have $(x\alpha)\gamma = (x\alpha)\mu$, where $\operatorname{dom}\alpha\gamma = \{x\} = \operatorname{dom}\alpha\mu$, whence $\alpha\gamma = \alpha\mu$. Therefore, α is left compatible. Finally, we assume that $\operatorname{dom}\alpha \subseteq Y$. Then $\operatorname{dom}\alpha\gamma \subseteq \operatorname{dom}\alpha \subseteq Y$. As $\gamma \subseteq \mu$ and $A(X,Y)$ is left compatible under $\subseteq$, we have that $\alpha\gamma \subseteq \alpha\mu$. So, by Theorem 6, we get $\alpha\gamma \leq \alpha\mu$. In all cases, we have shown that α is left compatible with respect to $\leq$, and the proof is complete. □

Next, we study the existence of the meet $\alpha \wedge \beta$ (or the greatest lower bound) and the join $\alpha \vee \beta$ (or the least upper bound) for $\alpha, \beta \in A(X,Y)$ under the natural partial order. Obviously, if $\alpha \leq \beta$, then $\alpha \wedge \beta = \alpha$ and $\alpha \vee \beta = \beta$. Therefore, we suppose that α and β are incomparable under $\leq$. In what follows, we refer to the equalizer of α and β which defined in [8] by $E(\alpha,\beta) = \{x \in \operatorname{dom}\alpha \cap \operatorname{dom}\beta : x\alpha = x\beta\}$. For convenience, we will denote $E(\alpha,\beta)$ by E and it is clear that $\alpha|_E = \beta|_E$.

Lemma 3. *Let $\alpha, \beta \in A(X,Y)$ which are incomparable with respect to $\leq$. If $\gamma \in A(X,Y)$ is a lower bound of $\{\alpha,\beta\}$, then* $\operatorname{dom}\gamma \subseteq E \cap Y$ *and* $\gamma \subseteq \alpha|_{E\cap Y} = \beta|_{E\cap Y}$.

Proof. Suppose that $\gamma \in A(X,Y)$ is a lower bound of $\{\alpha,\beta\}$ under $\leq$. If $\gamma = \emptyset$, then it is clear that $\operatorname{dom}\gamma = \emptyset \subseteq E \cap Y$ and $\gamma \subseteq \alpha|_{E\cap Y} = \beta|_{E\cap Y}$. If $\gamma \neq \emptyset$, then the conditions $\gamma \leq \alpha$ and $\gamma \leq \beta$ imply $\gamma \subseteq \alpha$ and $\gamma \subseteq \beta$. So $\emptyset \neq \operatorname{dom}\gamma \subseteq \operatorname{dom}\alpha \cap \operatorname{dom}\beta$ and $x\alpha = x\gamma = x\beta$ for all $x \in \operatorname{dom}\gamma$, whence $\operatorname{dom}\gamma \subseteq E$. Moreover, since α and β are incomparable, we have that $\alpha \neq \gamma$. Then by Theorem 6, as $\gamma \leq \alpha$, we have $\operatorname{dom}\gamma \subseteq Y$. Hence $\operatorname{dom}\gamma \subseteq E \cap Y$ and $\gamma \subseteq \alpha|_{E\cap Y} = \beta|_{E\cap Y}$ as required. □

Theorem 11. *Let $\alpha, \beta \in A(X,Y)$ which are incomparable with respect to $\leq$. Then the following statements hold.*

(*a*) *If $E \cap Y = \emptyset$, then $\alpha \wedge \beta = \emptyset$.*

(*b*) *If $E \cap Y \neq \emptyset$, then $\alpha \wedge \beta$ exists if and only if*

$$|X \setminus (E \cap Y)| = |X \setminus (E \cap Y)\alpha|.$$

In this case, $\alpha \wedge \beta = \alpha|_{E\cap Y} = \beta|_{E\cap Y}$.

Proof. To prove (*a*), suppose that $E \cap Y = \emptyset$. Then by Lemma 3, if $\gamma \leq \alpha$ and $\gamma \leq \beta$, then $\operatorname{dom}\gamma \subseteq E \cap Y = \emptyset$, that is $\gamma = \emptyset$. Thus, the only lower bound of $\{\alpha,\beta\}$ is $\emptyset$, whence $\alpha \wedge \beta = \emptyset$.

To prove (b), we suppose that $E \cap Y \neq \emptyset$ and $\alpha \wedge \beta = \gamma$ for some $\gamma \in A(X,Y)$. Then Lemma 3 implies that $\operatorname{dom}\gamma \subseteq E \cap Y$ and $\gamma \subseteq \alpha|_{E\cap Y} = \beta|_{E\cap Y}$. For each $x \in E \cap Y$, we define $\lambda_x = \begin{pmatrix} x \\ x\alpha \end{pmatrix} = \begin{pmatrix} x \\ x\beta \end{pmatrix}$, whence $\lambda_x \in A(X,Y)$. Moreover, $\lambda_x \subseteq \alpha$ and $\operatorname{dom}\lambda_x = \{x\} \subseteq Y$. Then by Theorem 6, $\lambda_x \leq \alpha$. Similarly, we get $\lambda_x \leq \beta$, whence λ_x is a lower bound under $\leq$ of $\{\alpha,\beta\}$. Since $\alpha \wedge \beta = \gamma$, we have $\lambda_x \leq \gamma$, which implies $\lambda_x \subseteq \gamma$ and so $\{x\} = \operatorname{dom}\lambda_x \subseteq \operatorname{dom}\gamma$. Therefore, $E \cap Y \subseteq \operatorname{dom}\gamma$ which implies $\operatorname{dom}\gamma = E \cap Y$ and $\gamma = \alpha|_{E\cap Y} = \beta|_{E\cap Y}$. Consequently, as $\gamma \in A(X,Y)$, we have $|X \setminus (E \cap Y)| = g(\gamma) = d(\gamma) = |X \setminus (E \cap Y)\alpha|$ as required.

Conversely, suppose that the conditions hold and let $\xi = \alpha|_{E\cap Y} = \beta|_{E\cap Y}$. We claim that $\alpha \wedge \beta = \xi$. It is clear that $X\xi = (E \cap Y)\alpha \subseteq Y$ and as $|X \setminus (E \cap Y)| = |X \setminus (E \cap Y)\alpha|$, we get $g(\xi) = d(\xi)$, whence $\xi \in A(X,Y)$. We also see that $\xi \subseteq \alpha$, $\xi \subseteq \beta$ and $\operatorname{dom}\xi \subseteq Y$. Therefore, ξ is a lower bound under $\leq$ of $\{\alpha,\beta\}$ by Theorem 6. To complete the proof, let $\mu \in A(X,Y)$ be such that $\mu \leq \alpha$ and $\mu \leq \beta$. Then Lemma 3 implies that $\operatorname{dom}\mu \subseteq E \cap Y$ and $\mu \subseteq \alpha|_{E\cap Y} = \beta|_{E\cap Y} = \xi$. Since α and β are incomparable, we have $\alpha \neq \mu$. Thus, the condition $\mu \leq \alpha$ and Theorem 6 imply $\operatorname{dom}\mu \subseteq Y$. Therefore, as $\mu \subseteq \xi$, we can deduce that $\mu \leq \xi$ by Theorem 6 again. Hence $\alpha \wedge \beta = \xi$. □

Next, to describe the least upper bound for $\alpha, \beta \in A(X,Y)$, we let $\alpha \cup \beta$ denote the mapping from $\operatorname{dom}\alpha \cup \operatorname{dom}\beta$ to $X\alpha \cup X\beta$ defined by

$$x(\alpha \cup \beta) = \begin{cases} x\alpha & \text{if } x \in \operatorname{dom}\alpha \setminus \operatorname{dom}\beta, \\ x\beta & \text{if } x \in \operatorname{dom}\beta \setminus \operatorname{dom}\alpha, \\ x\alpha = x\beta & \text{if } x \in \operatorname{dom}\alpha \cap \operatorname{dom}\beta. \end{cases}$$

Obviously, $\alpha \cup \beta$ is well defined if and only if $\operatorname{dom}\alpha \cap \operatorname{dom}\beta = \emptyset$ or $x\alpha = x\beta$ for all $x \in \operatorname{dom}\alpha \cap \operatorname{dom}\beta$, *i.e.*, $\operatorname{dom}\alpha \cap \operatorname{dom}\beta = E$. In this case, $\alpha \cup \beta$ is injective only when the sets $(\operatorname{dom}\alpha \setminus \operatorname{dom}\beta)\alpha$ and $(\operatorname{dom}\beta \setminus \operatorname{dom}\alpha)\beta$ are disjoint.

Theorem 12. *Let $\alpha, \beta \in A(X,Y)$ which are incomparable with respect to $\leq$. Then $\alpha \vee \beta$ exists if and only if the following conditions hold.*

(a) $\operatorname{dom}\alpha \cap \operatorname{dom}\beta = E$.

(b) $(\operatorname{dom}\alpha \setminus \operatorname{dom}\beta)\alpha \cap (\operatorname{dom}\beta \setminus \operatorname{dom}\alpha)\beta = \emptyset$.

(c) $|X \setminus (\operatorname{dom}\alpha \cup \operatorname{dom}\beta)| = |X \setminus (X\alpha \cup X\beta)|$.

(d) $\operatorname{dom}\alpha \subseteq Y$ *and* $\operatorname{dom}\beta \subseteq Y$.

In this case, $\alpha \vee \beta = \alpha \cup \beta$.

Proof. Suppose that $\alpha \vee \beta = \gamma$ for some $\gamma \in A(X, Y,)$. To show (a), it is clear that $E \subseteq \text{dom } \alpha \cap \text{dom } \beta$. For the equality, let $x \in \text{dom } \alpha \cap \text{dom } \beta$. As $\alpha \leq \gamma$ and $\beta \leq \gamma$, we get $\alpha \subseteq \gamma$ and $\beta \subseteq \gamma$. So $x\alpha = x\gamma = x\beta$ which implies $x \in E$, whence $\text{dom } \alpha \cap \text{dom } \beta = E$, that is (a) holds. To show (b), suppose for a contradiction that there exists $y \in (\text{dom } \alpha \setminus \text{dom } \beta)\alpha \cap (\text{dom } \beta \setminus \text{dom } \alpha)\beta$. Then $x\alpha = y = z\beta$ for some $x \in \text{dom } \alpha \setminus \text{dom } \beta$, $z \in \text{dom } \beta \setminus \text{dom } \alpha$. As $\alpha \subseteq \gamma, \beta \subseteq \gamma$, we get $x\alpha = x\gamma$ and $z\beta = z\gamma$, so $x\gamma = y = z\gamma$. It follows that $x = z$ since γ is injective, a contradiction. Thus $(\text{dom } \alpha \setminus \text{dom } \beta)\alpha \cap (\text{dom } \beta \setminus \text{dom } \alpha)\beta = \emptyset$, and this proves (b). Next, as $\alpha \subseteq \gamma, \beta \subseteq \gamma$, we have $\text{dom } \alpha \cup \text{dom } \beta \subseteq \text{dom } \gamma$, $X\alpha \cup X\beta \subseteq X\gamma$ and $(\text{dom } \gamma \setminus (\text{dom } \alpha \cup \text{dom } \beta))\gamma = X\gamma \setminus (X\alpha \cup X\beta)$, whence

$$\begin{aligned}
|X \setminus (\text{dom } \alpha \cup \text{dom } \beta)| &= |X \setminus \text{dom } \gamma| + |\text{dom } \gamma \setminus (\text{dom } \alpha \cup \text{dom } \beta)| \\
&= g(\gamma) + |(\text{dom } \gamma \setminus (\text{dom } \alpha \cup \text{dom } \beta))\gamma| \\
&= d(\gamma) + |X\gamma \setminus (X\alpha \cup X\beta)| \\
&= |X \setminus X\gamma| + |X\gamma \setminus (X\alpha \cup X\beta)| \\
&= |X \setminus (X\alpha \cup X\beta)|.
\end{aligned}$$

Hence, (c) holds. To prove (d), as $\alpha \leq \gamma$, $\beta \leq \gamma$ and α and β are incomparable, we have that α is not maximal under $\leq$ (otherwise, $\alpha \leq \gamma$ implies $\alpha = \gamma$ and so $\beta \leq \alpha$, a contradiction). Similarly, β is not maximal. Then Theorem 8 implies that $\text{dom } \alpha \subseteq Y$ and $\text{dom } \beta \subseteq Y$, whence (d) holds.

For the converse, suppose that the conditions $(a), (b), (c)$ and (d) hold. Let $\rho = \alpha \cup \beta$, we aim to show that $\alpha \vee \beta = \rho$. It follow from (a) and (b) that ρ is a well defined injective mapping from $\text{dom } \alpha \cup \text{dom } \beta$ to Y. In addition, the condition (c) implies that $g(\rho) = d(\rho)$, whence $\rho \in A(X, Y)$. It is clear that $\alpha \subseteq \rho$ and by the condition (d), $\text{dom } \alpha \subseteq Y$, whence $\alpha \leq \rho$ by Theorem 6. In a similar way, we can verify that $\beta \leq \rho$, so ρ is an upper bound of $\{\alpha, \beta\}$. Finally, let $\mu \in A(X, Y)$ be an upper bound under $\leq$ of $\{\alpha, \beta\}$, we aim to show that $\rho \leq \mu$. As $\alpha \leq \mu$ and $\beta \leq \mu$, we have $\alpha \subseteq \mu$ and $\beta \subseteq \mu$. Then $\rho = \alpha \cup \beta \subseteq \mu$, and from (d), we get $\text{dom } \rho = \text{dom } \alpha \cup \text{dom } \beta \subseteq Y$, whence $\rho \leq \mu$ by Theorem 6. Hence, $\alpha \vee \beta = \rho = \alpha \cup \beta$ as required. □

In the last part of this paper, we describe the existence of upper covers and lower covers of elements in $A(X, Y)$. We first recall the definitions of an upper cover and a lower cover in a partially ordered set $(X, \leq)$. Let a and b be distinct elements in X, we call a a lower cover of b if $a < b$ and there is no $c \in S$ such that $a < c < b$. When this occurs, b is called an upper cover of a.

Theorem 13. *Let $\alpha, \beta \in A(X, Y)$ be such that $\alpha < \beta$. Then β is an upper cover of α if and only if* $|\mathrm{dom}\ \beta \setminus \mathrm{dom}\ \alpha| = 1$ *or* $(\mathrm{dom}\ \beta \setminus \mathrm{dom}\ \alpha) \cap Y = \emptyset$. *In other words, in the event that this occurs, α is a lower cover of β.*

Proof. Suppose that $\alpha < \beta$ where β is an upper cover of α. By Theorem 6, we get $\mathrm{dom}\ \alpha \subseteq Y$ and $\alpha \subset \beta$. If $(\mathrm{dom}\ \beta \setminus \mathrm{dom}\ \alpha) \cap Y \neq \emptyset$, then there exists $a \in (\mathrm{dom}\ \beta \setminus \mathrm{dom}\ \alpha) \cap Y$. We define $\gamma = \alpha \cup \begin{pmatrix} a \\ a\beta \end{pmatrix}$. Since β is injective and $\alpha \subset \beta$, we get $a\beta \notin X\alpha$, whence γ is injective. We also see that $X\gamma = X\alpha \cup \{a\beta\} \subseteq Y$ and $g(\gamma) = |(X \setminus \mathrm{dom}\ \alpha) \setminus \{a\}| = |(X \setminus X\alpha) \setminus \{a\beta\}| = d(\gamma)$, whence $\gamma \in A(X, Y)$. Moreover, it is clear that $\alpha \subset \gamma \subseteq \beta$ and $\mathrm{dom}\ \gamma = \mathrm{dom}\ \alpha \cup \{a\} \subseteq Y$. It follows by Theorem 6 that, $\alpha < \gamma \leq \beta$. So, as β is an upper cover of α, we deduce that $\beta = \gamma$. Therefore, $\mathrm{dom}\ \beta = \mathrm{dom}\ \gamma = \mathrm{dom}\ \alpha \cup \{a\}$ and so $|\mathrm{dom}\ \beta \setminus \mathrm{dom}\ \alpha| = |\{a\}| = 1$.

Conversely, suppose that the conditions hold and let $\gamma \in A(X, Y)$ be such that $\alpha < \gamma \leq \beta$. Then $\alpha \subset \gamma \subseteq \beta$ and so $\mathrm{dom}\ \alpha \subset \mathrm{dom}\ \gamma \subseteq \mathrm{dom}\ \beta$. It follows that

$$\mathrm{dom}\ \beta \setminus \mathrm{dom}\ \alpha = (\mathrm{dom}\ \beta \setminus \mathrm{dom}\ \gamma) \,\dot{\cup}\, (\mathrm{dom}\ \gamma \setminus \mathrm{dom}\ \alpha). \tag{1}$$

If $|\mathrm{dom}\ \beta \setminus \mathrm{dom}\ \alpha| = 1$, then from (1), we obtain that $|\mathrm{dom}\ \beta \setminus \mathrm{dom}\ \gamma| = 0$ and $|\mathrm{dom}\ \gamma \setminus \mathrm{dom}\ \alpha| = 1$. This implies $\mathrm{dom}\ \gamma = \mathrm{dom}\ \beta$ and thus $\gamma = \beta$. Otherwise, if $(\mathrm{dom}\ \beta \setminus \mathrm{dom}\ \alpha) \cap Y = \emptyset$, then $(\mathrm{dom}\ \gamma \setminus \mathrm{dom}\ \alpha) \cap Y = \emptyset$ and therefore $\mathrm{dom}\ \gamma \nsubseteq Y$. Then by Theorem 8, γ is maximal under $\leq$. Thus the assumption $\gamma \leq \beta$ implies $\gamma = \beta$. In both cases, we conclude that β is an upper cover of α as required.

□

The descriptions of Green's relations for $A(X, Y)$ presented in this paper generalize the corresponding results for $A(X)$ in [6, Section 3]. In addition, by taking $X = Y$ in our results in Section 4, we obtain the description of the natural partial order and its various properties on $A(X)$ which have never been studied before.

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