© Algebra and Discrete Mathematics RESEARCH ARTICLE
Volume **38** (2024). Number 2, pp. 233–247
DOI:10.12958/adm2333

On the semigroup of monoid endomorphisms of the semigroup $\mathscr{C}_+(a,b)$

Oleg Gutik and Sher-Ali Penza

Communicated by V. Mazorchuk

Dedicated to Professor Yu. A. Drozd
on the occasion of his 80th birthday

Abstract. Let $\mathscr{C}_+(a,b)$ be the submonoid of the bicyclic monoid which is studied in [8]. We describe monoid endomorphisms of the semigroup $\mathscr{C}_+(a,b)$ which are generated by the family of all congruences of the bicyclic monoid and all injective monoid endomorphisms of $\mathscr{C}_+(a,b)$.

Introduction

We shall follow the terminology of [1,2,7]. By ω we denote the set of all non-negative integers, by $\mathbb{N}$ the set of all positive integers.

A semigroup S is called *inverse* if for any element $x \in S$ there exists a unique $x^{-1} \in S$ such that $xx^{-1}x = x$ and $x^{-1}xx^{-1} = x^{-1}$. The element x^{-1} is called the *inverse of* $x \in S$. If S is an inverse semigroup, then the function $\operatorname{inv}\colon S \to S$ which assigns to every element x of S its inverse element x^{-1} is called the *inversion*.

If S is a semigroup, then we shall denote the subset of all idempotents in S by $E(S)$. If S is an inverse semigroup, then $E(S)$ is closed under multiplication and we shall refer to $E(S)$ as a *band* (or the *band of* S). Then the semigroup operation on S determines the following partial order

2020 Mathematics Subject Classification: *20M15, 20M20.*
Key words and phrases: *endomorphism, injective, bicyclic semigroup, subsemigroup, direct product, semidirect product.*

$\preccurlyeq$ on $E(S)$: $e \preccurlyeq f$ if and only if $ef = fe = e$. This order is called the *natural partial order* on $E(S)$. A *semilattice* is a commutative semigroup of idempotents.

If S is an inverse semigroup then the semigroup operation on S determines the following partial order $\preccurlyeq$ on S: $s \preccurlyeq t$ if and only if there exists $e \in E(S)$ such that $s = te$. This order is called the *natural partial order* on S [9].

The bicyclic monoid $\mathscr{C}(p,q)$ is the semigroup with the identity 1 generated by two elements p and q subjected only to the condition $pq = 1$. The semigroup operation on $\mathscr{C}(p,q)$ is determined as follows:

$$q^k p^l \cdot q^m p^n = q^{k+m-\min\{l,m\}} p^{l+n-\min\{l,m\}}.$$

It is well known that the bicyclic monoid $\mathscr{C}(p,q)$ is a bisimple (and hence simple) combinatorial E-unitary inverse semigroup and every non-trivial congruence on $\mathscr{C}(p,q)$ is a group congruence [1].

Let $\mathfrak{h}\colon S \to T$ be a homomorphism of semigroups. Then for any $s \in S$ and $A \subseteq S$ by $(s)\mathfrak{h}$ and $(A)\mathfrak{h}$ we denote the images of s and A, respectively, under the homomorphism $\mathfrak{h}$. Also, for any $t \in T$ by $(s)\mathfrak{h}^{-1}$ we denote the full preimage of s under the map $\mathfrak{h}$. A homomorphism $\mathfrak{h}\colon S \to T$ of monoids which preserves the unit elements of S is called a *monoid homomorphism*. A homomorphism $\mathfrak{h}\colon S \to S$ of a semigroup (a monoid) is called an endomorphism (a monoid endomorphism) of S, and in the case when $\mathfrak{h}$ is an isomorphism then $\mathfrak{h}$ is said to be an *automorphism* of S.

It is well-known that every automorphism of the bicyclic monoid $\mathscr{C}(p,q)$ is the identity self-map of $\mathscr{C}(p,q)$ [1], and hence the group $\mathbf{Aut}(\mathscr{C}(p,q))$ of automorphisms of $\mathscr{C}(p,q)$ is trivial. In [6] all endomorphisms of the bicyclic semigroup are described and it is proved that the semigroups $\mathbf{End}(\mathscr{C}(p,q))$ of all endomorphisms of the bicyclic semigroup $\mathscr{C}(p,q)$ is isomorphic to the semidirect products $(\omega,+) \rtimes_{\varphi} (\omega,*)$, where $+$ and $*$ are the usual addition and the usual multiplication on ω.

Subsemigroups of then bicyclic monoid were studied in [3,4,8]. In [8] the following anti-isomorphic subsemigroups of the bicyclic monoid

$$\mathscr{C}_+(a,b) = \left\{b^i a^j \in \mathscr{C}(a,b) \colon i \leqslant j,\ i,j \in \omega\right\}$$

and

$$\mathscr{C}_-(a,b) = \left\{b^i a^j \in \mathscr{C}(a,b) \colon i \geqslant j,\ i,j \in \omega\right\}$$

are studied. In the paper [5] topologizations of the semigroups $\mathscr{C}_+(a,b)$ and $\mathscr{C}_-(a,b)$ are studied.

Later in this paper by $\mathfrak{End}(\mathscr{C}_+(a,b))$ we denote the semigroup of all monoid endomorphisms of the semigroup $\mathscr{C}_+(a,b)$.

In this paper we describe monoid endomorphisms of the semigroup $\mathscr{C}_+(a,b)$ which are generated by the family of all congruences of the bicyclic monoid and all injective monoid endomorphisms of $\mathscr{C}_+(a,b)$.

1. On monoid endomorphisms of $\mathscr{C}_+(a,b)$ which are restrictions of homomorphisms of the bicyclic monoid

In [6] was proved that every monoid endomorphism $\varepsilon\colon \mathscr{C}(a,b)\to\mathscr{C}(a,b)$ of the bicyclic monoid is one of the following forms:

(i) $\varepsilon=\lambda_k$ for some positive integer k, where $(b^ia^j)\lambda_k=b^{ki}a^{kj}$ for any $i,j\in\omega$;

(ii) $\varepsilon=\lambda_0$ is the annihilating endomorphism of $\mathscr{C}(a,b)$, i.e., $(b^ia^j)\lambda_0=1$ for any $i,j\in\omega$.

Simple verifications show that in the both cases each of these monoid endomorphisms of the bicyclic semigroup induces the monoid endomorphism of $\mathscr{C}_+(a,b)$, which we denote by the similar way:

$$\lambda_k\colon \mathscr{C}_+(a,b)\to\mathscr{C}_+(a,b), b^ia^j\mapsto b^{ki}a^{kj},\ i,j\in\omega$$

for some $k\in\omega$.

For any $k_1,k_2\in\omega$ we have that

$$(b^ia^j)(\lambda_{k_1}\circ\lambda_{k_2})=((b^ia^j)\lambda_{k_1})\lambda_{k_2}=(b^{k_1i}a^{k_1j})\lambda_{k_2}=b^{k_1k_2i}a^{k_1k_2j},\ i,j\in\omega.$$

This implies that $\lambda_{k_1}\circ\lambda_{k_2}=\lambda_{k_1k_2}$ for all $k_1,k_2\in\omega$, and hence the set $\{\lambda_k\colon k\in\omega\}$ of endomorphisms of $\mathscr{C}_+(a,b)$ is closed under the operation of composition.

By $\mathfrak{End}_{\langle\lambda\rangle}(\mathscr{C}_+(a,b))$ we denote the subsemigroup of $\mathfrak{End}(\mathscr{C}_+(a,b))$, which is generated by the family $\{\lambda_k\colon k\in\omega\}$ of endomorphisms of the monoid $\mathscr{C}_+(a,b)$.

Proposition 1. *The semigroup $\mathfrak{End}_{\langle\lambda\rangle}(\mathscr{C}_+(a,b))$ is isomorphic to the multiplicative semigroup $(\omega,*)$ of non-negative integers.*

Proof. We define the map $\mathfrak{I}\colon \mathfrak{End}_{\langle\lambda\rangle}(\mathscr{C}_+(a,b))\to(\omega,*)$ by the formula $(\lambda_k)\mathfrak{I}=k$ for any $\lambda_k\in\mathfrak{End}_{\langle\lambda\rangle}(\mathscr{C}_+(a,b))$. The above arguments and simple verifications show that so defined map $\mathfrak{I}$ is a semigroup isomorphism. $\square$

It is well known that any inverse semigroup S admits the *smallest* (*minimal*) *group congruence* $\mathfrak{C}_{\mathrm{mg}}$: $a\mathfrak{C}_{\mathrm{mg}}b$ if and only if there exists $e \in E(S)$ such that $ea = eb$ (see [7]). The smallest group congruence $\mathfrak{C}_{\mathrm{mg}}$ on the bicyclic semigroup $\mathscr{C}(a,b)$ is determined in the following way: $b^{i_1}a^{j_1}\mathfrak{C}_{\mathrm{mg}}b^{i_2}a^{j_2}$ if and only if $i_1-j_1=i_2-j_2$ [7]. Since the quotient semigroup $\mathscr{C}(a,b)/\mathfrak{C}_{\mathrm{mg}}$ is isomorphic to the additive group of integers $\mathbb{Z}(+)$, the natural homomorphism $\mathfrak{h}_{\mathfrak{C}_{\mathrm{mg}}}\colon \mathscr{C}(a,b) \to \mathscr{C}(a,b)/\mathfrak{C}_{\mathrm{mg}}$ generates the homomorphism $\mathfrak{h}_{\mathfrak{C}_{\mathrm{mg}}}\colon \mathscr{C}(a,b) \to \mathbb{Z}(+)$ by the formula $(b^ia^j)\mathfrak{h}_{\mathfrak{C}_{\mathrm{mg}}} = j-i$, $i,j \in \omega$. By $(\omega,+)$ we denote the additive group of non-negative integers. This implies that the restriction $\mathfrak{h}_{\mathfrak{C}_{\mathrm{mg}}}\restriction_{\mathscr{C}_+(a,b)}\colon \mathscr{C}_+(a,b) \to (\omega,+)$ of the homomorphism $\mathfrak{h}_{\mathfrak{C}_{\mathrm{mg}}}$ is a homomorphism, as well.

Lemma 1. *For any $i,j,k \in \omega$ with $j \geqslant i$ the set*

$$S_{i,j,k} = \left\{b^ia^i\right\} \cup \left\{(b^ja^{j+k})^n \colon n \in \mathbb{N}\right\}$$

with the induced semigroup operation from the bicyclic monoid $\mathscr{C}(a,b)$ is isomorphic to the semigroup $(\omega,+)$.

Proof. We define the mapping $\mathfrak{I}_{i,j,k}\colon (\omega,+) \to S_{i,j,k}$ by the formula

$$(n)\mathfrak{I}_{i,j,k} = \begin{cases} b^ia^i, & \text{if } n=0; \\ (b^ja^{j+k})^n, & \text{if } n>0. \end{cases}$$

Simple verifications show that such defined map $\mathfrak{I}_{i,j,k}$ is a bijective homomorphism. □

Definition 1. For arbitrary $l \in \omega$ and $m \in \mathbb{N}$ we define the map $\sigma_{l,m}\colon \mathscr{C}_+(a,b) \to S_{0,l,m}$ by the formula $(b^ia^j)\sigma_{l,m} = ((b^ia^j)\mathfrak{h}_{\mathfrak{C}_{\mathrm{mg}}})\mathfrak{I}_{0,l,m}$, $i \leqslant j$, $i,j \in \omega$. Since $\mathfrak{h}_{\mathfrak{C}_{\mathrm{mg}}}$ and $\mathfrak{I}_{0,l,m}\colon (\omega,+) \to S_{0,l,m}$ are homomorphisms, $\sigma_{l,m}$ is a homomorphism, too. Simple verifications show that

$$(b^ia^j)\sigma_{l,m} = \begin{cases} 1, & \text{if } i=j; \\ b^la^{l+m(j-i)}, & \text{if } i<j \end{cases}$$

for all $i,j \in \omega$.

We observe that every elements of the semigroup $\mathscr{C}_+(a,b)$ can be represented in the form b^ia^{i+j} for some $i,j \in \omega$. Then for any positive integer n we have that

$$\begin{aligned} (b^ia^{i+j})^n &= b^ia^{i+j} \cdot b^ia^{i+j} \cdot (b^ia^{i+j})^{n-2} = \\ &= b^ia^{i+2j} \cdot (b^ia^j)^{n-2} = \\ &= \cdots = \\ &= b^ia^{i+nj}. \end{aligned}$$

Lemma 2. *$\sigma_{l_1,m_1}\circ\sigma_{l_2,m_2}=\sigma_{l_2,m_1m_2}$ for arbitrary $l_1,l_2\in\omega$ and $m_1,m_2\in\mathbb{N}$.*

Proof. Fix an arbitrary $b^ia^j\in\mathscr{C}_+(a,b)$, $i,j\in\omega$. Then we have that

$$\begin{aligned}
((b^ia^j)\sigma_{l_1,m_1})\sigma_{l_2,m_2} &= \left\{\begin{array}{ll} (1)\sigma_{l_2,m_2}, & \text{if } i=j;\\ ((b^{l_1}a^{l_1+m_1})^{j-i})\sigma_{l_2,m_2}, & \text{if } i<j \end{array}\right. =\\
&= \left\{\begin{array}{ll} 1, & \text{if } i=j;\\ (b^{l_1}a^{l_1+(j-i)m_1})\sigma_{l_2,m_2}, & \text{if } i<j \end{array}\right. =\\
&= \left\{\begin{array}{ll} 1, & \text{if } i=j;\\ (b^{l_2}a^{l_2+m_2})^{(j-i)m_1}, & \text{if } i<j \end{array}\right. =\\
&= \left\{\begin{array}{ll} 1, & \text{if } i=j;\\ b^{l_2}a^{l_2+(j-i)m_1m_2}, & \text{if } i<j \end{array}\right. =\\
&= (b^ia^j)\sigma_{l_2,m_1m_2}.
\end{aligned}$$

$\square$

By $\mathfrak{End}_{\langle\sigma\rangle}(\mathscr{C}_+(a,b))$ we denote the subsemigroup of $\mathfrak{End}(\mathscr{C}_+(a,b))$, which is generated by the family $\{\sigma_{l,m}\colon l,m\in\omega,\ m>0\}$ of endomorphisms of the monoid $\mathscr{C}_+(a,b)$.

By $\mathfrak{RZ}(\omega)$ we denote the set ω with the right-zero multiplication, i.e., $xy=y$ for all $x,y\in\omega$, and by $(\mathbb{N},*)$ the multiplicative semigroup of positive integers. We define the map $\mathfrak{I}\colon\mathfrak{End}_{\langle\sigma\rangle}(\mathscr{C}_+(a,b))\to\mathfrak{RZ}(\omega)\times(\mathbb{N},*)$ by the formula $(\sigma_{l,m})\mathfrak{I}=(l,m)$, $l\in\omega$, $m\in\mathbb{N}$. Lemma 2 implies that such defined map $\mathfrak{I}$ is a semigroup homomorphism, and moreover $\mathfrak{I}$ is bijective. Hence we get the following proposition.

Proposition 2. *The semigroup $\mathfrak{End}_{\langle\sigma\rangle}(\mathscr{C}_+(a,b))$ is isomorphic to the direct product $\mathfrak{RZ}(\omega)\times(\mathbb{N},*)$.*

Fix an arbitrary $b^ia^j\in\mathscr{C}_+(a,b)$, $i,j\in\omega$, $j\geqslant i$. Then for any $\sigma_{l,m}\in\mathfrak{End}_{\langle\sigma\rangle}(\mathscr{C}_+(a,b))$ and any $\lambda_k\in\mathfrak{End}_{\langle\lambda\rangle}(\mathscr{C}_+(a,b))$ we have that

$$\begin{aligned}
((b^ia^j)\sigma_{l,m})\lambda_k &= \left\{\begin{array}{ll} (1)\lambda_k, & \text{if } i=j;\\ ((b^la^{l+m})^{j-i})\lambda_k, & \text{if } i<j \end{array}\right. =\\
&= \left\{\begin{array}{ll} 1, & \text{if } i=j;\\ (b^la^{l+m(j-i)})\lambda_k, & \text{if } i<j \end{array}\right. =\\
&= \left\{\begin{array}{ll} 1, & \text{if } i=j;\\ b^{kl}a^{kl+km(j-i)}, & \text{if } i<j \end{array}\right. =\\
&= (b^ia^j)\sigma_{kl,km}
\end{aligned}$$

and

$$\begin{aligned}
((b^ia^j)\lambda_k)\sigma_{l,m} &= (b^{ki}a^{kj})\sigma_{l,m} = \\
&= \begin{cases} (b^{ki}a^{ki})\sigma_{l,m}, & \text{if } ki = kj; \\ (b^la^{l+m})^{kj-ki}, & \text{if } ki < kj \end{cases} = \\
&= \begin{cases} (b^{ki}a^{ki})\sigma_{l,m}, & \text{if } i = j; \\ (b^la^{l+m})^{kj-ki}, & \text{if } i < j \end{cases} = \\
&= \begin{cases} 1, & \text{if } i = j; \\ b^la^{l+km(j-i)}, & \text{if } i < j \end{cases} = \\
&= (b^ia^j)\sigma_{l,km}.
\end{aligned}$$

This implies the following

Proposition 3. *$\sigma_{l,m}\lambda_k = \sigma_{kl,km}$ and $\lambda_k\sigma_{l,m} = \sigma_{l,km}$ for any $\sigma_{l,m} \in \mathfrak{End}_{\langle\sigma\rangle}(\mathscr{C}_+(a,b))$ and any $\lambda_k \in \mathfrak{End}_{\langle\lambda\rangle}(\mathscr{C}_+(a,b)) \setminus \{\lambda_0\}$.*

By $\mathfrak{End}_{\langle\lambda,\sigma\rangle}(\mathscr{C}_+(a,b))$ we denote the subsemigroup of $\mathfrak{End}(\mathscr{C}_+(a,b))$, which is generated by the families $\{\lambda_k : k\in\omega\}$ and $\{\sigma_{l,m} : l,m \in \omega, m > 0\}$ of endomorphisms of the monoid $\mathscr{C}_+(a,b)$. We summarise the results of this section in the following theorem.

Theorem 1. *(1) λ_1 is the identity element of the semigroup $\mathfrak{End}(\mathscr{C}_+(a,b))$, and hence it is the identity element of the semigroups $\mathfrak{End}_{\langle\lambda\rangle}(\mathscr{C}_+(a,b))$ and $\mathfrak{End}_{\langle\lambda,\sigma\rangle}(\mathscr{C}_+(a,b))$;*

(2) λ_0 is the zero of the semigroup $\mathfrak{End}(\mathscr{C}_+(a,b))$, and hence it is the zero of $\mathfrak{End}_{\langle\lambda\rangle}(\mathscr{C}_+(a,b))$ and $\mathfrak{End}_{\langle\lambda,\sigma\rangle}(\mathscr{C}_+(a,b))$;

(3) the set $I = \mathfrak{End}_{\langle\sigma\rangle}(\mathscr{C}_+(a,b)) \cup \{\lambda_0\}$ is an ideal of the semigroup $\mathfrak{End}_{\langle\lambda,\sigma\rangle}(\mathscr{C}_+(a,b))$.

Proof. Statements (1) and (1) are trivial.

(1) By Proposition 3 we have that $\sigma_{l,m}\lambda_k, \lambda_k\sigma_{l,m} \in \mathfrak{End}_{\langle\sigma\rangle}(\mathscr{C}_+(a,b))$ for any $\sigma_{l,m} \in \mathfrak{End}_{\langle\sigma\rangle}(\mathscr{C}_+(a,b))$ and $\lambda_k \in \mathfrak{End}_{\langle\lambda\rangle}(\mathscr{C}_+(a,b)) \setminus \{\lambda_0\}$. Since λ_0 is the zero of the semigroup $\mathfrak{End}(\mathscr{C}_+(a,b))$, the above arguments imply that I is an ideal of the semigroup $\mathfrak{End}_{\langle\lambda,\sigma\rangle}(\mathscr{C}_+(a,b))$. ☐

2. On monoid injective endomorphisms of $\mathscr{C}_+(a,b)$

Example 1. For arbitrary positive integer n and arbitrary $s=0,\ldots,n-1$ we define the mapping $\lambda_{n,s} : \mathscr{C}_+(a,b) \to \mathscr{C}_+(a,b)$ by the formula

$$(b^ia^j)\lambda_{n,s} = \begin{cases} a^{nj}, & \text{if } i = 0; \\ b^{ni-s}a^{nj-s}, & \text{if } i \neq 0 \end{cases}$$

for all $i,j\in\omega$.

Proposition 4. *For any positive integer n and any $s=0,\dots,n-1$ the map $\lambda_{n,s}$ is an injective monoid endomorphism of the monoid $\mathscr{C}_+(a,b)$.*

Proof. Fix any positive integers i,j,k,l such that $i\leqslant j$ and $k\leqslant l$, and non-negative integers m and q. Then we have that

$$(b^ia^j\cdot b^ka^l)\lambda_{n,s}=\begin{cases} (b^{i-j+k}a^l)\lambda_{n,s}, & \text{if } j<k;\\ (b^ia^l)\lambda_{n,s}, & \text{if } j=k;\\ (b^ia^{j-k+l})\lambda_{n,s}, & \text{if } j>k\end{cases} = \\ =\begin{cases} b^{n(i-j+k)-s}a^{nl-s}, & \text{if } j<k;\\ b^{ni-s}a^{nl-s}, & \text{if } j=k;\\ b^{ni-s}a^{n(j-k+l)-s}, & \text{if } j>k,\end{cases}$$

$$(b^ia^j)\lambda_{n,s}\cdot(b^ka^l)\lambda_{n,s}=b^{ni-s}a^{nj-s}\cdot b^{nk-s}a^{nl-s}= \\ =\begin{cases} b^{(ni-s)-(nj-s)+(nk-s)}a^{nl-s}, & \text{if } nj-s<nk-s;\\ b^{ni-s}a^{nl-s}, & \text{if } nj-s=nk-s;\\ b^{ni-s}a^{nj-s-(nk-s)+(nl-s)}, & \text{if } nj-s>nk-s\end{cases} = \\ =\begin{cases} b^{n(i-j+k)-s}a^{nl-s}, & \text{if } j<k;\\ b^{ni-s}a^{nl-s}, & \text{if } j=k;\\ b^{ni-s}a^{n(j-k+l)-s}, & \text{if } j>k,\end{cases}$$

$$\begin{aligned}(b^ia^j\cdot a^m)\lambda_{n,s}&=(b^ia^{j+m})\lambda_{n,s}=\\ &=b^{ni-s}a^{n(j+m)-s}=\\ &=b^{ni-s}a^{nj-s}\cdot a^{mn}=\\ &=(b^ia^j)\lambda_{n,s}\cdot(a^m)\lambda_{n,s},\end{aligned}$$

$$(a^m\cdot b^ia^j)\lambda_{n,s}=\begin{cases} (b^{i-m}a^j)\lambda_{n,s}, & \text{if } m<i;\\ (a^{m-i+j})\lambda_{n,s}, & \text{if } m\geqslant i\end{cases} = \\ =\begin{cases} b^{n(i-m)-s}a^{nj-s}, & \text{if } m<i;\\ a^{n(m-i+j)}, & \text{if } m\geqslant i,\end{cases}$$

$$(a^m)\lambda_{n,s}\cdot(b^ia^j)\lambda_{n,s}=a^{mn}\cdot b^{ni-s}a^{nj-s}= \\ =\begin{cases} b^{ni-s-nm}a^{nj-s}, & \text{if } mn<ni-s;\\ a^{mn-(ni-s)+nj-s}, & \text{if } mn\geqslant ni-s\end{cases} =$$

$$= \begin{cases} b^{n(i-m)-s}a^{nj-s}, & \text{if } m < i - s/n; \\ a^{n(m-i+j)}, & \text{if } m \geqslant i - s/n \end{cases} =$$
$$= \begin{cases} b^{n(i-m)-s}a^{nj-s}, & \text{if } m < i; \\ a^{n(m-i+j)}, & \text{if } m \geqslant i, \end{cases}$$

because $s = 0, \ldots, n-1$, and

$$(a^m \cdot a^q)\lambda_{n,s} = (a^{m+q})\lambda_{n,s} = a^{(m+q)n} = a^{mn} \cdot a^{qn} = (a^m)\lambda_{n,s} \cdot (a^q)\lambda_{n,s}.$$

Hence the map $\lambda_{n,s}$ is a monoid endomorphism. The condition that $s = 0, \ldots, n-1$ implies that $\lambda_{n,s}$ is an injective map. $\square$

By $\mathfrak{End}_{\langle\lambda^\infty\rangle}(\mathscr{C}_+(a,b))$ we denote the subset of $\mathfrak{End}(\mathscr{C}_+(a,b))$, which consists of the elements of the family $\{\lambda_{n,s} \colon n \in \omega,\ s = 0, \ldots, n-1\}$ of endomorphisms of the monoid $\mathscr{C}_+(a,b)$.

Let S and T be arbitrary semigroups. Let $\varphi \colon T \to \mathfrak{End}(S)$, $t \mapsto \varphi_t$ be a homomorphism from T into the semigroup $\mathfrak{End}(S)$ of endomorphisms of S. The *semidirect product* of S and T defined on the product $S \times T$ with the semigroup operation

$$(s_1, t_1) \cdot (s_2, t_2) = (s_1 \cdot (s_2)\varphi_{t_1}, t_1 \cdot t_2),$$

and it is denoted by $S \rtimes_\varphi T$.

Theorem 2. *The set $\mathfrak{End}_{\langle\lambda^\infty\rangle}(\mathscr{C}_+(a,b))$ is a submonoid of $\mathfrak{End}(\mathscr{C}_+(a,b))$ and $\mathfrak{End}_{\langle\lambda^\infty\rangle}(\mathscr{C}_+(a,b))$ is isomorphic to a submonoid of the semidirect product $(\mathbb{N}, *) \rtimes_\varphi (\omega, +)$, where $(p)\varphi_n = np$.*

Proof. Fix arbitrary positive integers n_1, n_2, $s_1 = 0, \ldots, n_1 - 1$ and $s_2 = 0, \ldots, n_2 - 1$. Then for any $b^i a^j \in \mathscr{C}_+(a,b)$ we have that

$$((b^i a^j)\lambda_{n_1,s_1})\lambda_{n_2,s_2} = \begin{cases} (a^{n_1 j})\lambda_{n_2,s_2}, & \text{if } i = 0; \\ (b^{n_1 i - s_1} a^{n_1 j - s_1})\lambda_{n_2,s_2}, & \text{if } i \neq 0 \end{cases} =$$
$$= \begin{cases} a^{n_1 n_2 j}, & \text{if } i = 0; \\ a^{n_2(n_1 j - s_1)}, & \text{if } i \neq 0 \text{ and } n_1 i - s_1 = 0; \\ b^{n_2(n_1 i - s_1) - s_2} a^{n_2(n_1 j - s_1) - s_2}, & \text{if } i \neq 0 \text{ and } n_1 i - s_1 \neq 0 \end{cases} =$$
$$= \begin{cases} a^{n_1 n_2 j}, & \text{if } i = 0; \\ b^{n_2 n_1 i - (s_1 n_2 + s_2)} a^{n_2 n_1 j - (s_1 n_2 + s_2)}, & \text{if } i \neq 0 \end{cases} =$$
$$= (b^i a^j)\lambda_{n_1 n_2, s_1 n_2 + s_2}.$$

Since $s_1 < n_1$ and $s_2 < n_2$, we have that

$$s_1 n_2 + s_2 \leqslant (n_1 - 1)n_2 + s_2 < (n_1 - 1)n_2 + n_2 = n_1 n_2 - n_2 + n_2 = n_1 n_2,$$

and hence $s_1n_2 + s_2 < n_1n_2$. This implies that $\mathfrak{End}_{\langle\lambda^\infty\rangle}(\mathscr{C}_+(a,b))$ is a subsemigroup of $\mathfrak{End}(\mathscr{C}_+(a,b))$. Since $\lambda_{n,0} = \lambda_n$ for any positive integer n, $\mathfrak{End}_{\langle\lambda\rangle}(\mathscr{C}_+(a,b))$ is a submonoid of $\mathfrak{End}_{\langle\lambda^\infty\rangle}(\mathscr{C}_+(a,b))$.

We define the map $\mathbf{\Phi}\colon \mathfrak{End}_{\langle\lambda^\infty\rangle}(\mathscr{C}_+(a,b)) \to (\mathbb{N},*) \rtimes_\varphi (\omega,+)$ by the formula $(\lambda_{n,s})\mathbf{\Phi} = (n,s)$. The above arguments imply that

$$\begin{aligned}
(\lambda_{n_1,s_1}\lambda_{n_2,s_2})\mathbf{\Phi} &= (\lambda_{n_1n_2,s_1n_2+s_2})\mathbf{\Phi} = \\
&= (n_1n_2, s_1n_2+s_2) = \\
&= (n_1n_2, (s_1)\varphi_{n_2}+s_2) = \\
&= (n_1,s_1)(n_2,s_2) = \\
&= (\lambda_{n_1,s_1})\mathbf{\Phi}(\lambda_{n_2,s_2})\mathbf{\Phi},
\end{aligned}$$

and hence $\mathbf{\Phi}$ is a homomorphism. □

Example 2. We define the map $\varsigma\colon \mathscr{C}_+(a,b) \to \mathscr{C}_+(a,b)$ by the formula

$$(b^ia^j)\varsigma = \begin{cases} 1, & \text{if } i=j=0; \\ b^{i+1}a^{j+1}, & \text{otherwise} \end{cases}$$

for any $i,j \in \omega$.

Lemma 3. *The map $\varsigma\colon \mathscr{C}_+(a,b) \to \mathscr{C}_+(a,b)$ is an injective monoid endomorphism.*

Proof. A simple verification shows that ς is an injective map. Obviously that it is sufficient to show that for any $b^ia^j \neq 1$ and $b^ka^l \neq 1$ the following equality $(b^ia^j \cdot b^ka^l)\varsigma = (b^ia^j)\varsigma \cdot (b^ka^l)\varsigma$ holds. Indeed,

$$\begin{aligned}
(b^ia^j \cdot b^ka^l)\varsigma &= \begin{cases} (b^{i-j+k}a^l)\varsigma, & \text{if } j<k; \\ (b^ia^l)\varsigma, & \text{if } j=k; \\ (b^ia^{j-k+l})\varsigma, & \text{if } j>k \end{cases} = \\
&= \begin{cases} b^{i-j+k+1}a^{l+1}, & \text{if } j<k; \\ b^{i+1}a^{l+1}, & \text{if } j=k; \\ b^{i+1}a^{j-k+l+1}, & \text{if } j>k \end{cases}
\end{aligned}$$

and

$$(b^ia^j)\varsigma \cdot (b^ka^l)\varsigma = b^{i+1}a^{j+1} \cdot b^{k+1}a^{l+1} =$$

$$= \begin{cases} b^{i+1-(j+1)+k+1}a^{l+1}, & \text{if } j+1<k+1; \\ b^{i+1}a^{l+1}, & \text{if } j+1=k+1; \\ b^{i+1}a^{j+1-(k+1)+l+1}, & \text{if } j+1>k+1 \end{cases} =$$

$$= \begin{cases} b^{i-j+k+1}a^{l+1}, & \text{if } j<k; \\ b^{i+1}a^{l+1}, & \text{if } j=k; \\ b^{i+1}a^{j-k+l+1}, & \text{if } j>k, \end{cases}$$

and hence ς is an injective monoid endomorphism of $\mathscr{C}_+(a,b)$. □

By $\mathfrak{End}_{\langle\varsigma\rangle}(\mathscr{C}_+(a,b))$ we denote the subsemigroup of $\mathfrak{End}(\mathscr{C}_+(a,b))$, which is generated by endomorphism ς of the monoid $\mathscr{C}_+(a,b)$. Also by $\mathfrak{End}^1_{\langle\varsigma\rangle}(\mathscr{C}_+(a,b))$ we denote the semigroup $\mathfrak{End}_{\langle\varsigma\rangle}(\mathscr{C}_+(a,b))$ with the adjoined unit. Without loss of generality we may assume that

$$\mathfrak{End}^1_{\langle\varsigma\rangle}(\mathscr{C}_+(a,b)) = \mathfrak{End}_{\langle\varsigma\rangle}(\mathscr{C}_+(a,b)) \cup \{\lambda_1\}.$$

Proposition 5. *The semigroup $\mathfrak{End}_{\langle\varsigma\rangle}(\mathscr{C}_+(a,b))$ is isomorphic to the additive semigroup of positive integers $(\mathbb{N},+)$, and hence $\mathfrak{End}^1_{\langle\varsigma\rangle}(\mathscr{C}_+(a,b))$ is isomorphic to the additive monoid of non-negative integers $(\omega,+)$.*

Proof. For any $b^ia^j \in \mathscr{C}_+(a,b)$ and any positive integer n by the definition of ς we have that

$$\begin{aligned}(b^ia^j)\varsigma^n &= ((b^ia^j)\varsigma)\varsigma^{n-1} = \\ &= \begin{cases} (1)\varsigma^{n-1}, & \text{if } i=j=0; \\ (b^{i+1}a^{j+1})\varsigma^{n-1}, & \text{otherwise} \end{cases} = \\ &= \ldots = \\ &= \begin{cases} (1)\varsigma, & \text{if } i=j=0; \\ (b^{i+n-1}a^{j+n-1})\varsigma, & \text{otherwise} \end{cases} = \\ &= \begin{cases} 1, & \text{if } i=j=0; \\ b^{i+n}a^{j+n}, & \text{otherwise.} \end{cases}\end{aligned}$$

The definition of the bicyclic monoid $\mathscr{C}(a,b)$ implies that $b^{k_1}a^{l_1} = b^{k_2}a^{l_2}$ in $\mathscr{C}_+(a,b)$ if an only if $k_1=k_2$ and $l_1=l_2$. This and above equalities imply that the endomorphism ς generates the infinite cyclic subsemigroup in $\mathfrak{End}(\mathscr{C}_+(a,b))$, and hence $\mathfrak{End}_{\langle\varsigma\rangle}(\mathscr{C}_+(a,b))$ is isomorphic to the additive semigroup of positive integers $(\mathbb{N},+)$. The last statement of the proposition is obvious. □

Lemma 4. *Let $\varepsilon\colon \mathscr{C}_+(a,b)\to\mathscr{C}_+(a,b)$ be an injective monoid endomorphism such that $(a)\varepsilon = a^n$ for some positive integer n. Then there exists $s\in\{0,\ldots,n-1\}$ such that $\varepsilon=\lambda_{n,s}$.*

Proof. We observe that if ε is an injective endomorphism of $\mathscr{C}_+(a,b)$ then for any idempotents $b^ia^i, b^ja^j \in \mathscr{C}_+(a,b)$ the inequality $b^ia^i \preccurlyeq b^ja^j$ implies that $(b^ia^i)\varepsilon \preccurlyeq (b^ja^j)\varepsilon$, because the equality $b^ia^i \cdot b^ja^j = b^ia^i$ implies that

$$(b^ia^i)\varepsilon \cdot (b^ja^j)\varepsilon = (b^ia^i \cdot b^ja^j)\varepsilon = (b^ia^i)\varepsilon.$$

Since ε is an injective monoid endomorphism of $\mathscr{C}_+(a,b)$, we conclude that $(b^ia^i)\varepsilon \neq (b^ja^j)\varepsilon$ and $(b^0a^0)\varepsilon = (1)\varepsilon = 1 = b^0a^0$. Hence there exists a strictly increasing sequence $\{s_i\}_{i\in\omega}$ in ω such that $(b^ia^i)\varepsilon = b^{s_i}a^{s_i}$ for any $i \in \omega$ and $s_0 = 0$. Then for any positive integer i we have that

$$\begin{aligned}
(b^ia^{i+1})\varepsilon &= (b^ia^i \cdot a)\varepsilon = \\
&= (b^ia^i)\varepsilon \cdot (a)\varepsilon = \\
&= b^{s_i}a^{s_i} \cdot a^n = \\
&= b^{s_i}a^{s_i+n}
\end{aligned}$$

and

$$\begin{aligned}
(b^ia^{i+1})\varepsilon &= (a \cdot b^{i+1}a^{i+1})\varepsilon = \\
&= (a)\varepsilon \cdot (b^{i+1}a^{i+1})\varepsilon = \\
&= a^n \cdot b^{s_{i+1}}a^{s_{i+1}} = \\
&= \begin{cases} a^n, & \text{if } n \geqslant s_{i+1}; \\ b^{s_{i+1}-n}a^{s_{i+1}}, & \text{if } n < s_{i+1}. \end{cases}
\end{aligned}$$

The injectivity of ε implies that $n < s_{i+1}$. Since $s_0 = 0$ and $s_i < s_{i+1}$ for any $i \in \omega$, the above equalities imply that $s_{i+1} = s_i + n$ for any positive integer i, and hence $s_{i+1} - s_i = n$ for $i \in \mathbb{N}$. By induction we have that $s_{i+1} = s_1 + in$ for all $i \in \mathbb{N}$. This implies that

$$(b^ia^i)\varepsilon = b^{(i-1)n+s_1}a^{(i-1)n+s_1}$$

for any positive integer i. Then we get that

$$\begin{aligned}
(b^ia^{i+l})\varepsilon &= (b^ia^i \cdot a^l)\varepsilon = \\
&= (b^ia^i)\varepsilon \cdot (a^l)\varepsilon = \\
&= b^{(i-1)n+s_1}a^{(i-1)n+s_1} \cdot ((a)\varepsilon)^l = \\
&= b^{(i-1)n+s_1}a^{(i-1)n+s_1} \cdot (a^n)^l = \\
&= b^{(i-1)n+s_1}a^{(i-1)n+s_1} \cdot a^{nl} = \\
&= b^{(i-1)n+s_1}a^{(i+l-1)n+s_1}
\end{aligned}$$

for any $i, l \in \omega$. Also, since

$$\begin{aligned} a^n &= (a)\varepsilon = \\ &= (a \cdot ba)\varepsilon = \\ &= (a)\varepsilon \cdot (ba)\varepsilon = \\ &= a^n \cdot b^{s_1} a^{s_1}, \end{aligned}$$

properties of semigroup operation of $\mathscr{C}_+(a,b)$ and the natural partial order on the set of idempotents of the monoid $\mathscr{C}_+(a,b)$ imply that $s_1 \leqslant n$, and by injectivity of ε we have that $s_1 > 0$. Put $s = n - s_1$. Then we obtain that $s \in \{0, \ldots, n-1\}$ and

$$\begin{aligned} (b^i a^{i+l})\varepsilon &= b^{(i-1)n+s_1} a^{(i+l-1)n+s_1} = \\ &= b^{in-n+s_1} a^{(i+l)n-n+s_1} = \\ &= b^{in-s} a^{(i+l)n-s}. \end{aligned}$$

This and Proposition 4 imply that $\varepsilon = \lambda_{n,s}$. □

Proposition 6. *Let S, T, and U be semigroups, let $\mathfrak{h}\colon S \to T$ be a homomorphism, and let $\mathfrak{g}\colon U \to T$ be an injective homomorphism. If $(S)\mathfrak{h} \subseteq (U)\mathfrak{g}$, then the mapping $\mathfrak{f}\colon S \to U$ which is defined by the formula $(s)\mathfrak{f} = ((s)\mathfrak{h})\mathfrak{g}^{-1}$ is a homomorphism. Moreover, if the homomorphism $\mathfrak{h}\colon S \to T$ is injective or a monoid homomorphism, then so is $\mathfrak{f}$, too.*

Proof. Since $\mathfrak{g}\colon U \to T$ is an injective homomorphism, the map $\mathfrak{f}\colon S \to U$ is well defined. Also, for arbitrary $s_1, s_2 \in S$ we have that

$$\begin{aligned} (s_1 \cdot s_2)\mathfrak{f} &= ((s_1 \cdot s_2)\mathfrak{h})\mathfrak{g}^{-1} = \\ &= ((s_1)\mathfrak{h} \cdot (s_2)\mathfrak{h})\mathfrak{g}^{-1} = \\ &= ((s_1)\mathfrak{h})\mathfrak{g}^{-1} \cdot ((s_2)\mathfrak{h})\mathfrak{g}^{-1} = \\ &= (s_1)\mathfrak{f} \cdot (s_2)\mathfrak{f}, \end{aligned}$$

because $\mathfrak{g}\colon U \to T$ is an injective homomorphism. Hence $\mathfrak{f}\colon S \to U$ is a homomorphism. The second statement of the proposition is obvious. □

Lemma 5. *Let $\varepsilon\colon \mathscr{C}_+(a,b) \to \mathscr{C}_+(a,b)$ be an injective monoid endomorphism such that $(a)\varepsilon = b^n a^{n+p}$ for some positive integers n and p. Then there exists $s \in \{0, \ldots, n-1\}$ such that $\varepsilon = \lambda_{p,s}\varsigma^n$.*

Proof. Since ε is an injective monoid endomorphism of $\mathscr{C}_+(a,b)$, arguments presented in the proof of Lemma 4 imply that there exists a strictly increasing sequence $\{s_i\}_{i\in\omega}$ in ω such that $(b^ia^i)\varepsilon = b^{s_i}a^{s_i}$ for any $i\in\omega$ and $s_0=0$. Then for any positive integers i and j we have that

$$\begin{aligned}
(b^ia^{i+j})\varepsilon &= (b^ia^i\cdot a^j)\varepsilon = \\
&= (b^ia^i)\varepsilon\cdot(a^j)\varepsilon = \\
&= b^{s_i}a^{s_i}\cdot(b^na^{n+p})^j = \\
&= b^{s_i}a^{s_i}\cdot b^na^{n+pj} = \\
&= \begin{cases} b^na^{n+pj}, & \text{if } n\geqslant s_i;\\ b^{s_i}a^{s_i+pj}, & \text{if } n< s_i.\end{cases}
\end{aligned}$$

This implies that $(\mathscr{C}_+(a,b))\varepsilon\subseteq(\mathscr{C}_+(a,b))\varsigma^n$. By Proposition 6 the mapping $\mathfrak{f}\colon\mathscr{C}_+(a,b)\to\mathscr{C}_+(a,b)$ which is defined by the formula $(b^ia^j)\mathfrak{f} = ((b^ia^j)\varepsilon)(\varsigma^n)^{-1}$ is an endomorphism of the monoid $\mathscr{C}_+(a,b)$. Simple verifications show that $(a)\mathfrak{f}=a^p$. By Lemma 4 we have that $\mathfrak{f}=\lambda_{p,s}$ for some $s\in\{0,\ldots,n-1\}$, i.e., $\lambda_{p,s}=\varepsilon(\varsigma^n)^{-1}$. Since ς is an injective monoid endomorphism of $\mathscr{C}_+(a,b)$, we conclude that $\varepsilon=\lambda_{p,s}\varsigma^n$. $\square$

Theorem 3 describes all injective endomorphisms of the semigroup $\mathscr{C}_+(a,b)$ and it follows from Lemmas 4 and 5.

Theorem 3. *Let $\varepsilon\colon\mathscr{C}_+(a,b)\to\mathscr{C}_+(a,b)$ be an injective monoid endomorphism. Then only one of the following statements holds:*

(1) there exist a positive integer p and $s\in\{0,\ldots,p-1\}$ such that $\varepsilon=\lambda_{p,s}$;

(2) there exist positive integers n, p, and $s\in\{0,\ldots,p-1\}$ such that $\varepsilon=\lambda_{p,s}\varsigma^n$.

It is natural to ask the following: what is a semigroup operation on the subsemigroup $\mathfrak{End}_{\overline{\langle\lambda\varsigma\rangle}}(\mathscr{C}_+(a,b))$ of $\mathfrak{End}(\mathscr{C}_+(a,b))$ of endomorphisms of $\mathscr{C}_+(a,b)$ which is generated by endomorphism of the form $\lambda_{p,s}\varsigma^n$, where $p,n\in\mathbb{N}$ and $s\in\{0,\ldots,n-1\}$. Theorem 4 describes the structure of the semigroup operation on the semigroup $\mathfrak{End}_{\overline{\langle\lambda\varsigma\rangle}}(\mathscr{C}_+(a,b))$ of injective monoid endomorphisms of $\mathscr{C}_+(a,b)$.

Theorem 4. *$\mathfrak{End}_{\overline{\langle\lambda\varsigma\rangle}}(\mathscr{C}_+(a,b))$ is a subsemigroup of $\mathfrak{End}(\mathscr{C}_+(a,b))$ and it is isomorphic to the subsemigroup of the Cartesian power $\mathbb{N}^3$ with the following semigroup operation*

$$(p_1,s_1,n_1)\cdot(p_2,s_2,n_2)=(p_2p_1,p_2s_1,p_2n_1-s_2+n_2).$$

Proof. Fix arbitrary $p_1, n_1, p_2, n_2 \in \mathbb{N}$, $s_1 \in \{0, \ldots, n_1 - 1\}$ and $s_2 \in \{0, \ldots, n_2 - 1\}$. Then for any $b^i a^j \in \mathscr{C}_+(a,b)$ we have that

$$
\begin{aligned}
&((((b^i a^j)\lambda_{p_1,s_1})\varsigma^{n_1})\lambda_{p_2,s_2})\varsigma^{n_2} = \\
&= \begin{cases} (((a^{p_1 j})\varsigma^{n_1})\lambda_{p_2,s_2})\varsigma^{n_2}, & \text{if } i=0; \\ (((b^{p_1 i - s_1} a^{p_1 j - s_1})\varsigma^{n_1})\lambda_{p_2,s_2})\varsigma^{n_2}, & \text{if } i \neq 0 \end{cases} = \\
&= \begin{cases} ((1)\lambda_{p_2,s_2})\varsigma^{n_2}, & \text{if } i=0 \text{ and } j=0; \\ ((b^{n_1} a^{p_1 j + n_1})\lambda_{p_2,s_2})\varsigma^{n_2}, & \text{if } i=0 \text{ and } j\neq 0; \\ ((b^{p_1 i - s_1 + n_1} a^{p_1 j - s_1 + n_1})\lambda_{p_2,s_2})\varsigma^{n_2}, & \text{if } i \neq 0 \end{cases} = \\
&= \begin{cases} (1)\varsigma^{n_2}, & \text{if } i=0 \text{ and } j=0; \\ (b^{p_2 n_1 - s_2} a^{p_2(p_1 j + n_1) - s_2})\varsigma^{n_2}, & \text{if } i=0 \text{ and } j\neq 0; \\ (b^{p_2(p_1 i - s_1 + n_1) - s_2} a^{p_2(p_1 j - s_1 + n_1) - s_2})\varsigma^{n_2}, & \text{if } i \neq 0 \end{cases} = \\
&= \begin{cases} 1, & \text{if } i=0 \text{ and } j=0; \\ b^{p_2 n_1 - s_2 + n_2} a^{p_2(p_1 j + n_1) - s_2 + n_2}, & \text{if } i=0 \text{ and } j\neq 0; \\ b^{p_2(p_1 i - s_1 + n_1) - s_2 + n_2} a^{p_2(p_1 j - s_1 + n_1) - s_2 + n_2}, & \text{if } i \neq 0 \end{cases} = \\
&= \begin{cases} 1, & \text{if } i=0 \text{ and } j=0; \\ b^{p_2 n_1 - s_2 + n_2} a^{p_2 p_1 j + p_2 n_1 - s_2 + n_2}, & \text{if } i=0 \text{ and } j\neq 0; \\ b^{p_2 p_1 i - p_2 s_1 + p_2 n_1 - s_2 + n_2} a^{p_2 p_1 j - p_2 s_1 + p_2 n_1 - s_2 + n_2}, & \text{if } i \neq 0 \end{cases} \\
&= (b^i a^j)\lambda_{p_2 p_1, p_2 s_1}\varsigma^{p_2 n_1 - s_2 + n_2}
\end{aligned}
$$

The above equalities imply the statement of the theorem. □

Remark 1. 1. The endomorphisms λ_k, $k \in \omega$, $\lambda_{p,s}$, $p, n \in \mathbb{N}$, $s \in \{0, \ldots, n-1\}$ and ς of the monoid $\mathscr{C}_-(a,b)$ are introduced by the same formulae.

2. Repeating the proofs of corresponding statements in Sections 1 and 2 we obtain that the same statements about the corresponding endomorphisms of the monoid $\mathscr{C}_-(a,b)$. Moreover, the corresponding semigroups of endomorphisms are pairwise isomorphic.

Acknowledgements

The authors acknowledge Alex Ravsky and the referee for they comments and suggestions.

References

[1] Clifford, A.H., Preston, G.B.: The Algebraic Theory of Semigroups. Vol. I., Amer. Math. Soc. Surveys 7, Providence, R.I. (1961)

[2] Clifford, A.H., Preston, G.B.: The Algebraic Theory of Semigroups. Vol. II., Amer. Math. Soc. Surveys 7, Providence, R.I. (1967)

[3] Descalço, L., Ruškuc, N.: Subsemigroups of the bicyclic monoid. Int. J. Algebra Comput. **15**(1), 37–57 (2005). https://doi.org/10.1142/S0218196705002098

[4] Descalço, L., Ruškuc, N.: Properties of the subsemigroups of the bicyclic monoid. Czech. Math. J. **58**(2), 311–330 (2008). https://doi.org/10.1007/s10587-008-0018-7

[5] Gutik, O.: On non-topologizable semigroups. Preprint at https://arxiv.org/abs/2405.16992 (2024)

[6] Gutik, O., Prokhorenkova, O., Sekh, D.: On endomorphisms of the bicyclic semigroup and the extended bicyclic semigroup. Visn. L'viv. Univ., Ser. Mekh.-Mat. **92**, 5–16 (2021) (in Ukrainian). https://doi.org/10.30970/vmm.2021.92.005-016

[7] Lawson, M.: Inverse Semigroups. The Theory of Partial Symmetries. World Scientific, Singapore (1998). https://doi.org/10.1142/3645

[8] Makanjuola, S.O., Umar, A.: On a certain subsemigroup of the bicyclic semigroup. Commun. Algebra. **25**(2), 509–519 (1997). https://doi.org/10.1080/00927879708825870

[9] Wagner, V.V.: Generalized groups. Dokl. Akad. Nauk SSSR. **84**, 1119–1122 (1952) (in Russian)

Contact information

O. Gutik,
Sh.-A. Penza

Ivan Franko National University of Lviv,
Universytetska 1, Lviv, 79000, Ukraine
E-Mail: oleg.gutik@lnu.edu.ua,
sher-ali.penza@lnu.edu.ua

Received by the editors: 22.08.2024
and in final form 16.11.2024.