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RESEARCH ARTICLE

Minimax equivalence method: initial ideas, first applications and new concepts

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ABSTRACT. In 2005, the author introduced for posets the notion of (min, max)-equivalence (later called minimax equivalence). This equivalence preserves $\mathbb{Z}$-equivalence of the corresponding Tits quadratic forms which play an important role in modern representation theory. The minimax equivalence method has been used to solving many classification problems. This method was first applied in the same year by the author together with his PhD student M. V. Styopochkina for classifying all posets with positive Tits quadratic form and all minimal posets with nonpositive Tits form. These results were often cited, but the corresponding publication is virtually inaccessible. The paper provides them (translated into English) and also the author's new ideas about the minimax equivalence method.

Introduction

The Tits quadratic forms of various algebraical objects play an important role in modern representation theory. They were first introduced by P. Gabriel for quivers [13].

Let $Q = (Q_0, Q_1)$ be a finite quiver with the set of vertices Q_0 and the set of arrows Q_1, and let n denote the order of Q_0. By definition, the Tits quadratic form of Q is an integer quadratic form $q_Q : \mathbb{Z}^n \to \mathbb{Z}$,

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given by the equality

$$q_Q(z) = \sum_{i \in Q_0} z_i^2 - \sum_{i \to j} z_i z_j,$$

where $i \to j$ runs through Q_1. P. Gabriel proved that the following conditions are equivalent:

(1q) Q is of finite representation type over a field k;

(2q) the Tits quadratic form of Q is positive.

He also described all the quivers of finite type in an explicit form. If one talks on the connected quivers (this is the main case), such quivers are exhausted by those quivers whose underlying graphs are (simply faced) Dynkin diagrams.

Now let S be a poset (without an element 0) and let $\mathbb{Z}^{0 \cup S}$ denote the set of all vectors $z = (z_i), i \in 0 \cup S$. The Tits quadratic form $q_S : \mathbb{Z}^{0 \cup S} \to \mathbb{Z}$ of S, which is given by the equality

$$q_S(z) = z_0^2 + \sum_{i \in S} z_i^2 + \sum_{i<j, i,j \in S} z_i z_j - z_0 \sum_{i \in S} z_i,$$

was first studied by Yu. A. Drozd in [12]. He proved that the following conditions are equivalent:

(1p) S is of finite representation type over a field k;

(2p) the Tits quadratic form of S is weakly positive[1].

There is essentially not an explicit description of posets of finite types since, up to isomorphism, there exists only five minimal posets of infinite representation type (in more details, see below 3.2.1).

The topic discussed in this paper was inspired by these results of P. Gabriel and Yu. A. Drozd, and for the first time, were studied by the author together with his PhD student M. V. Styopochkina.

In contrast to quivers, the posets with weakly positive and positive Tits forms do not coincide. The above results show that the posets with positive Tits quadratic form are analogs of the Dynkin diagrams.

In 2005, the author and M. V. Styopochkina classified all posets of width 2 and then in the general case with positive Tits quadratic form (resp. [3] and [4]) and all minimal posets with nonpositive Tits form [4]. In solving these problems the minimax equivalence method, introduced by the author in [1], was first applied. It has been subsequently used to solve many other classification problems (see, e.g. [5–10]).

[1]I.e. positive on the set of vectors with nonnegative coordinates.

Sections 1–3 of this paper is a translation from Russian of the main part of the paper [4] which was often cited but is virtually inaccessible. Sections 4 and 5 present some new the author's ideas about the minimax equivalence method.

1. Minimax equivalence method

Throughout the paper, all posets are finite of order $n > 0$ (without an element 0), but for formal convenience empty subposets are admitted. Instead $S = (P, \leq)$ we write simply S, and write $x \in S$, etc. All subposets are complete (i.e. with partial orders induced by those on the posets), and instead of subposets we often say simply subsets. One-element subsets are identified with the elements themselves. In the case, when the elements of a poset are numbered by integers, the partial order relation is denoted by $\preceq$ (and always is assumed that $i \prec j$ implies $i < j$).

The notation $T \cong S$ for posets means that T is isomorphic to S. When S is specific, we also say that T is of the form S. The dual poset for S is denoted by S^{op}; in other words, S^{op} and S are equal as usual sets, and $x < y$ in S^{op} if and only if $x > y$ in S. Posets T and S are called *antiisomorphic* if T and S^{op} are isomorphic.

We call a subposet X of a poset S *lower* (resp. *upper*) if $x \in X$ whenever $x < y$ (resp. $x > y$) and $y \in X$. For subposets X and Y of S, we write $X < Y$ if $x < y$ for each $x \in X, y \in Y$ (this is obviously true, when X or Y is empty). The notation $x \lessgtr y$ means that the elements x and y are incomparable.

1.1. (Min, max)-equivalence

The concept of (min, max)-equivalence of posets, which later was also called "minimax equivalence", has been introduced by the author in [1].

Let us define for a minimal (resp. maximal) element a of a poset S the poset $T = S_a^{\uparrow}$ (resp. $T = S_a^{\downarrow}$) in the following way: $T = S$ as usual sets, $T \setminus a = S \setminus a$ as posets, the element a is already maximal (resp. minimal) in T, and a is comparable with $x \neq a$ in T if and only if they are incomparable in S. Below we write $S_{xy}^{\uparrow\uparrow}$ instead of $(S_x^{\uparrow})_y^{\uparrow}$, $S_{xy}^{\uparrow\downarrow}$ instead of $(S_x^{\uparrow})_y^{\downarrow}$, etc. Let S and T be posets that are equal as usual sets. We write $T \simeq_{(\min,\max)} S$ and call T (min, max)-*equivalent* to S if there are posets $S_1, \ldots, S_p$ $(p \geq 0)$ such that, if one puts $S = S_0$ and $T = S_p$, for each $i = 0, 1, \ldots, p$ either $S_{i+1} = (S_i)_{y_i}^{\uparrow}$ or $S_{i+1} = (S_i)_{z_i}^{\downarrow}$. More compactly,

$T \simeq_{(\min,\max)} S$ if T is equal to some poset of the form $\overline{S} = S^{\varepsilon_1\varepsilon_2\ldots\varepsilon_p}_{x_1x_2\ldots x_p}$ $(p \geq 0)$, where $\varepsilon_1, \ldots, \varepsilon_p \in \{\uparrow, \downarrow\}$ and $x_1, \ldots, x_p$ as elements of S are not necessarily different.

Since $U^{\uparrow\downarrow}_{aa} = U$ and $U^{\downarrow\uparrow}_{bb} = U$ for a poset U, in this case one has $S = \overline{S}\,^{\varepsilon_p^{-1}\varepsilon_{p-1}^{-1}\ldots\varepsilon_2^{-1}\varepsilon_1^{-1}}_{x_px_{p-1}\ldots x_2x_1}$, where ε_i^{-1} denotes the arrow with the opposite direction compared to ε_i. Therefore, S is (min, max)-*equivalent* to T, and hence the relation $\simeq_{(\min,\max)}$ is an equivalence (transitivity is obvious).

The notion of (min, max)-equivalence $\simeq_{(\min,\max)}$ can be naturally extended to the notion of (min, max)-isomorphism $\cong_{(\min,\max)}$. Namely, a poset T is called (min, max)-*isomorphic* to a poset S if there exist posets U, V such that $T \cong U \simeq_{(\min,\max)} V \cong S$. When T is (min, max)-isomorphic to S^{op}, it is called (min, max)-*antiisomorphic* to S.

1.2. Tits quadratic form

Directly from the definition of the Tits quadratic form $q_S(z)$ for a poset S (see Introduction) we have the following general statement.

Proposition 1. *Let posets S and T be equal as usual set. When elements x and y are comparable in S if and only if they are comparable in T, then $q_S(z) = q_T(z)$.*

The following statement, proved by the author in [1], was the main motivation for introducing the notion of (min, max)-equivalence.

Proposition 2. *Let S and T be* (min, max)*-isomorphic posets. Then their Tits quadratic form are $\mathbb{Z}$-equivalent.*

The proposition follows from the following easily checfed equality for the Tits quadratic forms of posets X and $Y = X^{\uparrow}_a$ (resp. $Y = X^{\downarrow}_a$): $q_X(z) = q_Y(z')$, where $z'_0 = z_0 - z_a$, $z'_a = -z_a$ and $z'_x = z_x$ for any $x \neq a$.

From Propositions 1 and 2 follows the next corollary.

Corollary 1. *The Tits quadratic forms of* (min, max)*-equivalent or dual posets simultaneously are or are not positive* (*nonnegative*).

1.3. Min-equivalence

1.3.1. The main definitions. In the case when a poset T is (min, max)-equivalent to a poset S with $\overline{S} = S^{\uparrow\uparrow\ldots\uparrow}_{x_1x_2\ldots x_p}$ $(p \geq 0)$, we say that T is *min-equivalent* to S and write $T \simeq_{\min} S$. Recall that the elements x_i

are not necessarily different. This relation is also an equivalence relation (Proposition 6 below).

We call a sequence $\alpha = (x_1, x_2, \dots, x_p)$ of length $0 \le p < \infty$ with elements $x_i \in S$ min-*admissible* if the expression $\overline{S} = S^{\uparrow\uparrow\dots\uparrow}_{x_1 x_2 \dots x_p}$ makes sense (see above the definition of (min, max)-equivalence). In this case we also write $T = S^{\uparrow}_{\alpha}$. The length p is denoted by $d(\alpha)$. Note that the empty sequence α_0 is min-admissible (with $d(\alpha_0) = 0$). The set of all min-admissible sequences with elements in S is denoted by $\mathcal{P}(S)$. The multiplicity of occurrences of $x \in S$ in $\alpha \in \mathcal{P}(S)$ is denoted by $m_\alpha(x)$.

For a sequence $\alpha = (x_1, x_2, \dots, x_p) \in \mathcal{P}(S)$ and $0 \le i \le p$, we denote $\alpha_{(i)} = (x_1, x_2, \dots, x_i)$ and $\alpha^{(i)} = (x_i, x_{i+1}, \dots, x_p)$. Obviously, $\alpha_{(i)} \in \mathcal{P}(S)$ and $\alpha^{(i)} \in \mathcal{P}(S^{\uparrow}_{\alpha_{(i-1)}})$. Denote also $[\alpha]_S := \{x \in S \,|\, x = x_i$ for some $i\}$.

Lemma 1. *If $\alpha \in \mathcal{P}(S)$, then the subset $[\alpha]_S \subseteq S$ is lower.*

Indeed, if there were elements $a \notin [\alpha]_S$ and $b \in [\alpha]_S$ with $a < b$, and s denoted the smallest number such that $x_s = b$, then the inequality $a < b$ would also hold in $S^{\uparrow}_{\alpha_{(s-1)}}$, i.e. x_s would not be minimal in $S^{\uparrow}_{\alpha_{(s-1)}}$.

Lemma 2. *Let $\alpha = (x_1, x_2, \dots, x_p) \in \mathcal{P}(S)$ and X be a subposet of S. Let us denote by α_X a subsequence of α consisting of all $x_i \in X$. Then $\alpha_X \in \mathcal{P}(X)$ and $X^{\uparrow}_{\alpha_X}$ is a complete subposet of $S^{\uparrow}_{\alpha}$.*

Proof. We carry out the proof by induction on m; the case $m = 0$ is trivial. Let us consider the sequence $\alpha^{(2)}$ which, obviously, belongs to $\mathcal{P}(S^{\uparrow}_{x_1})$.

Let first $x_1 \ne X$. Then $\alpha^{(2)}_X = \alpha_X$ and, therefore, X is a subposet of $S^{\uparrow}_{x_1}$. By the induction hypothesis for $S' = S^{\uparrow}_{x_1}$, $\alpha' = \alpha^{(2)}$ and $X' = X$, we have that $\alpha^{(2)}_X \in \mathcal{P}(X)$ and $X^{\uparrow}_{\alpha^{(2)}_X}$ is a complete subposet in $(S^{\uparrow}_{x_1})^{\uparrow}_{\alpha^{(2)}}$. Consequently, $\alpha_X \in \mathcal{P}(X)$ and $X^{\uparrow}_{\alpha_X}$ is a complete subposet in $S^{\uparrow}_{\alpha}$ (since $\alpha^{(2)}_X = \alpha_X$ and $(S^{\uparrow}_{x_1})^{\uparrow}_{\alpha^{(2)}} = S^{\uparrow}_{\alpha}$).

Let now $x_1 \in X$. Then $X^{\uparrow}_{x_1}$ is a subposet of $S^{\uparrow}_{x_1}$, and by the induction hypothesis for $S' = S^{\uparrow}_{x_1}$, $\alpha' = \alpha^{(2)}$ and $X' = X^{\uparrow}_{x_1}$, we have that $\alpha^{(2)}_X \in \mathcal{P}(X^{\uparrow}_{x_1})$. And since $\alpha_X = (x_1, \alpha^{(2)}_X)$ with x_1 a minimal element of S, we conclude $\alpha_X \in \mathcal{P}(X)$. Further, due to the same inductive hypothesis $(X^{\uparrow}_{x_1})^{\uparrow}_{\beta}$ is a complete subset of $(S^{\uparrow}_{x_1})^{\uparrow}_{\alpha^{(2)}}$, where $\beta = \alpha^{(2)}_{X^{\uparrow}_{x_1}}$. Since X and $X^{\uparrow}_{x_1}$ are equal as usual sets, $\beta = \alpha^{(2)}_X$, and we have the equality

$(X^{\uparrow}_{x_1})^{\uparrow}_{\beta} = X^{\uparrow}_{\alpha_X}$. Besides, $(S^{\uparrow}_{x_1})^{\uparrow}_{\alpha^{(2)}} = S^{\uparrow}_{\alpha}$. Hence $X^{\uparrow}_{\alpha_X}$ is a complete subposet of $S^{\uparrow}_{\alpha}$. □

The set of all sequences $\alpha = (x_1, x_2, \ldots, x_p) \in \mathcal{P}(S)$ such that $m_\alpha(x) \leq k$ for arbitrary $x \in S$ is denoted by $\mathcal{P}_k(S)$.

1.3.2. The case of sequences without repetitions. A sequence from $\mathcal{P}(S)$ is called *without repetitions* if it belongs to $\mathcal{P}_1(S)$.

Proposition 3. *Let X be a subset of S. There exists a sequence $\alpha \in \mathcal{P}_1(S)$ with $[\alpha]_S = X$ if and only if the subset X is lower.*

Indeed, the necessity of this proposition follows from Lemma 1, and the sufficiency from the following obvious lemma.

Lemma 3. *Let S_1 denote the set of all minimal elements in S and, inductively, let S_i with $i > 1$ denote the set of all minimal elements in $S \setminus (\bigcup_{j=1}^{i-1} S_j)$. Let $h(x) = i$ for an element $x \in S$ mean that $x \in S_i$. Then there exists a sequence $(x_1, x_2, \ldots, x_p)$ of length $p = |S|$ satisfying $h(x_1) \leq h(x_2) \leq \ldots \leq h(x_p)$, and any such a sequence belongs to $\mathcal{P}_1(S)$.*

Proposition 4. *Let $\alpha = (x_1, x_2, \ldots, x_p)$ be a sequence from $\mathcal{P}_1(S)$, and let $a, b \in S$. Then $a < b$ in $\overline{S} = S^{\uparrow}_{\alpha}$ if and only if one of the following conditions holds:*

(a) *$a < b$ in S and either $a, b \in [\alpha]_S$ or $a, b \notin [\alpha]_S$;*

(b) *$a \not\lessgtr b$ in S and $b \in [\alpha]_S, a \notin [\alpha]_S$.*

Indeed, since $S^{\uparrow}_{x} \setminus x = S \setminus x$, we take into account only those steps (of the transition from S to $\overline{S}$) which are associated with elements a or b. Their number is 0, 1 or 2, and the proof is obvious.

Corollary 2. *If $\alpha \in \mathcal{P}_1(S)$ and $[\alpha]_S = S$, then $S^{\uparrow}_{\alpha} = S$.*

Corollary 3. *If $\alpha \in \mathcal{P}_1(S)$, then $[\alpha]_{S^{\uparrow}_{\alpha}}$ is an upper subset in $S^{\uparrow}_{\alpha}$, and $[\alpha]_{S^{\uparrow}_{\alpha}} = [\alpha]_S$.*

Proposition 4 shows that $S^{\uparrow}_{\alpha}$ does not depend on the order in which its members are arranged. More formally, we have the following statement.

Corollary 4. *If $\alpha, \beta \in \mathcal{P}_1(S)$ and $[\alpha]_S = [\beta]_S$, then $S^{\uparrow}_{\alpha} = S^{\uparrow}_{\beta}$.*

Lemma 3 and Corollary 4 imply the next statement.

Proposition 5. *When considering a poset $\overline{S}$, min-equivalent to a fix poset S and having the form $S^{\uparrow}_{\alpha}$ with $\alpha \in \mathcal{P}_1(S)$, one can take $\overline{S}$ in a more convenient form. Namely, in the form $\overline{S} = S^{\uparrow}_{X}$ with the lower subset $X = [\alpha]_S$ of S, the partial order for which is determined by Proposition 4, i.e. $a < b$ in $\overline{S} = S_X$ if and only if one of the following conditions holds:*

(a) *$a < b$ in S and either $a, b \in X$ or $a, b \notin X$;*

(b) *$a \not\gtrless b$ in S and $b \in X, a \notin X$.*

The proposition makes the invariants "more large" and clearer.

1.3.3. Three equivalences. At the beginning of this subsection we called a poset T min-equivalent to a poset S if T is (min, max)-equivalent to S with $\overline{S} = S^{\uparrow\uparrow\ldots\uparrow}_{x_1x_2\ldots x_p}$. In the dual case, when $\overline{S} = S^{\downarrow\downarrow\ldots\downarrow}_{x_1x_2\ldots x_p}$, we say that T is max-*equivalent* to S. Formally, the duality between these two notions is defined by the following statement.

Lemma 4. *Let X be a lower subset of a poset S. Then $S^{\uparrow}_{X} = S^{\downarrow}_{S\setminus X}$.*

Proof. From Proposition 5 it follows that $(S^{\uparrow}_{X})^{\uparrow}_{S\setminus X} = S$. Let us denote by γ a min-admissible sequence $\gamma = (z_1, z_2, \ldots, z_q)$ with $[\gamma]_{S^{\uparrow}_{X}} = S \setminus X$ (see Proposition 3). Then $(S^{\uparrow}_{X})^{\uparrow}_{\gamma} = (S^{\uparrow}_{X})^{\uparrow}_{S\setminus X} = S$ or, equivalently, $(S^{\uparrow}_{X})^{\uparrow\uparrow\downarrow}_{\gamma^{(q-1)}z_qz_q} = S^{\downarrow}_{z_q}$. Since $U^{\uparrow\downarrow}_{aa} = U$ for a poset U, it follows from the last equation that $(S^{\uparrow}_{X})^{\uparrow}_{\gamma^{(q-1)}} = S^{\downarrow}_{z_q}$. Continuing this process, after q steps we obtain that $S^{\uparrow}_{X} = S^{\downarrow}_{\overline{\gamma}}$ with $\overline{\gamma} = (z_q, \ldots, z_2, z_1)$. Since $[\gamma]_{S^{\uparrow}_{X}} = [\overline{\gamma}]_{S^{\uparrow}_{X}}$, the required equality is satisfied if one takes into account Corollary 4 for the dual situation. □

Using this lemma, one can easily reformulate the definitions and statements for min-equivalence to max-equivalence.

Proposition 6. *The following conditions are equivalent:*

(1) *T is* (min, max)*-equivalent to S;*

(2) *T is* min*-equivalent to S;*

(3) *T is* max*-equivalent to S.*

The implications (2) $\Rightarrow$ (1) and (3) $\Rightarrow$ (1) are obvious. The implications (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (3) follow, respectively, from the equality $Y^{\downarrow}_{x} = Y^{\uparrow}_{Y\setminus x}$ for a poset U (see Lemma 4) and dual to him equality $Z^{\uparrow}_{x} = Z^{\downarrow}_{Z\setminus x}$.

Corollary 5. *Both* min-*equivalence and* max-*equivalence are equivalence relations.*

These equivalence relations are denoted as $\simeq_{min}$ and $\simeq_{max}$.

1.3.4. The case of sequences from $\mathcal{P}_2(S)$. The above statements on sequences from $\mathcal{P}_1(S)$ can be generalized to sequences from $\mathcal{P}_2(S)$.

For $\alpha = (x_1, x_2, \dots, x_p) \in \mathcal{P}_2(S)$, denote by $[\alpha]_S^2$ the subset of $[\alpha]_S$ consisting of those elements from S which appear twice in α.

Proposition 7. *Let $\alpha \in \mathcal{P}_2(S)$ and $a, b \in S$. Then $a < b$ in $\overline{S} = S_\alpha^\uparrow$ if and only if one of the following conditions holds:*
(a) *$a < b$ in S and $m_\alpha(a) = m_\alpha(b)$;*
(b) *$b < a$ in S and $m_\alpha(a) = 0, m_\alpha(b) = 2$;*
(c) *$a \lessgtr b$ in S and $m_\alpha(b) = m_\alpha(a) + 1$.*

The proof of this proposition, which generalizes Proposition 4, also follows directly from the definitions.

Corollary 6. *If $\alpha, \beta \in \mathcal{P}_2(S)$ and β is obtained from α by rearranging its terms (or, in other words, $[\alpha]_S = [\beta]_S$ and $[\alpha]_S^2 = [\beta]_S^2$), then $S_\alpha^\uparrow = S_\beta^\uparrow$.*

Lemma 5. *If $\alpha \in \mathcal{P}_2(S)$, then $[\alpha]_S^2$ is a lower subset in $[\alpha]_S$ (consequently, in S) and $[\alpha]_S^2 < S \setminus [\alpha]_S$.*

Proof. Let us assume that $[\alpha]_S^2$ is not lower. Then there are elements $b \in [\alpha]_S^2$ and $a \notin [\alpha]_S^2$ with $a < b$. Let i and $j > i$ denote such numbers that $x_i = x_j = b$ and $x_q \neq b$ for $i < q < q$. Since the subset $[\alpha]_S$ is lower and $[\alpha]_S^2 \subseteq [\alpha]_S$, we have that $a \in [\alpha]_S$. Consequently, $a = x_s$ for some s, and since the element b is minimal in $S_{\alpha(i-1)}^\uparrow$, we have $s < i$, which means that $a \lessgtr b$ in $S_{\alpha(i-1)}^\uparrow$. Hence $a < b$ in $S_{\alpha(i)}^\uparrow$ and, therefore, in $S_{\alpha(j-1)}^\uparrow$ (taking into account that by the condition $a \notin [\alpha]_S^2$, among x_q with $i < q < j$ there is no a). But then the element b is not minimal in $S_{\alpha(j-1)}^\uparrow$. Thus, α is not min-admissible and we get a contradiction.

Next, assume that the inequality indicated in the condition of the lemma is not satisfied. Then there exist $a \in [\alpha]_S^2$ and $b \in S \setminus [\alpha]_S$ such that $a \lessgtr b$ (since $[\alpha]_S^2$ is a lower subset, the case $b < a$ is impossible). Let $a = x_i = x_j$, where $i < j$. Since $b \notin [\alpha]_S$, we have $a \lessgtr b$ in $S_{\alpha(i-1)}^\uparrow$ and, therefore, $b < a$ in $S_{\alpha(i)}^\uparrow$. Then $b < a$ in $S_{\alpha(j-1)}^\uparrow$, and hence in this poset $x_j = a$ cannot be a minimal element. Thus, α is not min-admissible and we again come to a contradiction. $\square$

We write (by analogy with sequences) $S^{\uparrow\uparrow}_{YX}$ instead $(S^{\uparrow}_{Y})^{\uparrow}_{X}$.

Lemma 6. *Let Y be a lower subset of S, X be a lower subset of Y, and $X < S \setminus Y$. Then the expression $S^{\uparrow\uparrow}_{YX}$ is correct.*

Indeed, the expression $S^{\uparrow}_{Y}$ is correct by Proposition 3, and $S^{\uparrow\uparrow}_{YX}$ is correct because due to the $X < S \setminus Y$ the subset X is lower in $S^{\uparrow}_{Y}$ too (see Proposition 5).

Proposition 8. *Let X and Y be subsets of S. There exists a sequence α in $\mathcal{P}_2(S)$ such that $[\alpha]_S = Y$ and $[\alpha]^2_S = X$ if and only if Y is a lower subset of S, X is a lower subset of Y and $X < S \setminus Y$.*

Indeed, the necessity of this proposition follows from Lemmas 1 and 5, and the sufficiency from the previous lemma and Proposition 5.

2. Min-equivalence algorithm

2.1. Additional statements

By Proposition 6 instead of (min, max)-equivalent posets we can consider min-equivalent ones.

Let S be an arbitrary poset and $n = |S|$.

Lemma 7. *Let α be a sequence of $\mathcal{P}(S)$ such that $[\alpha]_S = S$. Then there exists a sequence $\beta \in \mathcal{P}(S)$ of length $d(\beta) = d(\alpha) - n$ such that $S^{\uparrow}_{\beta} = S^{\uparrow}_{\alpha}$.*

Proof. We will carry out the proof by induction on $d(\alpha)$. The base of induction, namely the $d(\alpha) = n$, holds by Corollary 2 (as β one need to take the empty sequence).

Let now $\alpha = (x_1, x_2, \dots, x_p)$ with $p = d(\alpha) > n$ and denote by $s = s(\alpha)$ the smallest number such that $m_\alpha(x_s) > 1$.

First consider the case $s = 1$. Applying the inductive hypothesis to the poset $S' = S^{\uparrow}_{x_1}$ and to the sequence $\alpha' = \alpha^{(2)}$ of length $d(\alpha') = p - 1$, we obtain that there is a sequence $\beta' \in \mathcal{P}(S') = \mathcal{P}(S)$ of length $d(\beta') = p - 1 - n$ such that $(S')^{\uparrow}_{\alpha'} = (S')^{\uparrow}_{\beta'}$, i.e. $(S^{\uparrow}_{x_1})^{\uparrow}_{\alpha^{(2)}} = (S^{\uparrow}_{x_1})^{\uparrow}_{\beta'}$, from which $S^{\uparrow}_{\alpha} = (S^{\uparrow}_{x_1})^{\uparrow}_{\beta'}$. The last equality can be written as follows: $S^{\uparrow}_{\alpha} = S^{\uparrow}_{\beta}$, where $\beta = (x_1, \beta')$. Since $d(\beta) = d(\beta') + 1 = p - n$, the lemma is proven in this case.

Now consider the case $s > 1$. Then $m_\alpha(x_i) = 1$ for $1 \le i < s$. Show that x_j for $1 \le j < s$ and x_s are not comparable in S. Assume the opposite. Since $x_j \not> x_s$ in $S^{\uparrow}_{\alpha_{(j-1)}}$ according to the definition of min-admissible sequences, one has (taking into account the assumption) that $x_j < x_s$ in S. Therefore, $x_j < x_s$ in $S^{\uparrow}_{\alpha_{(j-1)}}$ and, consequently, in $S^{\uparrow}_{\alpha_{(s)}}$. But since x_j (as an element of S) does not occur in $\alpha^{(s+1)}$ and $x_s = x_t$ for some $t > s$, the element x_t cannot be minimal in $S^{\uparrow}_{\alpha_{(t-1)}}$. We come to a contradiction. So $x_1, \ldots, x_{s-1}$ are not comparable to x_s, and then by Corollary 4 $S^{\uparrow}_{\alpha} = S^{\uparrow}_{\alpha'}$, where $\alpha' = (x_s, x_1, \ldots, x_{s-1}, x_{s+1}, \ldots, x_p) \in \mathcal{P}(S)$. But since $m_{\alpha'}(x_s) = m_\alpha(x_s) > 1$ and $s(\alpha') = 1$, we come to the case already considered above. □

Proposition 9. *Let $T \simeq_{\min} S$. Then there exists a sequence $\beta \in \mathcal{P}_2(S)$ such that $T = S^{\uparrow}_{\beta}$.*

Proof. Fix a sequence $\alpha = (x_1, x_2, \ldots, x_p) \in \mathcal{P}(S)$ such that $T = S^{\uparrow}_{\alpha}$, and let $\alpha \notin \mathcal{P}_2(S)$. Show first that $[\alpha]_S = S$.

Assume the opposite and fix an element $a \in S \setminus [\alpha]_S$. Let x_s, x_t and x_r $(s < t < r)$ be equal members of the sequence α such that $m_{\alpha_{(x_r)}}(x_r) = 3$. By Lemma 1, $a \between x_s$ or $a > x_s$. Then in the first case $a < x_s = x_t$ in $S^{\uparrow}_{\alpha_{(s)}}$, and, therefore, in $S^{\uparrow}_{\alpha_{(t-1)}}$, but then it follows that x_t is not minimal in $S^{\uparrow}_{\alpha_{(t-1)}}$, and we get a contradiction. In the second case, $a \between x_s$ in $S^{\uparrow}_{\alpha_{(t-1)}}$, and therefore $a < x_s = x_t$ in $S^{\uparrow}_{\alpha_{(r-1)}}$, but then x_r is not minimal in $S^{\uparrow}_{\alpha_{(r-1)}}$, and we again get a contradiction.

So, $[\alpha]_S = S$, and then by Lemma 7 there is a sequence β of length $d(\beta) = d(\alpha) - n$ such that $S^{\uparrow}_{\beta} = S^{\uparrow}_{\alpha}$. If $\beta \notin \mathcal{P}_2(S)$, then we again apply the above reasonings, etc. In a finite number of steps we arrive to the desired sequence β. □

2.2. Description of the algorithm

If we take into account all what is stated above (especially Propositions 5, 8, and 9), then classification of all posets min-equivalent (or, equivalently, (min, max)-equivalent) to a fixed poset S can be carried out according to the following scheme:

I. To describe all lower subsets of $X \neq S$ in S, and for each of them to construct the poset $S^{\uparrow}_{X}$ ($X = \varnothing$ is not excluded).

II. To describe all pairs (Y, X) consisting of a proper lower subset of Y in S and a non-empty lower subset X in Y such that $X < S \setminus Y$; then for each such pair to build the poset $S^{\uparrow\uparrow}_{YX}$.

III. To choose among the obtained in I and II posets one representative from each class of isomorphic posets.

This algorithm is called the min-*equivalence algorithm.*

The first two steps of the algorithm can be simplified in the following way. Let us call the subsets X and X' of a poset S *strongly isomorphic* if there is an automorphism φ of S such that $\varphi(X) = X'$. Similarly, two pairs (Y, X) and (Y', X') of subposets of S is said to be *strongly isomorphic* if there is an automorphism φ such that $\varphi(Y) = Y'$ and $\varphi(X) = X'$. It is obviously that the subsets X in I and the pairs (Y, X) of subsets in II suffice to describe up to strong isomorphism.

The third step also can be simplified if the representatives are written out up to isomorphism and antiisomorphism or equivalently up to isomorphism and duality, etc.

Remark 1. If the representatives of the isomorphism-antiisomorphism classes are chosen, then there are natural representatives of the isomorphism classes. Namely, it is necessary to complement them with the dual posets that are not self-dual (but of course only those that occur in III). That is why we often say "up to isomorphism and duality" instead of "up to isomorphism and antiisomorphism". In this connection, see Theorem 2.

3. First classification results

3.1. Definitions about posets

The *width* of a poset S is defined as the maximum number of pairwise incomparable elements of S and is denoted by w(S). A linear ordered set is also called a *chain.* A poset with the only pair of incomparable elements is called an *almost chain.* We say that a poset S is a *sum of subposets* $A_1, A_2, \ldots, A_m$ and write $S = A_1 + A_2 + \cdots + A_m$ if $S = \bigcup\limits_{i \in S} A_i$ and $A_i \cap A_j = \varnothing$ for any i and $j \neq i$. If any two elements of different summands are incomparable, the sum is called *direct*; when we want to highlight this property, we write $\coprod$ instead of $+$. A poset S is called *connected* if it cannot be decomposed into a direct sum of its proper subposets. Obviously, a nonconnected poset is decomposed into a direct sum of connected ones uniquely up to permutation.

A sum $S = A_1 + A_2 + \cdots + A_m$ with $A_i \neq \varnothing$ is said to be *left* (resp. *right*) if $x < y$ for some $x \in A_i$, $y \in A_j$ with $i \neq j$ implies that $i < j$ (resp. $i > j$). Both left and right sums are called *one-sided.* Finally, a one-sided sum S is called *minimax* if $x < y$ with x and y belonging to different summands implies that x is minimal and y maximal in S.

An element of a poset S is called *nodal* if it is comparable to any other element. The set of all nodal elements of S (which is a chain) is denoted by S_0. For connected posets S and T, we say that T is 0-*isomorphic* S and write $T \cong_0 S$ if $T \setminus T_0$ is isomorphic to $S \setminus S_0$ and $|T_0| = |S_0|$. When T is 0-isomorphic to S^{op}, we say that T is 0-*antiisomorphic* to S. By Proposition 1 the Tits quadratic forms of 0-isomorphic or 0-anti-isomorphic posets are $\mathbb{Z}$-equivalent.

3.2. Classification of positive posets

The Tits quadratic form is called *positive* if it takes a positive value on each nonzero vector $z \in \mathbb{Z}^{0 \cup S}$, and *nonpositive* otherwise. A poset is called *positive* (resp. *nonpositive*) if so is its Tits form.

3.2.1. The case of posets of width 2. All positive posets of width 2 were classified in the paper [3]. Note that the posets of width 1, which are positive (see [3, Section 3]), were excluded from consideration.

The positive posets of width 2 and order $n \geq 8$ are classified by Theorem 1 [3].

Theorem 1. *Let S be a poset of width* 2 *and order at least* 8*. Then the Tits quadratic form of S is positive if and only if one of the following condition holds:*

(1) *S is a direct sum of two chains;*
(2) *S is an almost chain;*
(3) *S is a one-sided minimax sum of two chains.*

Obviously, the sufficiency of the theorem is also true for orders smaller than 8.

The positive posets of width 2 and order smaller than 8 are classified modulo Theorem 1 (considered for all orders) by Theorem 3 [3]. They can only be of the order 5, 6 or 7.

Theorem 2. *Let S be a poset of width* 2 *and order $n < 8$ that are not of the forms* (1)*–*(3)*. Then the Tits quadratic form of S is positive if and only if S_0 is the union of lower and upper subsets, and S is* 0*-isomorphic or* 0*-antiisomorphic to one and only one of the following posets* (*which consist of the elements* $1, \dots, n$ *and is a right sum of chains* $\{1 \prec \dots \prec i\}$ *and* $\{i+1 \prec \dots \prec n\}$*, where* $i \leq n/2$)[2]*:*

[2]For formal reasons, R_{13} in this list is dual to R_{13} from Theorem 3 [3].

(A) *of order* 5

$R_1(P_1) = \{2 \prec 3 \prec 4 \prec 5, 1 \prec 4\}$,
$R_2(P_4) = \{1 \prec 2 \prec 5, 3 \prec 4 \prec 5\}$,
$R_3(P_3) = \{1 \prec 2, 3 \prec 4 \prec 5, 1 \prec 4\}$,
$R_4(P_5) = \{1 \prec 2 \prec 5, 3 \prec 4 \prec 5, 1 \prec 4\}$;

(B) *of order* 6

$R_5(P_8) = \{2 \prec 3 \prec 4 \prec 5 \prec 6, 1 \prec 5\}$,
$R_6(P_6) = \{2 \prec 3 \prec 4 \prec 5 \prec 6, 1 \prec 4\}$,
$R_7(P_{13}) = \{1 \prec 2 \prec 6, 3 \prec 4 \prec 5 \prec 6\}$,
$R_8(P_{11}) = \{1 \prec 2, 3 \prec 4 \prec 5 \prec 6, 1 \prec 5\}$,
$R_9(P_{10}) = \{1 \prec 2, 3 \prec 4 \prec 5 \prec 6, 1 \prec 4\}$,
$R_{10}(P_{18}) = \{1 \prec 2 \prec 6, 3 \prec 4 \prec 5 \prec 6, 1 \prec 5\}$,
$R_{11}(P_{16}) = \{1 \prec 2 \prec 6, 3 \prec 4 \prec 5 \prec 6, 1 \prec 4\}$,
$R_{12}(P_{14}) = \{1 \prec 2 \prec 5, 3 \prec 4 \prec 5 \prec 6, 1 \prec 4\}$,
$R_{13}(P_{12}) = \{1 \prec 2 \prec 3, 4 \prec 5 \prec 6, 1 \prec 5\}$,
$R_{14}(P_{20}) = \{1 \prec 2 \prec 3, 4 \prec 5 \prec 6, 1 \prec 5, 2 \prec 6\}$;

(C) *of order* 7

$R_{15}(P_{24}) = \{2 \prec 3 \prec 4 \prec 5 \prec 6 \prec 7, 1 \prec 6\}$,
$R_{16}(P_{21}) = \{2 \prec 3 \prec 4 \prec 5 \prec 6 \prec 7, 1 \prec 4\}$,
$R_{17}(P_{28}) = \{1 \prec 2 \prec 7, 3 \prec 4 \prec 5 \prec 6 \prec 7\}$,
$R_{18}(P_{27}) = \{1 \prec 2, 3 \prec 4 \prec 5 \prec 6 \prec 7, 1 \prec 6\}$,
$R_{19}(P_{26}) = \{1 \prec 2, 3 \prec 4 \prec 5 \prec 6 \prec 7, 1 \prec 4\}$,
$R_{20}(P_{40}) = \{1 \prec 2 \prec 7, 3 \prec 4 \prec 5 \prec 6 \prec 7, 1 \prec 6\}$,
$R_{21}(P_{38}) = \{1 \prec 2 \prec 7, 3 \prec 4 \prec 5 \prec 6 \prec 7, 1 \prec 5\}$,
$R_{22}(P_{36}) = \{1 \prec 2 \prec 7, 3 \prec 4 \prec 5 \prec 6 \prec 7, 1 \prec 4\}$,
$R_{23}(P_{33}) = \{1 \prec 2 \prec 6, 3 \prec 4 \prec 5 \prec 6 \prec 7, 1 \prec 4\}$,
$R_{24}(P_{31}) = \{1 \prec 2 \prec 5, 3 \prec 4 \prec 5 \prec 6 \prec 7, 1 \prec 4\}$,
$R_{25}(P_{30}) = \{1 \prec 2 \prec 3, 4 \prec 5 \prec 6 \prec 7, 2 \prec 7\}$,
$R_{26}(P_{29}) = \{1 \prec 2 \prec 3, 4 \prec 5 \prec 6 \prec 7, 1 \prec 6\}$,
$R_{27}(P_{43}) = \{1 \prec 2 \prec 3, 4 \prec 5 \prec 6 \prec 7, 1 \prec 6, 2 \prec 7\}$,
$R_{28}(P_{42}) = \{1 \prec 2 \prec 3, 4 \prec 5 \prec 6 \prec 7, 1 \prec 5, 2 \prec 7\}$,
$R_{29}(P_{44}) = \{1 \prec 2 \prec 3 \prec 7, 4 \prec 5 \prec 6 \prec 7, 1 \prec 6\}$,
$R_{30}(P_{45}) = \{1 \prec 2 \prec 3 \prec 7, 4 \prec 5 \prec 6 \prec 7, 1 \prec 5, 2 \prec 6\}$.

The posets of Theorem 2 are included in a general criteria of [4] under other numbers; they are indicated in the formulation of the theorem in brackets.

Note that these results were obtained in [3] with using (min, max)-equivalence but without the min-equivalence algorithm. The algorithm was obtained in the same year but later – namely in [4].

In view of the following statement the case of positive posets of width 2 plays an impotent role in the consideration of the general case.

Proposition 10. *Any positive poset is min-equivalent to one of width* 2.

Proof. It is obvious that a poset S of width $w \geq 4$ is not positive[3]. Let $w(S) = 3$, $a \in S$ be a maximal element and $a^{<} := \{x \in S \,|\, x < a\}$. Then the poset $S^{\uparrow}_{a^{<}}$ is equal to $\{a\} \coprod U$ with $U = S \setminus a^{<}$ of width 2. Therefore, $(S^{\uparrow}_{a^{<}})^{\uparrow}_{a}$ also has the width 2. □

3.2.2. The general case. Positive posets in the general case were first classified in [4].

Theorem 3. *The Tits quadratic form of a poset S is positive if and only if one of the following condition holds:*

(I) *S is a direct sum of two chains;*

(II) *S is a direct sum of a chain and almost chain;*

(III) *S is a one-sided minimax sum of two chains;*

(IV) *S is (min, max)-isomorphic or (min, max)-antiisomorphic to one of the poset P_1, P_5, P_6, P_{14}, P_{21}, P_{31}, P_{33}, P_{45} of Theorem 2.*

Note that a chain in (I) or (II) can be empty.

We first prove two lemmas.

Lemma 8. *Let S be a poset of the form $(n) = (1), (2)$ or (3) of Theorem 1 and T be (min, max)-equivalent to S. Then T is of the form $(N) =$ (I), (II) or (III) of Theorem 3. Moreover, any poset of the form (II) is (min, max)-equivalent to a poset of the form (2).*

Taking into account that the posets of the form (n) are also of the form (N), and the forms themselves are self-dual, it is enough for the proving of the first part to consider only the cases $T = S^{\uparrow}_{a}$ with a being minimal. It is easy to see that T is of the form (I) for $(n) = (1)$, and of the form (II) or (III) for $(n) = (2), (3)$. To prove the second part one need to take, for a poset T of the form (III), the poset $T^{\uparrow}_{A}$ with A being the chained summand.

Lemma 9. *Let a poset T be isomorphic to a poset S from Theorem 2. Then T is (min, max)-isomorphic or (min, max)-antiisomorphic to one of the poset P_1, P_5, P_6, P_{14}, P_{21}, P_{31}, P_{33}, P_{45}.*

[3] For the poset U which consists of four incomparable elements 1, 2, 3, and 4, $q_U(2, 1, 1, 1, 1) = 0$.

The proof follows from the following isomorphisms:
$(P_2)^{\uparrow\uparrow}_{11} \simeq P_1$, $(P_3)^{\downarrow}_{2} \simeq P_1$, $(P_4)^{\uparrow}_{1} \simeq P_3^{\mathrm{op}}$; $(P_7)^{\uparrow\uparrow}_{11} \simeq P_6$, $(P_8)^{\uparrow}_{2} \simeq P_6^{\mathrm{op}}$, $(P_9)^{\uparrow\uparrow}_{11} \simeq P_8$, $(P_{10})^{\downarrow}_{2} \simeq P_6$, $(P_{11})^{\downarrow}_{2} \simeq P_8$, $(P_{12})^{\downarrow}_{3} \simeq P_{11}$, $(P_{13})^{\uparrow}_{1} \simeq P_{12}^{\mathrm{op}}$; $(P_{15})^{\uparrow\uparrow}_{11} \simeq P_{14}$, $(P_{16})^{\downarrow\downarrow}_{66} \simeq P_{17}$, $(P_{17})^{\downarrow}_{3} \simeq P_{14}$, $(P_{18})^{\downarrow\downarrow}_{66} \simeq P_{19}$, $(P_{19})^{\downarrow}_{3} \simeq P_{16}$, $(P_{20})^{\downarrow}_{3} \simeq P_{18}$; $(P_{22})^{\uparrow\uparrow}_{11} \simeq P_{21}$, $(P_{23})^{\uparrow\uparrow}_{11} \simeq P_{22}$, $(P_{24})^{\uparrow}_{1} \simeq P_{21}^{\mathrm{op}}$, $(P_{25})^{\uparrow\uparrow}_{11} \simeq P_{24}$, $(P_{26})^{\uparrow\downarrow}_{12} \simeq P_{24}^{\mathrm{op}}$, $(P_{27})^{\downarrow}_{2} \simeq P_{24}$, $(P_{28})^{\uparrow}_{1} \simeq P_{26}^{\mathrm{op}}$, $(P_{29})^{\downarrow}_{3} \simeq P_{27}$, $(P_{30})^{\downarrow}_{3} \simeq P_{28}$;
$(P_{32})^{\uparrow\uparrow}_{11} \simeq P_{31}$, $(P_{36})^{\downarrow\downarrow}_{77} \simeq P_{37}$, $(P_{37})^{\downarrow}_{3} \simeq P_{31}$, $(P_{40})^{\uparrow\uparrow}_{12} \simeq P_{31}^{\mathrm{op}}$,
$(P_{41})^{\uparrow\uparrow}_{11} \simeq P_{40}$, $(P_{43})^{\downarrow}_{3} \simeq P_{40}$; $(P_{34})^{\uparrow\uparrow}_{11} \simeq P_{33}$, $(P_{35})^{\uparrow\uparrow}_{11} \simeq P_{34}$,
$(P_{38})^{\downarrow\downarrow}_{77} \simeq P_{39}$, $(P_{39})^{\downarrow}_{3} \simeq P_{33}$, $(P_{42})^{\downarrow}_{3} \simeq P_{38}$, $(P_{44})^{\uparrow}_{4} \simeq P_{42}^{\mathrm{op}}$.

We will use Corollary 1 by default. Then the sufficiency of Theorem 3 follows: for (I)–(III), from Theorem 1 (sufficiency) and Lemma 8 (the second part); for (IV), from Theorem 2 (sufficiency). The necessity of Theorem 3 follows from Proposition 10, Theorems 1 and 2 (necessity), Lemma 8 (the first part) and Lemma 9.

A positive poset S is called *serial* if there is an infinite increasing sequence $S \subset S^{(1)} \subset S^{(2)} \subset \dots$ with positive terms, and *nonserial* if otherwise.

The conditions (I)–(III) of Theorem 3 classify all the serial positive posets. The nonserial posets were classified in [4], using (IV) and the algorithm of min-equivalence (see Section 2). There are, up to isomorphism and antiisomorphism, 108 nonserial positive posets. A list of such posets is also published in more accessible paper [11].

3.3. Classification of P-critical posets

A minimal nonpositive poset is called P-critical. More precisely, a nonpositive poset is called P-*critical* if any its proper subposet is positive.

3.3.1. Kleiner critical posets. In the paper [14], M. M. Kleiner proved that a poset is not of finite representation type if and only if it does not contain, up to isomorphism, the following subposets:

$\mathcal{K}_1 = \{1, 2, 3, 4\}$ with pairwise incomparable elements;
$\mathcal{K}_2 = \{1 \prec 2, 3 \prec 4, 5 \prec 6\}$;
$\mathcal{K}_3 = \{1, 2 \prec 3 \prec 4, 5 \prec 6 \prec 7\}$;
$\mathcal{K}_4 = \{1, 2 \prec 3, 4 \prec 5 \prec 6 \prec 7 \prec 8\}$;
$\mathcal{K}_5 = \{1 \prec 2, 3 \prec 4, 1 \prec 4, 5 \prec 6 \prec 7 \prec 8\}$.

Posets of the form $\mathcal{K}_i$ are called *Kleiner critical* (or simply Kleiner) *posets.*

Lemma 10. *All Kleiner posets are P-critical.*

Proof. The Kleiner posets are not positive by the just mentioned result from [14] and the conditions (2p) of Introduction.

Further, it is easy to see that each proper subposet of $\mathcal{K}_i$ is either a direct sum of two chains, or a direct sum of a chain and an almost chain, or isomorphic to a subposet of one of the following posets:

$U = \{1, 2 \prec 3, 4 \prec 5 \prec 6 \prec 7\}$; $V = \{1 \prec 2 \prec 3, 4 \prec 5, 6 \prec 7, 4 \prec 7\}$.

In the first two case it need to use Theorem 3. The posets U and V are positive by Theorem 2 (and Corollary 1) since $U_1^{\uparrow}$ and $V_{\{1,2,3\}}^{\uparrow}$ are, respectively, of the form R_{17} and R_{24}. □

3.3.2. The main theorem. The main result on P-critical posets is given by the following theorem from [4].

Theorem 4. *A poset S is P-critical if and only if it is (min, max)-equivalent to a Kleiner critical poset.*

This theorem allows to classify all P-critical posets by applying the algorithm of min-equivalence (see section 2) to the posets $\mathcal{K}_1$–$\mathcal{K}_5$. A complete table of such posets, up to isomorphism and antiisomorphism, is writen in [4]; see also more accessible papers [10], [11]. The table consists of 75 P-critical posets.

3.3.3. Proof of Theorem 4. The main statement in the proof of Theorem 4 is the following.

Proposition 11. *If each poset which is* (min, max)*-equivalent to a fixed poset S does not contain any Kleiner critical poset, then the Tits quadratic form of S is positive.*

Let a be a maximal element of S, and $S' = S_{a^<}^{\uparrow}$ with $a^< := \{x \in S \mid x < a\}$. Then $S' = A \coprod B$, where $A = a$ and $B = S \setminus a$. Fix a poset $T \simeq_{(\min,\max)} S$ of the same form with a linear ordered A of the maximal order and some B. If $B = \varnothing$ or $w(B) = 1$, then T is positive by Theorem 1 and, therefore, so is S (see Corollary 1). The case $w(B) \geq 3$ is impossible since S and, therefore, T contains a poset of the form $\mathcal{K}_1$. Thus, we should consider the case $w(B) = 2$.

Lemma 11. *Let $w(B) = 2$. Then*

(a) *B has two minimal elements, for example b and c, and two maximal elements, for example f and g;*

(b) *there are incomparable minimal and maximal elements in B;*

(c) *if $b < x < f$ and $c < y < g$, then x and y are not comparable.*

Indeed, if B has only one minimal (resp. maximal) element, for example h, then $T' = T_h^{\uparrow}$ (resp. $T' = T_h^{\downarrow}$) is equal to $(B \setminus h) \coprod (A \cup h)$, and since $w(A \cup h) = 1$, this contradicts the choice of T. Further, if the condition (b) is not satisfied, then (taking into account the condition (a)) we have that the subset in $S_{bc}^{\uparrow\uparrow}$, consisting of the elements b, c, f, g, is of the form $\mathcal{K}_1$, what contradicts the condition of the proposition. Finally, if the condition (c) is not satisfied (but the conditions (a) and (b) are satisfied), then the subset in $(S_{\{x\}^{\leq}}^{\uparrow})_c^{\uparrow}$, consisting of elements b, x, y, g, f, c, is of the form $\mathcal{K}_2$ and again we come to a contradiction.

We continue the proof of the proposition. If B is a direct sum of two subsets of width 1, then T is positive by Theorem 3 and Lemma 10. So, by the last lemma only the following two cases are possible:

(1) $T = A \coprod B$, where $A = \{z_1 < z_2 \ldots < z_r\}$ and $B = \{x_1 < x_2 \ldots < x_p, y_1 < y_2 \ldots < y_q, x_i < y_j\}$ with $r > 0$ and either $i = 1, j > 1$ or $i < p, j = q$;

(2) $T = A \coprod B$, where $A = \{z_1 < z_2 \ldots < z_r\}$ and $B = \{x_1 < x_2 \ldots < x_p, y_1 < y_2 < \ldots < y_q, x_1 < y_j, x_i < y_q\}$ with $r > 0$, $1 < i < p$, $1 < j < q$.

Note that in case (1) $p, q > 1$ and in case (2) $p, q > 2$.

(A) Let us first focus on the case (1) for $i = 1, j = q$. We assume that $p \leq q$ (the case of $p \geq q$ is considered in the dual way). Then $r < 4$, otherwise T contains $\mathcal{K} \cong \mathcal{K}_5$.

If $r = 2, 3$, then $p = q = 2$, otherwise T contains $\mathcal{K} \cong \mathcal{K}_2$; and then, by Lemma 10, T is positive. If $r = 1$ and $q = 2, 3$ or $p = 2, q = 4$, the poset T is positive since so is $T_{z_1}^{\uparrow}$ by Theorem 2 (see the posets R_4, R_{10}, S_{20} and R_{29}). The rest cases are impossible: if $p > 2, q = 4$, then T contains $\mathcal{K} \cong \mathcal{K}_3$, and if $q > 4$, then $T_{y_q}^{\uparrow}$ contains $\mathcal{K} \cong \mathcal{K}_5$.

(B) Let us now consider the case (1) for $(i, j) \neq (1, q)$. We assume that $i \neq 1, j = q$ (the case $i = 1, j \neq q$ is viewed in the dual way). Then $q = 2$ since otherwise the subset in $T_{y_q}^{\downarrow}$, consisting of elements $y_q, z_1, x_1, x_2, y_1, y_2$, is of the form $\mathcal{K}_2$. And if $r = 1$, then $(T_{\{x_i\}^{\leq}}^{\uparrow})_{y_1}^{\uparrow}$ is of the form discussed in (A); if $r > 1$, then $i = p - 1$ (otherwise the subset in T, consisting of elements $z_1, z_2, x_{p-1}, x_p, y_1, y_2$, is of the form $\mathcal{K}_2$) and, consequently, $T_{y_2}^{\downarrow}$ is also of the form discussed in (A). So, it all comes down to the already considered case (A).

(C) Let us finally consider case (2). We assume that $p \leq q$ (the case $p \geq q$ is viewed in the dual way). Then $r = 1$ since otherwise the subset in T, consisting of elements $z_1, z_2, x_i, x_p, y_1, y_j$, has the form $\mathcal{K}_2$. Next, $p = 3$, otherwise $T \setminus \{x_1, y_q\}$ contains $\mathcal{K} \cong \mathcal{K}_3$. Then $q = 3$ and the poset

T is positive since so is $T^{\uparrow}_{z_1}$ by Theorem 2 (see the poset R_{30}).

Proposition 11 is proved.

Proof of Theorem 4. Corollary 1 and Proposition 6 (the equivalence of (1) and (2)) are used by default.

Sufficiency. Let S be (min, max)-equivalent to a Kleiner poset $\mathcal{K}$. Then by Lemma 10 it is not positive. From Lemma 2 it is follows that any proper subposet T of S is positive. Indeed, otherwise K has a proper subposet Q (min-equivalent to T) which is not positive. But this contradicts to the fact that $\mathcal{K}$ is P-critical (by Lemma 10). So S is P-critical.

Necessity. Let S be P-critical. Then it is not positive. $S' \simeq_{(min,max)} S$ that contains a Kleiner poset $\mathcal{K} \cong \mathcal{K}_i$. But then by Lemma 2 $S' = \mathcal{K}$ (otherwise S has a proper nonpositive subposet), and thus S is (min, max)-equivalent to the Kleiner critical poset $\mathcal{K}_i$.

4. New minimax algorithms: philosophy and descriptions

Let us denote by Min-$\mathbb{ALG}$ the algorithm of classifying all posets min-equivalent to a fixed poset (introduced in [4] and considered above in Subsection 2.2). The dual algorithm, i.e. one with respect to max-equivalence (see 1.3.3) is denoted by Max-$\mathbb{ALG}$.

In this section we introduce a series of algorithms of such type as Min-$\mathbb{ALG}$ and Max-$\mathbb{ALG}$, but which are (in some sense) more symmetrical. The basic among them are the two following, which are dual to one another and which are denoted by $M_\bullet\mathbb{ALG}$ and $M^\bullet\mathbb{ALG}$.

Step I for $M_\bullet\mathbb{ALG}$ (resp. $M^\bullet\mathbb{ALG}$).
To describe, up to strongly isomorphism, all lower (resp. upper) subsets $X \neq S$ in S, and for each of them to construct the poset $S^{\uparrow}_{X}$ (resp. $S^{\downarrow}_{X}$); $X = \varnothing$ is not excluded.

Step II for $M_\bullet\mathbb{ALG}$ and $M^\bullet\mathbb{ALG}$.
To describe, up to strongly isomorphism, all pairs (X, Y) consisting of a proper lower subset $X \subset S$ and a proper upper subset $Y \subset S$ such that $X < Y$; then for each such pair to build the poset $S^{\uparrow\,\downarrow}_{XY} (= S^{\downarrow\,\uparrow}_{YX})$.

Step III for $M_\bullet\mathbb{ALG}$ and $M^\bullet\mathbb{ALG}$.
To choose among the obtained in I and II posets one representative from each class of isomorphic posets.

The new algorithms $M_\bullet\mathbb{ALG}$ and $M^\bullet\mathbb{ALG}$ are equivalent to the "old" ones Min-$\mathbb{ALG}$ and Max-$\mathbb{ALG}$ in the sense that one can choose, up to isomorphism, the same representatives of the isomorphism classes of obtained posets. This follows from Lemma 4 (used in Step II).

Step II in the algorithms $M_\bullet\mathbb{ALG}$ and $M^\bullet\mathbb{ALG}$ (unlike Step II in Min-$\mathbb{ALG}$ and Max-$\mathbb{ALG}$) is symmetric with respect to lower and upper subposets.

Step I also can be modified to symmetric. For instance, one can take all lower subsets of order $0 \le s \le [n/2]$ and all upper subsets of order $1 \le s \le [n-1/2]$. This follows from Lemma 4.

Consider now two examples which illustrate other possibilities for modification of Step I–III in $M_\bullet\mathbb{ALG}$ (resp. $M^\bullet\mathbb{ALG}$).

Example 1. Let S be a self-dual poset of order n, and the posets in Step III are considered up to isomorphism and duality. Then in Step I one can take not all lower (resp. upper) posets X but only of order $0 \le s \le [n/2]$, and in Step II only with the condition $|X| \le |Y|$. This follows from the equality $S^{\uparrow}_X = (S^{\uparrow}_{S\setminus X})^{\mathrm{op}}$ which is true by Proposition 5.

Example 2. The notion of 0-isomorphism of connected posets (see Subsection 3.1) was generalized for all posets in [3]. An element of a poset S is called *local nodal*, if it is nodal in a direct summand of S. A bijective map between posets that preserve in both directions the comparability of elements is called 0-*isomorphism* if it induces an isomorphism between their largest subposets without local nodal elements. When a poset S is primitive (i.e., a direct sum of chains) and in Step III the posets are considered up to 0-isomorphism, Step II can be ignored since it does not give new posets. This follows from the main result of [2].

If we talk about the modification of Step III in general, one can write out the representatives up to not isomorphism but up to more powerful equivalence relations. Note that the classification of nonserial positive posets of width 2 [3] was obtained up to 0-isomorphism and 0-antiisomorphism, and of width 3 [4] up to isomorphism and antiisomorphism. It is obvious that in each of such cases one can easily to write out representatives of all classes of general isomorphism (see Remark 1 in 2.2).

Remark 2. The use of 0-isomorphism (as the strongest isomorphisms in our situation) together with 0-antiisomorphisms (or, equivalently, 0-antiisomorphisms and the duality – see again Remark 1) is justified by the smallest number of classes in Step III. For example, the number of nonserial positive posets (resp. P-critical posets) up to isomorphism and duality is equal to 108 (resp. 75), while up to 0-isomorphism and duality is equal to 88 (resp. 59).

By an MM-$\mathbb{ALG}$ we meant an algorithm which is a modification of $M_\bullet\mathbb{ALG}$ or $M^\bullet\mathbb{ALG}$) (including themselves) with respect to the steps

I–III in the sense which was discussed above.

Note that the orders of classes in Step III are small for each of our algorithms. The question of when each class consists of only one poset remains open even in the case when the posets in Step III are considered only up to usual isomorphism (then the classes consist of the smallest number of posets).

Despite the fact that all such algorithms are equivalent, the examples show that their diversity can be useful in exploring various specific problems.

In conclusion of this section we highlight one feature of all new algorithms. If the (min, max)-equivalences that occur in Steps I and II are in the initial terms (see Subsection 1.1), we obtained only sequences without repetitions. Recall that for the case of the algorithm Min-$\mathbb{ALG}$ or Max-$\mathbb{ALG}$, all the corresponding min-admissible or max-admissible sequences in Step II twice contain some elements. This feature can to create not only technical but also aesthetic advantages.

5. Minimax systems of generators

The classifications of the non-serial positive posets (see 3.2.2), P-critical posets (see 3.3), NP-critical posets (see [5] and [6]), etc., are carried out up to isomorphism and antiisomorphism according to the following scheme:

(1) it is determined some subset of a fixed class of posets (for example, for the class of all P-critical posets) such that any poset from this class is (min, max)-isomorphic or (min, max)-antiisomorphic to a poset from the chosen subset;

(2) to the posets of this set is applied the algorithm described in 3.3[4].

In this part of the paper the author introduces new notions that not only formalize this process but also lead to new ideas.

Let $\mathcal{K}$ be a class of finite posets closed under isomorphism and duality (or, equivalently, isomorphism and antiisomorphism), and let $U = \{U_i\}$ be a set of posets $U_i \in \mathcal{K}$ with i running through a (finite or infinite) set I. We say that U is a *minimax system of generators* of $\mathcal{K}$ if any $X \in \mathcal{K}$ is minimax isomorphic to a poset U_i for some $i \in I$. In the case when any proper subset of U is not a minimax system of generators, the system of generators U is said to be *minimal*.

[4] Other algorithms are discussed in details in the previous section.

Further, the set U we call *minimax d-system of generators* if any $X \in \mathcal{K}$ is minimax isomorphic to a poset U_i or U_i^{op} for some $i \in I$. A special case of such system is a system for which every poset U_i is self-dual (i.e. isomorphic to its dual). We call it *self-dual*.

Finally, a minimal minimax system (or d-system) $U = \{U_i\}$ of generators of $\mathcal{K}$ is called *semicanonical* if every U_i is the only in its minimax isomorphism class, which satisfies a fixed property α_i of posets. In the case when all α_i are the same, the system U is called *canonical*.

Let us give three example.

Example 3. The posets P_1, P_5, P_6, P_{14}, P_{21}, P_{31}, P_{33}, P_{45} from Lemma 9 form a minimal minimax d-system of generators for all non-serial positive posets.

Example 4. The Kleiner posets $\mathcal{K}_i$, $1 \le i \le 5$ (see 3.2.1) form a canonical self-dual minimax system of generators for all minimal nonpositive posets.

Example 5. Let n_i denote the order of the Kleiner poset $\mathcal{K}_i$ $(1 \le i \le 5)$. Put $\mathcal{N}_0 = \mathcal{K}_1 \coprod \{n_1+1\}$ and $\mathcal{N}_i = \mathcal{K}_i \cup \{n_i+1\}$ with $n_i \prec n_i+1$. We call $\mathcal{N}_j$ $(0 \le j \le 5)$ *Nazarova critical posets* (they are critical with respect to tameness of posets [15]). From the main result of the paper [5] it follows that these posets form a canonical self-dual minimax system of generators for all minimal nonnegative posets.

Remark 3. It should be emphasized that in mathematics, for a set of objects S with an equivalence relation R, a canonical form is defined as a choosing system of representatives of the equivalence classes. In practice, one tries to choose the canonical objects to be “the most simple”, but in general case this property is formally indefinable. The definitions of semi–canonical and canonical systems proposed by the author also are not entirely correct (since a set of valid properties is not specified), but even its intuitive perception is useful and leds to new ideas and results.

References

[1] Bondarenko, V.M.: On (min, max)-equivalence of posets and applications to the Tits forms. Bull. of Taras Shevchenko National University of Kyiv. Series: Physics and Mathematics. **1**, 24–25 (2005).

[2] Bondarenko, V.M.: Minimax isomorphism algorithm and primitive posets. Algebra Discrete Math. **12**(2), 31–37 (2011).

[3] Bondarenko, V.M., Styopochkina, M.V.: On posets of width two with positive Tits form. Algebra Discrete Math. **4**(2), 585–606 (2005).

[4] Bondarenko, V.M., Stepochkina, M.V.: (Min, max)-equivalence of partially ordered sets and the Tits quadratic form. Zb. Pr. Inst. Mat. NAN Ukr.: Problems of Analysis and Algebra. **2**(3), 18–58 (2005) (in Russian).

[5] Bondarenko, V.M., Stepochkina, M.V.: (Min, max)-equivalence of posets and nonnegative Tits forms. Ukr. Math. J. **60**(9), 1349–1359 (2008). https://doi.org/10.1007/s11253-009-0147-7

[6] Bondarenko, V.M., Styopochkina, M.V.: Description of posets critical with respect to the nonnegativity of the quadratic Tits form. Ukr. Math. J. **61**(5), 734–746 (2009). https://doi.org/10.1007/s11253-009-0245-6

[7] Bondarenko, V.M., Styopochkina, M.V.: Strengthening of a theorem on Coxeter-Euclidean type of principal partyally ordered sets. Bull. of Taras Shevchenko National University of Kyiv. Series: Physics and Mathematics. **61**(4), 8–15 (2018).

[8] Bondarenko, V.M., Styopochkina, M.V.: The classification of serial posets with the non-negative quadratic Tits form being principal. Algebra Discrete Math. **27**(2), 202–211 (2019).

[9] Bondarenko, V.M., Styopochkina, M.V.: On classifying the non-Tits P-critical posets. Algebra Discrete Math. **32**(2), 185–196 (2021). http://dx.doi.org/10.12958/adm1912

[10] Bondarenko, V.M., Styopochkina, M.V.: On the transitivity coefficients for minimal posets with nonpositive quadratic Tits form. Math. Sci. **274**(5), 583–593 (2023). https://doi.org/10.1007/s10958-023-06624-6

[11] Bondarenko, V.M., Styopochkina, M.V.: Combinatorial properties of non-serial posets with positive Tits quadratic form. Algebra Discrete Math. **36**(1), 1–13 (2023). http://dx.doi.org/10.12958/adm2151

[12] Drozd, Ju.A.: Coxeter transformations and representations of partially ordered sets. Funct. Anal. Appl. **8**(3), 219–225 (1974). https://doi.org/10.1007/BF01075695

[13] Gabriel, P.: Unzerlegbare Darstellungen I. Manuscripta Math. **6**, 71–103 (1972). https://doi.org/10.1007/BF01298413

[14] Kleiner, M.M.: Partially ordered sets of finite type. Zap. Nauch. Semin. LOMI. **28**, 32–41 (1972) (in Russian). English trans.: J. Soviet math. **3**, 607–615 (1975).

[15] Nazarova, L.A.: Partially ordered sets of infinite type. Izv. Akad. Nauk SSSR. Series: Mathematics. **39**(5), 963–991 (1975) (in Russian). English trans.: Math. of the USSR-Izvestiya. **9**(5), 911–938 (1975). https://doi.org/10.1070/im1975v009n05abeh001511

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