© Algebra and Discrete Mathematics RESEARCH ARTICLE
Volume **41** (2026). Number 1, pp. 110–119
DOI:10.12958/adm2331

A note on squarefree monomial ideals and matroid ports

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Communicated by I. Protasov

Abstract. A matroid port is a clutter determined by the circuits of a matroid that contain a fixed point. A monomial ideal associated to a (binary) matroid port can be characterized by using its minimal generators and its minimal prime ideals. In this note we point out this characterization and we demonstrate that any full-supported squarefree monomial ideal is the intersection of finitely many ideals associated to matroid ports. In the binary case, the matroid port decomposition is related to the irredundant primary decomposition of the ideal.

1. Introduction

Let K be a field and $R = K[x_1, \ldots, x_n]$ be the polynomial ring in n variables over K. The *support of a monomial* $m = \prod_{i=1}^{n} x_i^{a_i} \in R$ is the set $\operatorname{supp}(m) = \{x_i : a_i > 0\}$, and the *support of a monomial ideal* $J \subseteq R$ is $\operatorname{supp}(J) = \bigcup_{m \in G(J)} \operatorname{supp}(m)$ where $G(J)$ is the set of monomials

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The author wishes to express his deepest gratitude to Rosa Maria Cuevas for her invaluable support and unwavering encouragement. This work was partially supported by AGAUR, Generalitat de Catalunya, from the Catalan Government under project SGR-Cat 2021 00595 and by Universitat Politècnica de Catalunya under funds AGRUP-UPC.

2020 Mathematics Subject Classification: *13C99, 05B99.*

Key words and phrases: *monomial ideal, matroid port, primary decomposition.*

that minimally generate J. It is said that the ideal J is *full-supported* if $\mathrm{supp}(J) = \{x_1, \ldots, x_n\}$. (See [3] for topics on monomial ideals.)

Let $I \subseteq R$ be a squarefree monomial ideal. Let $m \in G(I)$ and let $\mathfrak{p} \in \mathrm{Min}(I)$ be a minimal prime of the ideal I. Denote by $i(m, \mathfrak{p})$ the size of the non-empty intersection $\sup(m) \cap \sup(\mathfrak{p})$. Observe that $i(m, \mathfrak{p}) \geqslant 1$ and that for all $\mathfrak{p} \in \mathrm{Min}(I)$ there exists $m \in G(I)$ such that $i(m, \mathfrak{p}) = 1$. We say that the ideal I has *odd intersection* if $i(m, \mathfrak{p})$ is odd for all $m \in G(I)$ and $\mathfrak{p} \in \mathrm{Min}(I)$. Otherwise we say that the ideal I has *even intersection*.

The full-supported ideals with odd intersection are related to *matroid ports*, a combinatorial object introduced by Lehman [2] to solve the Shannon switching game. In Section 2 we recall the definitions and properties about *clutters*, *matroids* and *matroid ports* that we will use, and we focus on the relation between ideals with odd intersection and matroid ports, (Proposition 1). It is worth noting that from a combinatorial point of view matroid ports play a key role in secret sharing schemes (see [8]).

The present note deals with full-supported ideals with odd intersection and with ideals associated to matroid ports, and explores how with these ideals it is possible to *decompose* any squarefree monomial ideal.

It is clear that the irredundant primary decomposition $I = \mathfrak{p}_1 \cap \ldots \cap \mathfrak{p}_s$ of a full-supported squarefree monomial ideal $I \subseteq R$ provides a decomposition of I as an intersection of ideals with odd intersection. However, the odd intersection ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$ are not full-supported (except for $\mathfrak{p}_i = (x_1, \ldots, x_n)$). So, a natural question that arises at this point is to determine whether it is possible to obtain an irredundant decomposition $I = J_1 \cap \cdots \cap J_s$ of I as an intersection of finitely many full-supported ideals $J_1, \ldots, J_s$ with odd intersection.

The goal of this note is to show that such a decomposition exists. Actually we present a more general result involving matroid ports. Namely we demonstrate that any full-supported squarefree monomial ideal I can be decomposed as an intersection $I = J_1 \cap \cdots \cap J_s$ of finitely many ideals $J_1, \ldots, J_s$ associated to ports of representable matroids over any field $\mathbb{K}$. In the case $\mathbb{K} = \mathbb{Z}_2$, the irredundant matroid port decomposition of the ideal I provides a decomposition of I as the intersection of full-supported ideals with odd intersection.

Our decomposition result is presented in Section 3 both for monomial ideals (Theorem 1) and clutters (Corollary 1). We also prove that there are no similar results if we replace odd intersection by even intersection, or if we replace matroid ports by matroids (other decompositions of clutters with matroids have been studied in [7]).

2. Matroid ports and ideals with odd intersection

As mentioned in the introduction, ideals with odd intersection are closely related to ports of matroids. In this section we point out this relation in Proposition 1. For completeness we recall the different concepts involved in this proposition. (We refer to [9] for general definitions about matroids.)

A *clutter* Δ on a finite set Ω is a collection of non-comparable subsets of Ω, that is, $\Delta = \{A_1, \ldots, A_r\}$ where $A_i \subseteq \Omega$ and $A_i \not\subseteq A_j$ if $i \neq j$. A clutter Δ on a set Ω is said to be *full-supported* if $\Omega = \bigcup_{A \in \Delta} A$. Clutters are also known as *antichains* or *Sperner systems*. Observe that if $I \subseteq K[x_1, \ldots, x_n]$ is a squarefree monomial ideal then, we can consider its associated clutter Δ_I on the finite set $\Omega = \{x_1, \ldots, x_n\}$ whose elements are the subsets corresponding to the support of the monomial that minimally generate I; that is, $\Delta_I = \{\sup(m) : m \in G(I)\}$. Clearly, I is full-supported if and only if Δ_I is full-supported.

A *matroid* is a combinatorial object that generalizes the properties of linear dependence among a finite set of vectors. There are many different equivalent definitions of a matroid. The one we use is based on the axioms of the *circuits*, that is, the *minimal dependent sets* of the matroid. Namely, a *matroid* $\mathcal{M}$ is an ordered pair $\mathcal{M} = (E(\mathcal{M}), \mathcal{C}(\mathcal{M}))$ consisting of a finite set $E(\mathcal{M})$, called the *ground set* of the matroid, and a clutter $\mathcal{C}(\mathcal{M})$ of nonempty subsets of $E(\mathcal{M})$ which satisfy the *weak circuit elimination property*: if C_1 and C_2 are distinct members of $\mathcal{C}(\mathcal{M})$ and $x \in C_1 \cap C_2$, then there is a member C_3 of $\mathcal{C}(\mathcal{M})$ such that $C_3 \subseteq (C_1 \cup C_2) \setminus \{x\}$. The members of the clutter $\mathcal{C}(\mathcal{M})$ are the *circuits* of the matroid $\mathcal{M}$, and the *dependent sets* of the matroid are the supersets of the circuits. A matroid $\mathcal{M}$ is called *representable* over a field $\mathbb{K}$ if there exists some matrix A with entries in the field $\mathbb{K}$ such that $\mathcal{M}$ is the *column matroid* of A; that is, the ground set of $\mathcal{M}$ are the columns of A and the dependent sets of $\mathcal{M}$ are those sets of columns of the matrix A that are linearly dependent as sets of vectors (so the circuits of $\mathcal{M}$ are the inclusion-minimal sets of columns of the matrix A that are linearly dependent as sets of vectors). A matroid $\mathcal{M}$ is called *binary* if it is representable over the finite field $\mathbb{Z}_2$. A matroid $\mathcal{M}$ is said to be *connected* if every two points $x, y \in E(\mathcal{M})$ lie in a common circuit; that is, there exists $C \in \mathcal{C}(\mathcal{M})$ with $x, y \in C$.

Given a matroid $\mathcal{M}$ and a point $w_0 \in E(\mathcal{M})$ in the ground set of $\mathcal{M}$, the *port of the matroid* $\mathcal{M}$ *at the point* w_0 is the clutter $\mathcal{M}_{w_0}$ on the set $E(\mathcal{M}) \setminus \{w_0\}$ defined by $\mathcal{M}_{w_0} = \{C \setminus \{w_0\} : C \in \mathcal{C}(\mathcal{M}) \text{ with } w_0 \in C\}$; that is, $\mathcal{M}_{w_0}$ is the clutter on $E(\mathcal{M}) \setminus \{w_0\}$ defined by the circuits of $\mathcal{M}$ containing the point w_0. From [2], a connected matroid $\mathcal{M}$ is determined by any of its ports $\mathcal{M}_{w_0}$. Clearly, all ports of a connected matroid are full-supported. Moreover, as a consequence of [9, Proposition 4.1.2], a matroid $\mathcal{M}$ is connected if and only if at least one of its ports $\mathcal{M}_{w_0}$ is a full-supported clutter (and in this case all ports of $\mathcal{M}$ are full-supported).

A full-supported clutter Δ on a finite set Ω is said to be a *matroid port* if it is the port of a matroid; that is, if there exists a matroid $\mathcal{M}$ with ground set $E(\mathcal{M}) = \Omega \cup \{w_0\}$, where $w_0 \notin \Omega$, such that $\Delta = \mathcal{M}_{w_0}$. Since Δ is full-supported, from the above it follows that if Δ is a matroid port then there exists a unique connected matroid $\mathcal{M}_\Delta$ with $\Delta = (\mathcal{M}_\Delta)_{w_0}$. An explicit description of the circuits of the matroid $\mathcal{M}_\Delta$ can be found in [2] or in [9, Theorem 4.3.2].

There are several characterizations of matroid ports which range from combinatorial properties and excluding minors [5, 11, 12] to independent sequences and bounds on the optimal information rate in secret-sharing schemes [8]. From the ones in [12] we get the following description of the ideals with odd intersection.

Proposition 1. *Let $I \subsetneq K[x_1, \dots, x_n]$ be a full-supported squarefree monomial ideal. Then I is an ideal with odd intersection if and only if the clutter Δ_I is the port of a binary matroid.*

Proof. Let Δ be a clutter on Ω. A blocking set of Δ is a subset of Ω that intersects every member of Δ. The clutter $b(\Delta)$ of all inclusion minimal blocking sets of Δ is the blocker of Δ. From the definition of blocker we get that if I is a squarefree monomial ideal then, $b(\Delta_I) = \{\sup(\mathfrak{p}) : \mathfrak{p} \in M(I)\}$ (in other words, $b(\Delta_I)$ is the clutter $\Delta_{I^\vee}$ associated to the Alexander dual $I^\vee$ of I). So the ideals with odd intersection are those associated to clutters Δ such that $|A \cap B|$ is odd for $A \in \Delta$ and for $B \in b(\Delta)$. From [12, Theorem, page 358], these clutters are the ports of binary matroids. $\square$

It is worth noting that this proposition provides an easy tool to construct ideals with odd intersection. Namely, given a binary matrix with

$n+1$ columns, the ports at each one of its columns determine monomial ideals with odd intersection.

Example 1. The ideal $I = (xy, xz, yt, zt) \subseteq K[x, y, z, t]$ has odd intersection because $\mathrm{Min}(I) = \{(x, t), (y, z)\}$. Observe that the associated clutter Δ_I is the port (at the first column) of the binary matroid defined by the matrix

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \end{pmatrix}.$$

Up to isomorphism, the ideal (xz, xt, y) is the odd intersection ideal corresponding to the ports at each one of the other four columns of the binary matrix A.

The relation between ideals with odd intersection and ports of matroids leads us to the following definitions.

Definition 1. Let $I \subseteq K[x_1, \ldots, x_n]$ be a full-supported squarefree monomial ideal. We say that the ideal I is a *matroid port ideal* if the clutter Δ_I is a matroid port and, if $\mathbb{K}$ is a field, we say that I is a $\mathbb{K}$-*representable matroid port ideal* if the clutter Δ_I is the port of a $\mathbb{K}$-representable matroid. Furthermore, we say that the ideal I is a *matroidal ideal* if the clutter Δ_I is the set of circuits of a matroid.

So, the odd intersection ideals with full support are the $\mathbb{Z}_2$-representable matroid port ideals. Monomial ideals associated to matroids and polymatroids have been studied by several authors (see for instance [1,4,10] for arithmetic properties of these ideals). The following remark is about matroid ports ideals and the unmixedness property of their associated matroidal ideals.

Remark 1. If $I \subseteq K[x_1, \ldots, x_n]$ is a matroid port ideal then there exists a unique matroidal ideal $I_0 \subseteq K[x_0, x_1, \ldots, x_n]$ such that I is the port of I_0 at x_0; that is, such that $m \in G(I)$ if and only if $x_0 m \in G(I_0)$. Since $\Delta_{I_0} = \mathcal{C}(\mathcal{M}_{\Delta_I})$, the ideal I_0 can be determined from the explicit description of the circuits of the matroid $\mathcal{M}_{\Delta_I}$. From [9, Corollary 2.1.17], the prime ideals of I_0 are exactly the prime ideals defined by the basis of the dual matroid $(\mathcal{M}_{\Delta_I})^*$. In particular, the ideal I_0 is height unmixed. For example, for the ideal $I = (xy, xz, yt, zt) \subseteq K[x, y, z, t]$ we have that $I_0 = (x_0xy, x_0xz, x_0yt, x_0zt, yz, xt) \subseteq K[x_0, x, y, z, t]$ which is unmixed of height three.

3. A decomposition result

It was already mentioned in the introduction that the irredundant primary decomposition $I=\mathfrak{p}_1\cap\cdots\cap\mathfrak{p}_s$ of a full-supported squarefree monomial ideal $I\subseteq K[x_1,\dots,x_n]$ provides a decomposition of the ideal as an intersection of ideals with odd intersection. However, if $I\neq(x_1,\dots,x_n)$ then the minimal primes $\mathfrak{p}_1,\dots,\mathfrak{p}_s$ are not full-supported. Our goal is to prove the existence of a decomposition of the ideal I with full-supported ideals with odd intersection. This is done in Theorem 1 within the more general framework of representable matroid ports ideals (recall that matroid ports are considered to be full-supported). Unlike what happens with the irredundant primary decomposition, we observe that in general the odd intersection decomposition of the ideal I is not unique (see Example 2). We will also show that it is not possible to obtain a general decomposition result by using ideals with even intersection (see Remark 2). To finish, in Corollary 1 we present our result in the context of clutters, and we stress that this corollary is similar but not equivalent to the results stated in [7] by using matroids. (General results concerning decomposition in lattices of clutters has been studied in [6].)

Theorem 1. *Let* $I\subseteq K[x_1,\dots,x_n]$ *be a full-supported and squarefree monomial ideal and let* $\mathbb{K}$ *be a field. Then, there are full-supported* $\mathbb{K}$*-representable matroid port ideals* $J_1,\dots,J_s\subseteq K[x_1,\dots,x_n]$ *such that* $I=J_1\cap\cdots\cap J_s$*.*

Proof. Set $\mathrm{Port}_{\mathbb{K}}(I)=\{J\subseteq K[x_1,\dots,x_n] : I\subseteq J$ and J is a $\mathbb{K}$-representable matroid port ideal$\}$. Observe that the clutter $\{\{x_1\},\dots,\{x_n\}\}$ is the port of the uniform matroid $\mathcal{U}_{1,n+1}$ which is representable over any field (see [9]). Therefore we get that $(x_1,\dots,x_n)\in\mathrm{Port}_{\mathbb{K}}(I)$, hence $\mathrm{Port}_{\mathbb{K}}(I)\neq\emptyset$, and thus we can consider the set $\{J_1,\dots,J_s\}\subseteq\mathrm{Port}_{\mathbb{K}}(I)$ of the inclusion minimal elements of $(\mathrm{Port}_{\mathbb{K}}(I),\subseteq)$. Since $I\subseteq J_1\cap\cdots\cap J_s$, the proof will be completed by showing that $J_1\cap\cdots\cap J_s\subseteq I$. We are going to prove it by contradiction.

Let us assume that $J_1\cap\cdots\cap J_s\not\subseteq I$.

Set $G(I)=\{m_1,\dots,m_r\}$. Since $J_1,\dots,J_s$ are squarefree monomial ideals, the intersection $J_1\cap\cdots\cap J_s$ is too. So, there exists a monomial $a\in G(J_1\cap\cdots\cap J_s)$ such that $a\notin(m_i)$ for $i=1,\dots,r$. Since $\prod_{j=1}^{n}x_j\in(m_i)$, hence $1\leqslant|\mathrm{supp}(a)|\leqslant n-1$. Without loss of generality we may assume that $\operatorname{supp}(a) = \{x_1, \ldots, x_t\}$; that is, we may assume that $a = \prod_{j=1}^{t} x_j$.

Now let us consider the full-supported and squarefree monomial ideal $J_a \subseteq K[x_1, \ldots, x_n]$ defined as follows: $J_a = (x_1x_n, \ldots, x_{n-1}x_n)$ if $t = n-1$, whereas $J_a = (x_1x_n, \ldots, x_tx_n, x_{t+1}, \ldots, x_{n-1})$ if $t < n-1$.

At this point observe that the ideal J_a is a $\mathbb{K}$-representable matroid port ideal. Namely it is not hard to check that the clutters defined by the ideals $(x_1x_n, \ldots, x_{n-1}x_n)$ and $(x_1x_n, \ldots, x_tx_n, x_{t+1}, \ldots, x_{n-1})$ are the ports at the first column of the $\mathbb{K}$-representables matroids defined by the matrices $A_1, A_2 \in \mathcal{M}_{2,n+1}(\mathbb{K})$ where

$$A_1 = \begin{pmatrix} 1 & 1 & \overset{n-1)}{\cdots\cdots} & 1 & 0 \\ 1 & 0 & \overset{n-1)}{\cdots\cdots} & 0 & 1 \end{pmatrix},$$

$$A_2 = \begin{pmatrix} 1 & 1 & \overset{t)}{\cdots\cdots} & 1 & 1 & \overset{n-t-1)}{\cdots\cdots} & 1 & 0 \\ 1 & 0 & \overset{t)}{\cdots\cdots} & 0 & 1 & \overset{n-t-1)}{\cdots\cdots} & 1 & 1 \end{pmatrix}.$$

Now let us prove that $I \subseteq J_a$; that is, we must demonstrate that $m_1, \ldots, m_r \in J_a$. Let $i \in \{1, \ldots, r\}$. Since $a \notin (m_i)$ then, there exists $j_i > t$ with $m_i \in (x_{j_i})$. So if $t < n-1$ we conclude that $m_i \in (x_{j_i}) \subseteq J_a$. Now, assume that $t = n-1$. In such a case $j_i = n$ and so $m_i \in (x_n)$. Observe that if $m_i = x_n$ then $I = (x_n)$ which is not possible because I is an ideal with full support. Therefore there exists $k_i \leqslant n-1$ with $m_i \in (x_{k_i}x_n)$. Hence $m_i \in (x_{k_i}x_n) \subseteq J_a$.

From the above we have that $J_a \in \operatorname{Port}_{\mathbb{K}}(I)$. Hence it follows that there exists a minimal element $J_0 \in \{J_1, \ldots, J_s\}$ such that $J_0 \subseteq J_a$. So we get that $J_1 \cap \cdots \cap J_s \subseteq J_0 \subseteq J_a$. Therefore we conclude that $a \in J_a$ (recall that $a \in G(J_1 \cap \cdots \cap J_m)$). In the case $t = n-1$ we have that $a = \prod_{j=1}^{n-1} x_j$ and that $J_a = (x_1x_n, \ldots, x_{n-1}x_n)$, while in the case $t < n-1$ we have that $a = \prod_{j=1}^{t} x_j$ and $J_a = (x_1x_n, \ldots, x_tx_n, x_{t+1}, \ldots, x_{n-1})$. In any case we get that $a \notin J_a$, a contradiction. □

By considering $\mathbb{K} = \mathbb{Z}_2$, the above theorem provides a decomposition of any full-supported squarefree monomial ideal $I \subseteq K[x_1, \ldots, x_n]$ as an intersection of full-supported ideals with odd intersection. Namely, given the ideal I, set $\mathcal{O}(I) = \{J \subseteq K[x_1, \ldots, x_n] : I \subseteq J$ and J is full-supported and has odd intersection$\}$. From the proof of Theorem 1 we get that $\mathcal{O}(I) \neq \emptyset$ and that $I = J_1 \cap \cdots \cap J_s$ where $\{J_1, \dots, J_s\} = \operatorname{Min}\big(\mathcal{O}(I)\big)$ are the inclusion minimal elements of $\mathcal{O}(I)$. It is clear that if $I = J'_1 \cap \cdots \cap J'_t$ is an irredundant decomposition with $J'_i \in \mathcal{O}(I)$, then there exists an irredundant decomposition $I = J_{i_1} \cap \cdots \cap J_{i_r}$ of I with $\{J_{i_1}, \dots, J_{i_r}\} \subseteq \operatorname{Min}\big(\mathcal{O}(I)\big)$ and with $1 \leqslant r \leqslant \min\{s, t\}$, (observe that if $r < s$ then $\{J_{i_1}, \dots, J_{i_r}\}$ determine I and so $\operatorname{Min}\big(\mathcal{O}(I)\big)$). The following example shows that, in general, the irredundant decomposition with minimal elements is not unique.

Example 2. In the polynomial ring $K[x, y, z, t]$, let us consider the ideal $I = (xyz, xyt, zt)$. This ideal has not odd intersection because $(z, t) \in \operatorname{Min}(I)$. By exploring all the squarefree monomials ideals J with $I \subsetneq J$ we conclude that $\operatorname{Min}\big(\mathcal{O}(I)\big) = \{J_1, J_2, J_3\} = \{(xyz, t), (xyt, z), (xy, zt)\}$. It is not difficult to check that $I = J_1 \cap J_2 \cap J_3 = J_1 \cap J_2 = J_1 \cap J_3 = J_2 \cap J_3$. So I has three different irredundant decompositions with minimal elements. (As mentioned above, observe that if $\{i, j, k\} = \{1, 2, 3\}$ then, the ideals J_i and J_j determine I and so J_k.)

Remark 2. It is worth mentioning that there is not a general result concerning decomposition with ideals with even intersection. Let us show some examples in three different situations. In the polynomial ring $K[x, y, z, t]$ let us consider the ideals $I_1 = (x, y, zt)$, $I_2 = (xt, yt, zt)$ and $I_3 = (xy, zt)$. Now set $\mathcal{E}(I_i) = \{J \subseteq k[x, y, z, t] \,:\, I_i \subseteq J$ and J is full-supported and has even intersection$\}$. As before, by exploring all the squarefree monomials ideals J with $I_i \subsetneq J$ we conclude that $\mathcal{E}(I_1) = \emptyset$ and that $\mathcal{E}(I_2) = \{(xy, xz, yz, t)\}$. So, for I_1 and I_2 there does not exist a decomposition by using ideals with even intersection. However, such a decomposition exists for I_3. Namely $I_3 = (xy, xz, zt) \cap (xy, yz, zt)$ and $(xy, xz, zt), (xy, yz, zt) \in \mathcal{E}(I_3)$ (in fact, it is not hard to check that (xy, xz, zt) and (xy, yz, zt) are two minimal elements of $\mathcal{E}(I_3)$).

To finish, in Corollary 1 we present our decomposition result in the framework of clutters. We stress that this corollary is similar to the results stated in [7] for matroids (in fact, by using the operation $\sqcap^+$ of [7] we have that $\min\{A_1 \cup \cdots \cup A_s \,:\, A_i \in \Delta_i\} = \Delta_1 \sqcap^+ \cdots \sqcap^+ \Delta_s$). However, here we consider clutters with full support, and without this restriction this result is not true for matroids (see Example 3).

Corollary 1. *Let Δ be a full-supported clutter on Ω. The following statements hold:*

1. *If* Δ *is not a matroid port, then there are full-supported matroid ports* $\Delta_1, \ldots, \Delta_s$ *on* Ω *such that* $\Delta = \min\{A_1 \cup \cdots \cup A_s \ : \ A_i \in \Delta_i\}$*.*
2. *If* Δ *is not a binary matroid port, then there are full-supported binary matroid ports* $\Delta_1, \ldots, \Delta_s$ *on* Ω *such that* $\Delta = \min\{A_1 \cup \cdots \cup A_s \ : \ A_i \in \Delta_i\}$*.*

Proof. Set $\Omega = \{x_1, \ldots, x_n\}$ and let $I \subseteq K[x_1, \ldots, x_n]$ be the squarefree monomial ideal with $\Delta_I = \Delta$. The two statements of the corollary follows by applying Theorem 1 to the ideal I and by considering $\Delta_i = \Delta_{J_i}$. □

Example 3. On the finite set $\Omega = \{x, y, z, t\}$ let us consider the clutter $\Delta = \{\{x, y, z\}, \{x, y, t\}\}$. This clutter does not satisfy the weak circuit elimination property. Therefore Δ is not the set of circuits of any matroid. We are going to show that there does not exist full-supported clutters $\Delta_1, \ldots, \Delta_s$ on Ω such that Δ_i is the set of circuits of a matroid and $\Delta = \min\{A_1 \cup \cdots \cup A_s \ : \ A_i \in \Delta_i\}$. Otherwise, let $I, J_1, \ldots, J_s$ be the squarefree monomial ideals with $\Delta_I = \Delta$ and $\Delta_{J_i} = \Delta_i$. Then $I = J_1 \cap \cdots \cap J_s$. Let $I_0 = (xyz, xyt, xzt, yzt)$ which is a matroidal ideal. It is clear that $I \subseteq I_0$ and $I_0 \subseteq J_i$ because J_i is a full-supported matroidal ideal. So $I = I_0$, a contradiction.

References

[1] Chiang-Hsieh, H.J.: Some arithmetic properties of matroidal ideals. Comm. Algebra **38**(3), 944–952 (2010). https://doi.org/10.1080/00927870902887120

[2] Lehman, A.: A solution of the Shannon switching game. J. Soc. Indust. Appl. Math. **12**(4), 687–725 (1964). https://doi.org/10.1137/0112059

[3] Herzog, J., Hibi, T.: Monomial ideals. Grad. Texts Math., vol. **260**. Springer, London (2011). https://doi.org/10.1007/978-0-85729-106-6

[4] Herzog, J., Hibi, T.: Cohen–Macaulay polymatroidal ideals. Eur. J. Comb. **27**(4), 513–517 (2006). https://doi.org/10.1016/j.ejc.2005.01.004

[5] Lehman, A.: Matroids and ports. Notices Amer. Math. Soc. **12**, 356–360 (1965)

[6] Martí-Farré, J., de Mier, A.: Approximation and decomposition in lattices of clutters. Discrete Math. **346**(8), 113445 (2023). https://doi.org/10.1016/j.disc.2023.113445

[7] Martí-Farré, J., de Mier, A.: Transformation and decomposition of clutters into matroids. Advances Math. **312**, 286–314 (2017). https://doi.org/10.1016/j.aim.2017.03.022

[8] Martí-Farré, J., Padró, C.: On secret sharing schemes, matroids and polymatroids. J. Math. Cryptol. **4**, 95–120 (2010). https://doi.org/10.1515/jmc.2010.004

[9] Oxley, J.G.: Matroid Theory. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York (1992)

[10] Saremi, H., Mafi, A.: Unmixedness and arithmetic properties of matroidal ideals. Arch. Math. **114**, 299–304 (2020). https://doi.org/10.1007/s00013-019-01402-w

[11] Seymour, P.D.: A forbidden minor characterization of matroid ports. Quart. J. Math. **27**(4), 407–413 (1976). https://doi.org/10.1093/qmath/27.4.407

[12] Seymour, P.D.: The forbidden minors of binary clutters. J. London Math. Soc. **12**(2), 356–360 (1976). https://doi.org/10.1112/jlms/s2-12.3.356

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Received by the editors: 19.08.2024
and in final form 19.01.2026.