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Extending properties of *z*-closed projection invariant submodules

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Abstract. In this article, we define a module M to be ZPG if and only if for each zp-submodule X of M there exists a direct summand D such that $X \cap D$ is essential in both X and D. We investigate structural properties of ZPG modules and locate the implications between the other extending properties. Our focus is the behavior of the ZPG modules with respect to direct sums and direct summands. We obtain the property is closed under right essential overring and rational hull.

Introduction

Throughout this paper, all rings are associative with unity and R stands for such a ring. All modules are unital right R-modules. For a module M, consider the following relations on the set of submodules of M: (i) $X\alpha Y$ if and only if there exists $A \leq M$ such that $X \leq_e A$ and $Y \leq_e A$; (ii) $X\beta Y$ if and only if $X \cap Y \leq_e X$ and $X \cap Y \leq_e Y$. Note that β is an equivalence relation and is equivalent to a relation defined in Goldie [5]. Recall that a module is called CS (or, *extending*) if every submodule is essential in a direct summand. Equivalently, every complement submodule is a direct summand (see [3, 13]). Many authors have studied various generalizations of CS-modules [1, 2, 7, 10, 11, 12]. In particular, a module M is called CLS if every z-closed submodule of M is a direct summand of M [12].

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Recall that a submodule N of M is called z-closed provided that M/N is nonsingular. A useful generalization of CS-modules, namely, Goldie extending modules, was investigated [1]. A module M is *Goldie extending* if and only if for every submodule X of M there exists a direct summand D of M such that $X \cap D$ is essential in both X and D. Following [10, 11] a module is called a C_{11}-module (or satisfies C_{11}) if every submodule of the module has a complement which is a direct summand. In addition, a module is called C_{11}^z-module (or satisfies C_{11}^z) if each z-closed submodule has a complement which is a direct summand [7]. Good sources for literature about CS-modules and a forementioned generalizations of extending modules are [3, 13]. To this end, a submodule N of M is called *projection invariant*, if $f(N) \subseteq N$ for all $f^2 = f \in End(M_R)$. Then a module M is called PI-extending if every projection invariant submodule is essential in a direct summand of M (see [2]).

This paper studies the behavior of z-closed projection invariant (zp) submodules of the condition G-extending. We obtain basic results about zp-submodules. We call M_R is a ZPG-module if for every zp-submodule X of M there exists a direct summand D of M such that $X \cap D$ is essential in both X and D. We show that a direct sum of ZPG-modules is also ZPG-module. Furthermore, we investigate conditions which provide the inheritance of ZPG-modules by direct summands. In the last section, we focus on right essential overring of a right ZPG ring enjoys with the ZPG property. We show that if M is ZPG, then $\widetilde{E}(M)$ is ZPG where $\widetilde{E}(M)$ is the rational hull of M.

Let R be a ring and M a right R-module. If $X \subseteq M$, then $X \leq M$, $X \leq_e M$, $Z(M)$, $E(M)$, $\widetilde{E}(M)$ and $End(M_R)$ denote X is a submodule of M, X is an essential submodule of M, the singular submodule of M, the injective hull of M, the rational hull of M, and the ring of endomorphisms of M, respectively. For R, $T_m(R)$ and $M_m(R)$ symbolize the ring of $m \times m$ upper triangular matrices over R and the ring of $m \times m$ full matrices over R, respectively. Other terminology and notation can be found in [3, 4, 6, 8, 9, 13].

1. Basic results

This section is devoted to the fundamental properties of the set of zp-submodules. We call M is ZPG-module if and only if for each zp-submodule X of M, there exists a direct summand D of M such that $X \cap D$

is essential both X and D. It is clear that *zp*-submodules are building bricks to the establishment of ZPG-modules. We begin by mentioning some basic facts projection invariant submodules of a module.

Lemma 1. *Let M be a module.*

(i) Any intersection of projection invariant submodules of M is a projection invariant submodule of M.

(ii) If $X \leq Y \leq M$ such that X is projection invariant in Y and Y is projection invariant in M, then X is projection invariant in M.

(iii) If $M = \bigoplus_{i \in J} X_i$ and S is a projection invariant submodule of M, then $S = \bigoplus_{i \in J} \pi_i(S) = \bigoplus_{i \in J} (X_i \cap S)$, where π_i is the i^{th} projection homomorphism of M.

Proof. It is straightforward to check. □

Lemma 2. *Let M_R be a module.*

(i) Any intersection of zp-submodules of M_R is a zp-submodule of M_R.

(ii) Let X, Y be submodules of M_R such that $X \leq Y$. If X is a zp-submodule of Y and Y is a zp-submodule of M_R, then X is a zp-submodule of M_R.

(iii) Let $M = M_1 \oplus M_2$, where M_1 is a zp-submodule of M_R. For any zp-submodule N of M_2, $M_1 \oplus N$ is a zp-submodule of M_R.

Proof. (i) Assume A, B are *zp*-submodules of M_R. Then A, B are projection invariant in M_R and $Z(M/A) = Z(M/B) = 0$. By Lemma 1, $A \cap B$ is projection invariant in M_R. Now, define the homomorphism $\alpha : M \to (M/A) \oplus (M/B)$ by $\alpha(m) = (m + A, m + B)$ where $m \in M_R$. It is clear that $M/(A \cap B) \cong \alpha(M) \leq (M/A) \oplus (M/B)$. Hence $Z(M/(A \cap B)) = 0$. So $A \cap B$ is a *zp*-submodule of M_R.

(ii) Let X be a *zp*-submodule of Y and Y be a *zp*-submodule of M_R. Then X be a projection invariant submodule of Y and Y be a projection invariant submodule of M_R. Also, we have that $Z(M/Y) = Z(Y/X) = 0$. By Lemma 1, X is projection invariant in M_R. Since, $M/Y \cong (M/X)/(Y/X)$, Y/X is z-closed submodule in M/X. Therefore

$Z(M/X) \leq Y/X$. Then $Z(Z(M/X) = Z(M/X) = (M/X) \cap (Y/X) = 0$. It follows that X is a zp-submodule of M_R.

(iii) By [13, Lemma 4.122], $M_1 \oplus N$ is a projection invariant submodule of M. Since $M/M_1 \oplus N \cong M_2/N$ and $Z(M_2/N) = 0$, $M_1 \oplus N$ is a z-closed submodule of M. Hence $M_1 \oplus N$ is a zp-submodule of M. □

Now, we locate the ZPG-module with respect to several known generalizations of the extending property.

Proposition 1. *Consider the following conditions of a module M_R:*

(i) M is G-extending;

(ii) M is C_{11};

(iii) M is PI-extending;

(iv) M is ZPG.

Then $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv)$, but these implications are not reversible, in general.

Proof. $(i) \Rightarrow (ii) \Rightarrow (iii)$ are clear.

$(iii) \Rightarrow (iv)$. Let X be a zp-submodule of M. There exists $c^2 = c \in End(M_R)$ such that $X \leq_e cM$. Therefore, M is a ZPG-module.

To see that $(ii) \not\Rightarrow (i)$, let $R = T_2(A)$ where A is a right Ore domain that is not a division ring. By [1, Proposition 1.6], R is a right C_{11}-module, but R_R is not a right G-extending module.

We conjecture that $(iii) \not\Rightarrow (ii)$ in Proposition 1; but, at this time, we have no example to support this conjecture.

$(iv) \not\Rightarrow (iii)$. Let F be any field and V_F be a vector space over F with $dim_F V \geq 2$. Let R be the trivial extension of F with V_F, i.e., $R = \begin{bmatrix} F & V \\ 0 & F \end{bmatrix} = \left\{ \begin{bmatrix} f & v \\ 0 & f \end{bmatrix} : f \in F, v \in V \right\}$. Thus R is an indecomposable R-module. Note that R is not uniform, hence R is not a PI-extending module. R has only three nontrivial submodules namely, $I_1 = \begin{bmatrix} 0 & v_1F \\ 0 & 0 \end{bmatrix}$, $I_2 = \begin{bmatrix} 0 & v_2F \\ 0 & 0 \end{bmatrix}$ and $I_3 = \begin{bmatrix} 0 & V \\ 0 & 0 \end{bmatrix}$ where $v_1, v_2 \in V$. Since $Z(R_R) \neq 0$, 0 is not z-closed ideal. On the other hand, I_3 is essential in R_R. Thus

R/I_3 is singular which gives that I_3 is not a z-closed submodule of R. Moreover $I_3^2 = 0$ so $I_3^2 \leq I_1$. However I_3 is not contained in I_1. Thus I_1 is not a z-closed submodule of R, similarly I_2 is not a z-closed submodule of R (see [13, Example 5.59]). It follows that R is the only z-closed submodule in itself. Since every submodule of an indecomposable module is projection invariant, R is a zp-submodule. Hence R_R is a ZPG-module. □

Corollary 1. *Let M be an indecomposable. If M is a ZPG-module then M is a C_{11}^z-module.*

Proof. Let $0 \neq X$ be a z-closed submodule of M. Since every submodule of an indecomposable module is projection invariant, X is a zp-submodule. There exists a direct summand D submodule of M, such that $X\beta D$. By [13, Lemma 5.58], X is a complement in M. So, $X = M$, Hence, M is a C_{11}^z-module. □

Proposition 2. *Let M be a module such that $End(M_R)$ is Abelian and X be a z-closed submodule of M implies $X = \sum_{i\in I} h_i(M)$, where $h_i \in End(M_R)$. Then M is a ZPG-module if and only if M is a CLS-module.*

Proof. Assume M is a ZPG-module and X is a z-closed submodule of M. There exists $e = e^2 \in End(M_R)$ such that $eX = e\sum_{i\in I} h_i(M) = \sum_{i\in I} h_i(eM) \subseteq X$. So X is a zp-closed submodule of M. By the hypothesis, $X\beta eM$. Then $X = eX \oplus (1-e)X$. Since X is a projection invariant submodule of M, by Lemma 1, $X = (X \cap eM) \oplus (X \cap (1-e)M)$. Hence $X \cap eM = eX$ is essential both in eM and X. Therefore, $X \cap (1-e)M = 0$. Thus $X = eX$. Since X is a z-closed submodule of M, $eM/X = Z(eM/X) \subseteq Z(M/X) = 0$ implies that $X = eM$. It follows that M is a CLS-module. For the converse, let M be a CLS-module and Y be a zp-submodule of M. Then by [13, Lemma 5.58], Y is a direct summand of M. Hence M is a ZPG-module. □

In next results we obtain characterizations of the ZPG-module for arbitrary module.

Proposition 3. *Let M be a module. The following condition are equivalent.*

(i) *M is a ZPG-module.*

(ii) *For each zp-submodule Y of M, there exists $X \leq M$ and a direct summand D of M such that $X \leq_e Y$ and $X \leq_e D$.*

(iii) *For each zp-submodule Y of M, there exists a complement L of Y and a complement K of L such that $Y\beta K$ and every homomorphism $f : K \oplus L \to M$ extends to a homomorphism $\overline{f} : M \to M$.*

Proof. $(i) \Rightarrow (ii)$. Let Y be a zp-submodule of M. Hence there exists a direct summand D of M such that $Y\beta D$. Now take $X = Y \cap D$.

$(ii) \Rightarrow (iii)$. From (ii), there exists D, $D^{'}$ such that $Y \cap D \leq_e Y$, $Y \cap D \leq_e D$ and $M = D \oplus D^{'}$. Take $D = K$ and $D^{'} = L$.

$(iii) \Rightarrow (i)$. Let Y be a zp-submodule of M. Then by [13, Lemma 5.58], Y is a complement in M. By [13, Lemma 3.97], K is a direct summand of M. Hence M is a ZPG-module. □

Theorem 1. *Let M be a module. The following condition are equivalent.*

(i) *M is ZPG-module.*

(ii) *For each zp-submodule X of M, there exists $e^2 = e \in End(E(M))$ such that $X\beta e(E(M))$ and $eM \leq M$.*

Proof. $(i) \Rightarrow (ii)$. Let X be a zp-submodule of M. Since M is a ZPG-module, there exists a direct summand D of M such that $X \cap D$ is essential in X and D. Assume $M = B \oplus D$ for some B is a complement of D. Then, $E(M) = E(B) \oplus E(D)$. Let $e : E(M) \to E(D)$ be the canonical projection. For $m \in M$, if $m = b + d$ with $b \in B$ and $d \in D$, then $e(m) = e(d) = d$. Hence $X \cap D \leq X \cap E(D) \leq X$ and $X \cap E(D)$ is essential in X. Since $X \cap D \leq_e D \leq_e E(D)$, $X \cap D$ is essential in $E(D)$. So $X\beta E(D) = X\beta e(E(M))$. In addition $eM \subset D \subseteq M$.

$(ii) \Rightarrow (i)$. Let X be a zp-submodule of M and $X\beta e(E(M))$ and $eM \subset M$ for some $e^2 = e \in End(E(M))$. Then there exists a direct summand D of M, $X \cap E(D)$ is essential in both X and $E(D)$. Since $X \cap D \leq_e X \cap E(D)$, $X \cap D$ is essential in both X and $E(D)$. So $X \cap D$ is essential in D. Hence $X\beta D$ and X is a ZPG-module. □

Proposition 4. *Let M be an indecomposable module. If M is a ZPG-module and X is a zp-submodule of M then M/X is a ZPG-module.*

Proof. Let Y/X be a zp-submodule of M/X. Since M is indecomposable and $M/Y \cong (M/X)/(Y/X)$, Y is a zp-submodule of M. By hypothesis, there exists $e^2 = e \in End(M_R)$ such that $Y\beta eM$. Since X is projection invariant in M, $X = (X \cap eM) \oplus (X \cap (1-e)M)$. Then $X \cap eM = X \cap (Y \cap eM) \leq_e X \cap Y = X$. So $X \leq eM$. Hence eM/X is a direct summand of M/X. Since $Y\beta eM$ and X is a zp-submodule of M, by [6, Proposition 1.4], we obtain $(Y/X)\beta(eM/X)$. Therefore, M/X is a ZPG-module. □

2. Decompositions

In this section, we deal with direct sums and summands of the class of ZPG-modules. We show that any finite direct sum of modules with ZPG is also a ZPG-module. To this end, we obtain several results on the inheritance of ZPG property on direct summands.

The next result shows that a finite direct sum of ZPG-modules is a ZPG-module.

Theorem 2. *Let $M = \bigoplus_{i=1}^{n} M_i$ for some submodules M_i of M $(1 \leq i \leq n)$. If each M_i is a ZPG-module, then M is a ZPG-module.*

Proof. We prove the result for $n = 2$ and then apply the induction argument on n. Let $M = M_1 \oplus M_2$ and Y be any zp-submodule of M. By Lemma 2, $Y \cap M_1$ is a zp-submodule of M_1 and $Y \cap M_2$ is a zp-submodule of M_2. Then there are direct summands D_1 of M_1 and D_2 of M_2 such that $(Y \cap M_1) \cap D_1 = Y \cap D_1 \leq_e D_1$ and $Y \cap D_1 \leq_e Y \cap M_1$ similarly $Y \cap D_2 \leq_e D_2$ and $Y \cap D_2 \leq_e Y \cap M_2$. So, $(Y \cap D_1) \oplus (Y \cap D_2) \leq Y \cap (D_1 \oplus D_2) \leq D_1 \oplus D_2$, $Y \cap (D_1 \oplus D_2) \leq_e D_1 \oplus D_2$. It follows that, $(Y \cap D_1) \oplus (Y \cap D_2) \leq Y \cap (D_1 \oplus D_2) \leq (Y \cap M_1) \oplus (Y \cap M_2) = Y$, then $Y \cap (D_1 \oplus D_2) \leq_e Y$. Since $D_1 \oplus D_2$ is a direct summand of M, M is a ZPG-module. □

Lemma 3. *Let M be a module and X be a zp-submodule of M. If M is a ZPG-extending then X is a ZPG-module.*

Proof. Assume S is a zp-submodule of X. By Lemma 2. S is a zp-submodule of M. Then there exists a direct summand D of M such that $S\beta D$. Let $\pi : M \to D$ be the projection endomorphism. Then $S \cap D = \pi(S \cap D) \leq \pi(X) \cap D = \pi(X)$. Hence $S \cap \pi(X)$ is essential in S and $\pi(X)$. So $S\beta\pi(X)$. By Lemma 1, $X = (D \cap X) \oplus (D^{'} \cap X)$ where $M = D \oplus D^{'}$. Thus $\pi(X) = D \cap X$. Therefore, $\pi(X)$ is a direct summand of X. Hence, X is a ZPG-module. □

Corollary 2. *Let $M = M_1 \oplus M_2$ be a direct sum of uniform modules M_1 and M_2. Then every direct summand of M is a ZPG-module.*

Proof. Let $0 \neq D$ be a direct summand of M. If $D = M$, then D is a ZPG-module by Proposition 1. If $D \neq M$, then D is uniform and hence it is a ZPG-module. □

Recall that the decomposition $M = X \oplus Y$ is said to be *exchangeable* if for any direct summand K, there exist $X' \leq X$ and $Y' \leq Y$such that $M = K \oplus X' \oplus Y'$ (see [9, Definition 4]).

Proposition 5. *If M is ZPG and the decomposition $M = M_1 \oplus M_2$ is exchangeable and M_1 is a zp-submodule of M, then M_1 and M_2 are ZPG-modules.*

Proof. By Theorem 2, M_1 is a ZPG-module. Let X be a zp-submodule of M_2. By Lemma 2(iii), $M_1 \oplus X$ is zp-submodule of M. Since M is ZPG-module, there exist a D direct summand of M and essential submodule Y of D such that $Y \leq_e M_1 \oplus X$ and $M_i' \leq M_i$ $(i = 1, 2)$. Since M is exchangeable, $M = D \oplus M_1' \oplus M_2'$. Therefore, $Y \cap M_1' = 0$ and $M_1' = 0$. So, $M_2 = M_2' \oplus (D \cap M_2)$. Now $Y \leq_e D$ and $Y \leq_e M_1 \oplus X$ yield that $Y \cap M_2 \leq_e D \cap M_2$ and $Y \cap M_2 \leq_e (M_1 \oplus X) \cap M_2 = X \oplus (M_1 \cap M_2) = X$. It follows that M_2 is ZPG-module. □

Before proving our main result on direct summands of a ZPG-module recall the following conditions (see [13]). A module M is said to have;

(i) the C_2 *property* if $X \leq M$ is isomorphic to a direct summand of M, then X is a direct summand of M;

(ii) the C_3 *property* if whenever M_1 and M_2 are direct summands of M such that $M_1 \cap M_2 = 0$, then $M_1 \oplus M_2$ is a direct summand of M.

Now we prove our main result on direct summand of a ZPG-module.

Proposition 6. *Let $M = M_1 \oplus M_2$ be a direct sum of modules M_1 and M_2 such that M_1 is a zp-submodule of M. If M is a ZPG-module which satisfies C_3 condition, then M_1 and M_2 are ZPG-modules.*

Proof. By Lemma 3, M_1 is a ZPG-module. Let X be any zp-submodule of M_2. By Lemma 2(iii), $M_1 \oplus X$ is a zp-submodule of M. By hypothesis, there exists a direct summand Y of M such that $(M_1 \oplus X) \cap Y$ is essential both $M_1 \oplus X$ and Y. Since M satisfies C_3 condition, $M_1 \oplus Y$ is a direct summand of M. Let $\pi : M_1 \to M_2$ be the canonical projection. So by [13, Lemma 2.71], $M_1 \oplus Y = M_1 \oplus \pi(Y)$. Then $\pi(Y)$ is a direct summand of M_2. For any, $0 \neq y \in \pi(Y)$, $y = \pi(x)$ for some $0 \neq x \in Y$. There exists a $r \in R$ such that $0 \neq xr \in (M_1 \oplus X) \cap Y$. So $xr = m_1 + x_1 = x_2$, where $x_1 \in X$, $m_1 \in M_1$ and $x_2 \in Y$. Hence, $0 \neq xr = \pi(xr) = x_1 = \pi(x_2) \in X \cap \pi(Y)$. It follows that $X \cap \pi(Y) \leq_e \pi(Y)$. Then, $\pi(Y) = M_2 \cap (M_1 \oplus \pi(Y)) = M_2 \cap (M_1 \oplus Y)$. Also, $X \cap \pi(Y) = X \cap (M_1 \oplus Y) \leq_e X$. Thus M_2 is a ZPG-module. □

Corollary 3. *Let $M = M_1 \oplus M_2$ be a direct sum of modules M_1 and M_2 such that M_1 is a zp-submodule of M. If M is a ZPG-module which satisfies C_2 condition, then M_1 and M_2 are ZPG-modules.*

Proof. Since C_2 condition implies the C_3 condition, the proof follows immediately by Proposition 6. □

Proposition 7. *Let M be a module and X is a zp-submodule of M. If M is ZPG, then there exists $e^2 = e \in End(M_R)$ such that $M = M_1 \oplus M_2$ and $X \leq_e M_2$.*

Proof. Assume that M is a ZPG-module and X an zp-submodule of M. Then there exists $e^2 = e \in End(M_R)$ such that $X\beta eM$. Now $X = eX \oplus (1-e)X$, $eX = X \cap eM$, and $(1-e)X = X \cap (1-e)M$ because X is projection invariant in M. Since $X\beta eM$, $eX \leq_e eM$ and $eX \leq_e X$. It follows that $X \cap (1-e)M = 0$. Thus $X = eX \leq_e eM$. Now, let $M_1 = (1-e)M$, and $M_2 = eM$. □

3. Extensions

In this section, we investigate essential extensions of modules or rings under the ZPG condition. We show that if a ring satisfies the right ZPG condition, then so is its essential overring. To this end, we obtain that the ZPG property is inherited by its rational hull. Recall that S is called a *right essential overring* of a ring R if S is an overring of R such that R_R is essential in S_R (see [8, 13]).

Theorem 3. *Let S be a right essential overring of R. If R_R is a ZPG-module then S_R and S_S are ZPG-module.*

Proof. Let Y_R be a zp-submodule of S_R and $X = Y \cap R$. By Proposition 3, there exists $K_R \leq R_R$ and $e^2 = e \in R$ such that K_R is essential in X_R and K_R is essential in eR_R. Then $K_R \leq_e X_R = Y \cap R \leq_e Y \cap S = Y_R$. So K_R is essential in Y_R. Let $0 \neq es \in eS$. Then there exists $r_1 \in R$ such that $0 \neq esr_1 \in eR$. So there exists $r_2 \in R$ such that $D \neq esr_1r_2 \in K$. Hence K_R is essential in eS_R. By Proposition 3, S_R is a ZPG-module. A similar proof to the above shows that KS_S is essential both in Y_S and eS_S. Thus S_S is a ZPG-module. □

Corollary 4. *Let $T = T_m(R)$ and $M = M_m(R)$. If T_T is ZPG, then M_T and M_M are ZPG.*

Proof. This result is a consequence of Theorem 3 and the fact that M_T is a rational extension of T_T. □

The next theorem shows that if a module satisfies ZPG property then so its rational hull. Recall that for a module M, the rational hull of M is defined as the following submodule of $E(M)$.

$$\widetilde{E}(M) = \{x \in E(M) : h(M) = 0 \Rightarrow h(x) = 0\}$$

for all $h \in End(M)$. Note that $E(M) \cong \widetilde{E}(M)$ whenever M is nonsingular (see [8]).

Theorem 4. *If M is a ZPG-module, then $\widetilde{E}(M)$ is a ZPG-module.*

Proof. Let K be a zp-submodule of $\widetilde{E}(M)$. Then by Lemma 1(ii), $X = K \cap M$ is a zp-submodule of M. By hypothesis, there exists $Y \leq M_R$ and $e^2 = e \in End(M_R)$ such that $Y \leq_e X$ and $Y \leq_e eM$. Notice that $Y \leq_e K$. By [8], there exists $f \in End(\widetilde{E}(M))$ such that $f|_M = e$. Since $E(M)$ is injective, there exists $\overline{e} \in End(\widetilde{E}(M))$ such that $\overline{e}|_{\widetilde{E}(M)} = f$.

Let $m \in M$. Then $[\overline{e} - \overline{e}^2](m) = (e - e^2)(m) = 0$. From the definition of $\widetilde{E}(M)$, $[\overline{e} - \overline{e}^2](y) = 0$ for all $y \in \widetilde{E}(M)$. Hence $f = f^2$. Assume $k \in K$ such that $fk - k \neq 0$. There exists $r \in R$ such that $0 \neq (fk - k)r$ and $kr \in M$. Then $kr \in X$. So $(fk - k)r = fkr - kr = ekr - kr = 0$, a contradiction. Hence $K \leq f\widetilde{E}(M)$. Let $0 \neq fr \in \widetilde{E}(M)$ There exists $s \in R$ such that $0 \neq frs$ and $rs \in M$. Then $0 \neq frst \in X \leq K$. Therefore, $K \leq_e f\widetilde{E}(M)$, so $\widetilde{E}(M)$ is a ZPG-module. $\square$

References

[1] Akalan, E., Birkenmeier, G.F., Tercan, A.: Goldie extending modules. Comm. Algebra **37**(2), 663–683 (2009). https://doi.org/10.1080/00927870802254843

[2] Birkenmeier, G.F., Tercan, A., Yücel, C.C.: The extending condition relative to sets of submodules. Comm. Algebra **42**(2), 764–778 (2014). https://doi.org/10.1080/00927872.2012.723084

[3] Birkenmeier, G.F., Park, J.K., Rizvi, S.T.: Extensions of Rings and Modules. Springer New York, NY (2013). https://doi.org/10.1007/978-0-387-92716-9

[4] Dung, N.V., Huynh, D.V., Smith, P.F., Wisbauer, R.: Extending modules. Pitman Research Notes in Mathematics Series, vol. **313**. Longman Scientific and Technical, Harlow (1994). https://doi.org/10.1201/9780203756331

[5] Goldie, A.W.: Semi-prime rings with maximum condition. Proc. London Math. Soc. **10**, 201–220 (1960)

[6] Goodearl, K.R.: Ring theory, pure and applied mathematics, vol. **33**. Marcel Dekker, Inc., New York (1976)

[7] Kara, Y., Tercan, A.: When some complement of a z-closed submodule is a summand. Comm. Algebra **46**(7), 3071–3078 (2018). https://doi.org/10.1080/00927872.2017.1404080

[8] Lam, T.Y.: Lectures on Modules and Rings. In: Graduate Texts in Mathematics, vol. **189**. Springer New York, NY (1999). https://doi.org/10.1007/978-1-4612-0525-8

[9] Mohamed, S.H., Müller, B.J.: Ojective modules. Comm. Algebra **30**(4), 1817–1827 (2002). https://doi.org/10.1081/AGB-120013218

[10] Smith, P.F., Tercan, A.: Generalizations of CS-modules. Comm. Algebra **21**(6), 1809–1847 (1993). https://doi.org/10.1080/00927879308824655

[11] Smith, P.F., Tercan, A.: Direct summands of modules which satisfy C_{11}. Algebra Collog. **11**, 231–237 (1993)

[12] Tercan, A.: On CLS-modules. Rocky Mount. J. Math. **25**(4), 1557–1564 (1995). https://doi.org/10.1216/rmjm/1181072161

[13] Tercan, A., Yücel, C.C.: Module Theory. Extending Modules and Generalizations. In: Frontiers in Mathematics. Birkhäuser Basel (2016). https://doi.org/10.1007/978-3-0348-0952-8

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