

On the Cartesian product of the Menger algebras of terms and relational formulas

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ABSTRACT. A relational formula which is a first-order formula that only uses relation symbols and terms of arbitrary type is one of the important concepts in the study of algebras and algebraic systems. In this paper, necessary and sufficient conditions for any element in a semigroup whose universe arises from the Cartesian product of the Menger algebras of terms and relational formulas to be idempotent and 2-potent are given. By the formula for counting the occurrence of all variables in a formula F , we further show that the order of such pairs is 1, 2, or infinite.

1. Introduction and preliminaries

An algebraic system of type (τ, τ') , a triplet comprising of a nonempty set A , a family of operations defined on A , and a family of relations on A , playing a key role in both mathematics and theoretical computer science aspects. In this matter, the type of algebraic systems is a pair (τ, τ') where τ is a sequence of the arity of each operation on A , which τ' is a sequence of the arity of each relation on A . Among recent contributions on algebraic systems are [6, 9, 12]. Basically, any order semigroup may be considered as an algebraic system of type $(2, 2)$. To study algebraic properties of algebraic systems, the notion of terms and formulas are essential. For details, one can refer the reader to the comprehensive

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monograph [2]. Let $X = \{x_1, x_2, \dots\}$ be a countably infinite set of symbols called variables. The type $\tau = (n_i)_{i \in I}$ is a sequence of each n_i -ary operation symbol f_i where $n_i \in \mathbb{N} := \{1, 2, \dots\}$. Formally, a term of type τ is defined as follows: (1) each variable $x \in X$ is a term of type τ and (2) $f_i(t_1, \dots, t_{n_i})$ is also a term of type τ if $t_1, \dots, t_{n_i}$ are terms of type τ previously. The set of all terms of type τ is denoted by $W_\tau(X)$, which is the smallest set containing X closed under finite application of (2). Moreover, $\text{var}(t)$ stands for the set of all variables in a term t . There are various directions for studying terms, for instance, see [1, 10, 14, 15]. Actually, terms also play an important role in the construction of free algebras, in particular in varieties of algebras with two, three, or more binary associative operations (see, e.g., [16–20]). Recall from [2] that a formula of type (τ, τ') constructed from terms, an equation symbol $\approx$, relation symbols, a negation $\neg$, a connector $\vee$ and a quantifier $\exists$, is defined in the following way:

- (1) If t_1, t_2 are terms of type τ , then the equation $t_1 \approx t_2$ is a formula of type (τ, τ') .
- (2) If $j \in J$ and $t_1, \dots, t_{n_j}$ are terms of type τ and γ_j is an n_j -ary relation symbol, then $\gamma_j(t_1, \dots, t_{n_j})$ is a formula of type (τ, τ') .
- (3) If F is a formula of type (τ, τ') , then $\neg F$ is a formula of type (τ, τ') .
- (4) If F_1 and F_2 are formulas of type (τ, τ') , then $F_1 \vee F_2$ is a formula of type (τ, τ') .
- (5) If F is a formula of type (τ, τ') and $x_i \in X_n$, then $\exists x_i(F)$ is a formula of type (τ, τ') .

Let $\mathcal{F}_{(\tau, \tau')}(W_\tau(X))$ be the set of all formulas of type (τ, τ') . Normally, we use $\text{var}(F)$ to denote the set of all variables that appears in a formula F . In particular, the formula of the forms (1) or (2) is called atomic. The symbol $F_{(\tau, \tau')}^*(X)$ stands for the set of all formulas of the form (2), and whose elements are called relational formulas. Note that the equation symbol $\approx$ in (1) differs from the relation symbol γ_j for all $j \in J$. Let see the following concrete example. Let $(\tau, \tau') = ((3), (2))$ be the type with a ternary operation symbol $+$ and a binary relation symbol ρ . Then the following are relational formulas in the set $F_{((3), (2))}^*(X)$: $\rho(x_3, x_2)$, $\rho(x_1, x_1)$ and $\rho(+ (x_5, + (x_2, x_4, x_3), x_1), + (x_4, x_6, x_8))$. For the study of algebras of formulas, we refer to [5, 7].

For $n \in \mathbb{N}$, applying the generalized superposition S^n on $W_\tau(X)$, i.e.,

the operation $S^n : W_\tau(X)^{n+1} \rightarrow W_\tau(X)$, which is defined by the following steps: for $t \in W_\tau(X)$

- (1) if t is a variable x_j , $1 \leq j \leq n$, then $S^n(t, t_1, \dots, t_n) = t_j$,
- (2) if t is a variable x_j from $X \setminus X_n$, then $S^n(t, t_1, \dots, t_n) = x_j$,
- (3) if $t = f_i(s_1, \dots, s_{n_i})$ for any $s_1, \dots, s_{n_i} \in W_\tau(X)$, then
 $S^n(t, t_1, \dots, t_n) = f_i(S^n(s_1, t_1, \dots, t_n), \dots, S^n(s_{n_i}, t_1, \dots, t_n))$,

the operation on the set $F_{(\tau, \tau')}^*(X)$ of all relational formulas is defined as a mapping

$$R^n : (W_\tau(X) \cup F_{(\tau, \tau')}^*(X)) \times (W_\tau(X))^n \rightarrow W_\tau(X) \cup F_{(\tau, \tau')}^*(X)$$

and is defined by

- (1) if $t \in W_\tau(X)$, then $R^n(t, s_1, \dots, s_n) := S^n(t, s_1, \dots, s_n)$,
- (2) if $\gamma_j(t_1, \dots, t_{n_j}) \in F_{(\tau, \tau')}^*(X)$, then $R^n(\gamma_j(t_1, \dots, t_{n_j}), s_1, \dots, s_n)$ is a relational formula $\gamma_j(R^n(t_1, s_1, \dots, s_n), \dots, R^n(t_{n_j}, s_1, \dots, s_n))$.

The semigroup whose universe set is the Cartesian product of the set of terms and relational formulas was introduced in [5]. To attain this, some notations are required. Let (τ, τ') be any type and let

$$W \times F_{(\tau, \tau')}^*(X) := \{(t, F) \mid t \in W_\tau(X), F \in F_{(\tau, \tau')}^*(X)\}.$$

Then for any $n \in \mathbb{N}$ the binary operation

$$+_n : (W \times F_{(\tau, \tau')}^*(X))^2 \rightarrow W \times F_{(\tau, \tau')}^*(X)$$

can be defined by

$$(t, F) +_n (t', F') = (S^n(t', t, \dots, t), R^n(F', t, \dots, t)).$$

It was mentioned in [5] that the operation $+_n$ is associative due to the superassociativity of S^n and R^n , see [2], i.e.,

$$\begin{aligned} & S^n(S^n(a, b_1, \dots, b_n), d_1, \dots, d_n) = \\ & S^n(a, S^n(b_1, d_1, \dots, d_n), \dots, S^n(b_n, d_1, \dots, d_n)), \\ & \text{and } R^n(R^n(F, b_1, \dots, b_n), d_1, \dots, d_n) = \\ & R^n(F, R^n(b_1, d_1, \dots, d_n), \dots, R^n(b_n, d_1, \dots, d_n)) \end{aligned}$$

for every $a, b_j, d_j \in W_\tau(X)$, $F \in F_{(\tau, \tau')}^*(W_\tau(X))$, $j = 1, \dots, n$, and thus the semigroup $(W \times F_{(\tau, \tau')}^*(X), +_n)$ is formed. In this way, the alge-

ras $(W_\tau(X), S^n)$ and $(F_{(\tau, \tau')}^*(X), R^n)$ are called the Menger algebras of terms and relational formulas, respectively. More backgrouds and developments in Menger algebras and algebras of multiplace functions are collected in the monograph [3]. Green's relations $\mathcal{L}, \mathcal{R}, \mathcal{J}, \mathcal{D}$ and $\mathcal{H}$ of this semigroup were studied in [5]. Moreover, the semigroups of terms and formulas under the generalized superposition also connect with the Menger algebras because any semigroup may be regarded as a particular class of Menger algebras. See the papers [4, 8, 11] for this topic.

The present study is an attempt to characterize the sets of idempotent and 2-potent elements in the semigroup $(W \times F_{(\tau, \tau')}^*(X), +_n)$ where $n \in \mathbb{N}$. Conditions for which the order of each pair $(t, F) \in W \times F_{(\tau, \tau')}^*(X)$ on a cyclic subsemigroup generated by (t, F) is finite or infinite are determined.

2. Main Results

This section begins with the definition of idempotent elements in the semigroup $(W \times F_{(\tau, \tau')}^*(X), +_n)$ for every natural number n . For every $n \in \mathbb{N}$, let E_{+_n} be the set of all idempotent elements in $(W \times F_{(\tau, \tau')}^*(X), +_n)$, i.e.,

$$E_{+_n} = \{(t, F) \in W \times F_{(\tau, \tau')}^*(X) \mid (t, F) +_n (t, F) = (t, F)\}.$$

Alternatively, by the definition of $+_n$, an element (t, F) in $W \times F_{(\tau, \tau')}^*(X)$ is called idempotent (in $(W \times F_{(\tau, \tau')}^*(X), +_n)$) if $t = S^n(S^n(t, t, \dots, t))$ and $F = R^n(F, t, \dots, t)$.

To provide a characterization of idempotent elements in the semigroup $(W \times F_{(\tau, \tau')}^*(X), +_n)$, we need a technical lemma.

Lemma 1. *For any $(t, F) \in W \times F_{(\tau, \tau')}^*(X)$, the following statements hold:*

- (1) $S^n(t, t, \dots, t) = t$ if and only if $t \in X$ or $t \in W_\tau(X) \setminus X$ such that $\text{var}(t) \cap X_n = \emptyset$;
- (2) $R^n(F, t, \dots, t) = F$ if and only if $\text{var}(F) \cap X_n = \emptyset$;
- (3) if $t \in X$ and $\text{var}(F) \cap X_n \neq \emptyset$, $(S^n(t, t, \dots, t), R^n(F, t, \dots, t)) \neq (t, F)$;
- (4) if $\text{var}(t) \cap X_n = \emptyset$ and $\text{var}(F) \cap X_n = \emptyset$, then (t, F) is right zero;
- (5) if $t \in X_n, t \notin \text{var}(F)$ and $\text{var}(F) \cap X_n \neq \emptyset$, then $(t, F) +_n (t, F) \neq (t, F)$;

- (6) if $t \in X_n, t \in \text{var}(F)$ and $\text{var}(F) \cap X_n \neq \emptyset$, then $(t, F) +_n (t, F) \neq (t, F)$;
- (7) if $t \in X \setminus X_n$ and $\text{var}(F) \cap X_n \neq \emptyset$, then $(t, F) +_n (t, F) \neq (t, F)$;
- (8) if $t \in W_\tau(X) \setminus X, \text{var}(t) \cap X_n \neq \emptyset$ and $\text{var}(F) \cap X_n \neq \emptyset$, then $(t, F) +_n (t, F) \neq (t, F)$;
- (9) if $t \in W_\tau(X) \setminus X, \text{var}(t) \cap X_n \neq \emptyset$ and $\text{var}(F) \cap X_n = \emptyset$, then $(t, F) +_n (t, F) \neq (t, F)$;
- (10) if $t \in W_\tau(X) \setminus X, \text{var}(t) \cap X_n = \emptyset$ and $\text{var}(F) \cap X_n \neq \emptyset$, then $(t, F) +_n (t, F) \neq (t, F)$;
- (11) if $t \in W_\tau(X) \setminus X$ and $\text{var}(F) \cap X_n \neq \emptyset$, then $(t, F) +_n (t, F) \neq (t, F)$.

Proof. To prove (1), suppose that $t \notin X$. Then t is a composed term of the form $f_i(t_1, \dots, t_{n_i})$. The assumption and a structure of a term t allow us to conclude that the sets of all variables in t and X_n are different. The opposite direction is clear. Next, assume that $\text{var}(F) \cap X_n = \emptyset$. It follows from the operation R^n that the equation $R^n(F, t, \dots, t) = F$ occurs. The converse is obvious. Moreover, the statement (3) follows directly from (1) and (2). Besides, (4) is a consequence of the definitions of S^n and R^n . Clearly, the proofs of (5) and (6) are obtained from a direct calculation. To prove (7), since $\text{var}(F) \cap X_n \neq \emptyset$ and $t = x_j$ for $j > n$, then by the definition of $+_n$, we have $R^n(F, t, \dots, t) \neq t$, which implies $(S^n(t, \dots, t), R^n(F, t, \dots, t)) \neq (t, F)$. From now on, let $t = f_i(t_1, \dots, t_{n_i})$. It is not hard to verify that $(t, F) +_n (t, F) \neq (t, F)$ if $\text{var}(t) \cap X_n \neq \emptyset$ and $\text{var}(F) \cap X_n \neq \emptyset$, which proves (8). Similarly, if $\text{var}(t) \cap X_n \neq \emptyset$ but $\text{var}(F) \cap X_n = \emptyset$, then by the definition of the generalized operation S^n and (1), the statement (9) holds. Again, by (2) and (3) we obtain the equality $(t, F) +_n (t, F) \neq (t, F)$ if $\text{var}(t) \cap X_n = \emptyset$ and $\text{var}(F) \cap X_n \neq \emptyset$. Thus, (10) holds. Finally, without loss of generality, we may assume that $x_i \in \text{var}(F) \cap X_n$ for some $i \in \{1, \dots, n\}$. Then $(t, F) +_n (t, F) = (S^n(t, t, \dots, t), R^n(F, t, \dots, t)) \neq (t, F)$ because a variable $x_i \in \text{var}(F)$ is substituted by a term $S^n(t, t, \dots, t)$. This is the proof of (11). $\square$

As a consequence, we prove:

Theorem 1. *Let (t, F) be any element in the semigroup $W \times F_{(\tau, \tau')}^*(X)$. Then (t, F) is idempotent (with respect to $+_n$) if and only if it satisfies one of the following conditions:*

- (1) $t \in X_n$ and $\text{var}(F) \subseteq \{t\} \cup (X \setminus X_n)$;

- (2) $t \in X \setminus X_n$ and $\text{var}(F) \cap X_n = \emptyset$;
 (3) $t \in W_\tau(X) \setminus X$ and $(\text{var}(t) \cup \text{var}(F)) \cap X_n = \emptyset$.

Proof. Assume first that (t, F) satisfies the condition (1). It is clear that (t, F) is idempotent. Since t is a variable in $X \setminus X_n$ and $\text{var}(F) \cap X_n = \emptyset$, we have $S^n(t, t, \dots, t) = t$ and $R^n(F, t, \dots, t) = F$, respectively. This means that $(t, F) +_n (t, F) = (t, F)$. Finally, assume that (t, F) satisfies the condition (3). Again by our assumption, i.e., $(\text{var}(t) \cup \text{var}(F)) \subseteq X \setminus X_n = \emptyset$ and $t \in W_\tau(X) \setminus X$, it is not hard to verify that $S^n(t, t, \dots, t) = t$ and $R^n(F, t, \dots, t) = F$. These imply the equation $(t, F) +_n (t, F) = (t, F)$ holds. Conversely, suppose that a pair (t, F) does not satisfy all conditions (1)–(3). This means that we have (t, F) from the set $W \times F_{(\tau, \tau')}^*(X)$ without conditions (1)–(3). As a result, we divide our proof in a few cases. The first case (t, F) satisfies $t \in X \setminus X_n$ and there exists $\{x_{k_1}, \dots, x_{k_p}\} \subseteq X_n \setminus \{t\}$ such that $t_{k_r} \in \text{var}(F)$ for some $r \in \{1, \dots, p\}$, applying Lemma 1(7), we conclude that (t, F) is not idempotent. Next, we consider when $t \in X \setminus X_n$ and $\text{var}(F) \cap X_n \neq \emptyset$. It follows from Lemma 1(7) that $(t, F) +_n (t, F) \neq (t, F)$. It is a contradiction. Finally, we consider t is a composed term $f_i(t_1, \dots, t_{n_i})$, i.e., $t \in W_\tau(X) \setminus X$. If $\text{var}(t) \cap X_n \neq \emptyset$ and $\text{var}(F) \cap X_n \neq \emptyset$, then by Lemma 1(8), we obtain that $(t, F)^2 \neq (t, F)$. Actually, by a routine calculation, (t, F) is not idempotent if $\text{var}(t) \cap X_n \neq \emptyset$ and $\text{var}(F) \cap X_n = \emptyset$. We finish the proof in the case when $\text{var}(t) \subseteq X \setminus X_n$ and $\text{var}(F) \cap X_n \neq \emptyset$. Applying Lemma 1(10), thus (t, F) is not idempotent. $\square$

We now consider the following sets:

$$\begin{aligned} E_{+,n,1} &:= \{(t, F) \in W \times F_{(\tau, \tau')}^*(X) \mid t \in X_n \text{ and } \text{var}(F) \subseteq \{t\} \cup (X \setminus X_n)\}, \\ E_{+,n,2} &:= \{(t, F) \in W \times F_{(\tau, \tau')}^*(X) \mid t \in X \setminus X_n \text{ and } \text{var}(F) \cap X_n = \emptyset\}, \\ E_{+,n,3} &:= \{(t, F) \in W \times F_{(\tau, \tau')}^*(X) \mid t \in W_\tau(X) \setminus X \text{ and } (\text{var}(t) \cup \text{var}(F)) \cap X_n = \emptyset\}. \end{aligned}$$

Corollary 1. *The following assertions hold:*

- (1) $E_{+n} = E_{+,n,1} \cup E_{+,n,2} \cup E_{+,n,3}$;
 (2) the set E_{+n} is a subsemigroup of $(W \times F_{(\tau, \tau')}^*(X), +_n)$.

Proof. It follows from Theorem 1 that (1) is obtained. Clearly, $\emptyset \neq E_{+n} \subseteq W \times F_{(\tau, \tau')}^*(X)$. To prove (2), let (t_1, F_1) and (t_2, F_2) be two pairs in E_{+n} . We separate our consideration into a few cases. We first consider when $(t_1, F_1) \in E_{+,n,1}$ and $(t_2, F_2) \in E_{+n} \setminus E_{+,n,1}$. If $(t_2, F_2) \in E_{+,n,2}$,

we have $(t_1, F_1) +_n (t_2, F_2) \in E_{+n,2} \subset E_{+n}$. Assume that $(t_2, F_2) \in E_{+n,3}$. From the definitions of generalized operations S^n and R^n and Lemma 1(4), we have that $(t_1, F_1) +_n (t_2, F_2)$ is equal to (t_2, F_2) because $(\text{var}(t_2) \cup \text{var}(F_2)) \cap X_n = \emptyset$. Secondly, the case $(t_1, F_1) \in E_{+n,2}$ and $(t_2, F_2) \in E_{+n} \setminus E_{+n,2}$ is proved. If $(t_2, F_2) \in E_{+n,1}$, and $\text{var}(F) = \{t\}$, then $(t_1, F_1) +_n (t_2, F_2) \in E_{+n,2}$. If $(t_2, F_2) \in E_{+n,1}$, and $\text{var}(F) \subseteq X \setminus X_n$, then $(t_1, F_1) +_n (t_2, F_2) \in E_{+n,2}$. Similarly, if there exists $t \in \text{var}(F)$, then $(t_1, F_1) +_n (t_2, F_2)$ is also contains in $E_{+n,2}$. Assume that $(t_2, F_2) \in E_{+n,3}$. Since there is no any replacement for t_2 and F_2 , then by Lemma 1(4), we get $(t_1, F_1) +_n (t_2, F_2) \in E_{+n,3}$. Finally, suppose that $(t_1, F_1) \in E_{+n,3}$ and $(t_2, F_2) \in E_{+n} \setminus E_{+n,3}$. If $(t_2, F_2) \in E_{+n,1}$ and $\text{var}(F) = \{t\}$, then $(t_1, F_1) +_n (t_2, F_2) \in E_{+n,3}$. If $(t_2, F_2) \in E_{+n,1}$ and $\text{var}(F) \subseteq X \setminus X_n$, then $(t_1, F_1) +_n (t_2, F_2) \in E_{+n,3}$. If $(t_2, F_2) \in E_{+n,1}$ and there exists $t \in \text{var}(F)$, then $(t_1, F_1) +_n (t_2, F_2) \in E_{+n,3}$. If $(t_2, F_2) \in E_{+n,2}$, then $(t_1, F_1) +_n (t_2, F_2) \in E_{+n,2}$. Thus, in all cases, the set E_{+n} is a subsemigroup of $W \times F_{(\tau, \tau')}^*(X)$ (under $+_n$). $\square$

Example 1. On the Cartesian product $W \times F_{((2),(3))}^*(X)$, which equals the set $\{(t, F) \mid t \in W_{(2)}(X), F \in F_{((2),(3))}^*(X)\}$, then the following are some examples of idempotent elements in that set (with respect to $+_2$):

$$(x_2, \rho(x_2, x_2, x_2)), (x_1, \rho(x_1, f(x_4, x_1), x_5)), (x_2, \rho(f(x_3, x_3), x_4, x_5)), \\ (x_4, \rho(x_3, x_4, x_5)), (f(f(x_4, x_3), x_5), \rho(f(x_6, x_6), x_4, x_3)).$$

On the other hand, some examples of pairs in $W \times F_{((2),(3))}^*(X)$ which are not idempotent with respect to $+_2$ are listed as follows:

$$(x_1, \rho(x_2, x_4, x_2)), (x_1, \rho(x_1, x_2, x_5)), (x_3, \rho(x_2, x_2, x_2)), (x_5, \rho(x_1, x_3, x_3)), \\ (f(x_1, x_3), \rho(x_2, x_2, x_2)), (f(x_5, f(x_7, x_7)), \rho(f(x_1, x_4), x_2, x_6)).$$

It is interesting to note from Corollary 1 and Propositions 2.1 and 2.3 in the paper [5] that every pair (t, F) in the set E_{+n} is a left identity for an $\mathcal{R}$ -class $R_{(t,F)}$ containing (t, F) and a right identity for an $\mathcal{L}$ -class $L_{(t,F)}$ containing (t, F) .

Recall that the index and period of an element a of any given semi-group S are the smallest values of $m \in \mathbb{N}$ and $r \in \mathbb{N}$ such that $a^{m+r} = a^m$. An element $a \in S$ is called m -potent element if whose index is m and period is 1, i.e., $a^{m+1} = a$ and $a^{k+1} \neq a^k$ for all $k < m$. Particularly, we remark that a is idempotent if and only if it is 1-potent.

It is known that the set E_{+n} of all idempotent elements, i.e., 1-potent (in $(W \times F_{(\tau, \tau')}^*(X), +_n)$) is completely described in Corollary 1. Thus, it

is possible to characterize m -potent elements where $m = 2$. The following theorem provides a characterization for any pair in $(W \times F_{(\tau, \tau')}^*(X), +_n)$ to be 2-potent.

Theorem 2. *Let (t, F) be any element in the semigroup $W \times F_{(\tau, \tau')}^*(X)$ with respect to $+_n$. Then (t, F) is 2-potent if and only if $t \in X$ and $\text{var}(F) \cap X_n \neq \emptyset$.*

Proof. Assume that (t, F) satisfies $t \in X$ and $\text{var}(F) \cap X_n \neq \emptyset$. Particularly, we have the following two cases. If t is a variable in X and $\text{var}(F) \subseteq X_n$, then we have $(t, F)^2 = (S^n(t, t, \dots, t), R^n(F, t, \dots, t)) = (t, Q)$ where $\text{var}(Q) = \text{var}(t)$. From this $(t, Q) +_n (t, F) = (t, F)$ from the definition of $+_n$ and $\text{var}(F) \subseteq X_n$. Therefore, $(t, F)^3 = (t, F)^2$ and a couple (t, F) and $(t, F)^2$ are different. Secondly, we consider $t \in X_n$ and $\text{var}(F) \cap X_n \neq \emptyset$. Thus, $(t, F)^2 = (S^n(t, t, \dots, t), R^n(F, t, \dots, t)) = (t, Q)$ where $\text{var}(Q) = \text{var}(t) \cup \text{var}(F)$. Continuously, $(t, Q) +_n (t, F) = (t, F)$ because it does not depend on Q but rely on the definition of $+_n$. As a result, $(t, F)^3 = (t, F)^2$ and both (t, F) and $(t, F)^2$ are distinct. From these two cases, (t, F) is 2-potent (with respect to $+_n$). For the opposite direction, suppose towards a contradiction that $t \notin X$ or $\text{var}(F) \cap X_n = \emptyset$. In first case, i.e., $t \notin X$, then by Lemma 1(1), we have $S^n(t, t, \dots, t) \notin t$. By a calculation with $+_n$, we obtain $(t, F)^3 \neq (S^n(t, t, \dots, t), R^n(F, t, \dots, t)) = (t, F)^2$, which shows $(t, F)^3 \neq (t, F)^2$. It is contradicts to the fact that (t, F) is 2-potent. If $\text{var}(F) \cap X_n = \emptyset$, it follows from Lemma 1(2) that $R^n(F, t, \dots, t) = F$. From this, we obtain $(t, F)^3 \neq (t, F)^2$. Indeed, $(t, F) +_n (t, F) +_n (t, F) = (S^n(S^n(t, t, \dots, t), t, \dots, t), R^n(F, t, \dots, t)) = (S^n(S^n(t, t, \dots, t), t, \dots, t), F) \neq (t, F)^2$, which contradicts to the fact that (t, F) is 2-potent, especially, $(t, F)^3 = (t, F)^2$. This completes the proof. $\square$

Example 2. The following are some examples of 2-potent elements (under the operation $+_2$) in $W \times F_{((2), (2))}^*(X)$ where f is a binary operation symbol and α is a binary relation symbol: $(x_1, \alpha(x_1, x_2)), (x_4, \alpha(x_2, x_1)), (x_2, \alpha(x_1, x_3)), (x_4, \alpha(f(x_2, x_3), f(x_5, x_1)))$.

However, there are many pairs in $W \times F_{((2), (2))}^*(X)$ which are not 2-potent with respect to $+_2$. Some of them are now revealed:

$$(x_3, \alpha(x_4, x_3)), (x_3, \alpha(f(x_3, x_5), x_5)), (f(x_1, x_1), \alpha(x_2, x_2)).$$

On the Cartesian product $W \times F_{(\tau, \tau')}^*(X)$ with respect to $+_n$, the set of all 2-potent elements is denoted by $2 - \text{potent}_{+_n}$.

Theorem 3. *The following assertions hold:*

- (1) $2\text{-potent}_{+n} = \{(t, F) \in W \times F_{(\tau, \tau')}^*(X) \mid t \in X \text{ and } \text{var}(F) \cap X_n \neq \emptyset\}$;
 (2) 2-potent_{+n} forms a subsemigroup of $(W \times F_{(\tau, \tau')}^*(X), +_n)$.

Proof. By Theorem 2, we have that (t, F) is 2-potent if and only if $t \in X$ and $\text{var}(F) \cap X_n \neq \emptyset$, which proves the first part. It remains to show that 2-potent_{+n} is closed under $+_n$. Let (t, F) and (s, Q) be elements in 2-potent_{+n} . Then t and s are variables from X and $\text{var}(F) \cap X_n \neq \emptyset$ and $\text{var}(Q) \cap X_n \neq \emptyset$. Thus $(t, F) +_n (s, Q) = (S^n(s, t, \dots, t), R^n(Q, t, \dots, t))$, which implies that a term $S^n(s, t, \dots, t)$ is a variable in X and $\text{var}(R^n(Q, t, \dots, t)) \cap X_n \neq \emptyset$. Hence, $(t, F) +_n (s, Q)$ belongs to 2-potent_{+n} . Hence, 2-potent_{+n} is a subsemigroup of $W \times F_{(\tau, \tau')}^*(X)$ (under $+_n$). $\square$

Corollary 2. *Let $m \in \mathbb{N}$ be fixed. If $m \leq 2$, the set $m\text{-potent}_{+n}$ of all m -potent elements in the semigroup $(W \times F_{(\tau, \tau')}^*(X), +_n)$ is non-empty.*

Proof. Let m be a natural number. Suppose that $m \leq 2$. If $m = 1$, then by Theorem 1 we have that $1\text{-potent}_{+n} = E_{+n} \neq \emptyset$. Applying Corollary 3, we obtain $2\text{-potent}_{+n} \neq \emptyset$. $\square$

In closing this paper we briefly examine some of the consequences of Corollary 1, Theorem 2 and Corollary 2. Recall that the order of an element a in any semigroup S is defined as the cardinality of $\langle a \rangle = \{a, a^2, \dots\}$, a cyclic subsemigroup of S generated by a .

To obtain description of the order of each element in the semigroup $(W \times F_{(\tau, \tau')}^*(X), +_n)$, some preparations are needed.

Recall from [13] that the variables count of a term t is the total number of occurring variables in t , and is denoted by $\text{vb}(t)$. If t is a variable, then $\text{vb}(t) = 1$ and if $t = f_i(t_1, \dots, t_{n_i})$, then $\text{vb}(t) = \sum_{j=1}^{n_i} \text{vb}(t_j)$.

The x_k -variable count of t for $k \in \{1, \dots, n\}$, denoted by $\text{vb}_k(t)$, is defined by $\text{vb}_k(x_k) = 1$; if t is a variable or if x_k does not occur in t , then $\text{vb}_k(t) = 0$, and $\text{vb}_k(t) = \sum_{j=1}^{n_i} \text{vb}_k(t_j)$ if $t = f_i(t_1, \dots, t_{n_i})$ and $t_1, \dots, t_{n_i} \in W_\tau(X_n)$.

The variable count of a relation formula F , denoted by $\text{vb}(F)$, is the total number of occurring variables in $F = \gamma_j(t_1, \dots, t_{n_j})$. This can be defined inductively by $\text{vb}(F) = \sum_{k=1}^{n_j} \text{vb}(t_k)$. Finally, the x_k -variable count of a relational formula $F = \gamma_j(t_1, \dots, t_{n_j})$, denoted by $\text{vb}_k(F)$, is

the total number of occurrences of variable x_k , $1 \leq k \leq n$, in F and is inductively defined by $\text{vb}_k(F) = \sum_{m=1}^{n_j} \text{vb}_k(t_m)$.

Lemma 2 ([13, Proposition 3.1]). *Let $s, s_1, \dots, s_n \in W_\tau(X)$. Then*

$$\text{vb}(S^n(s, s_1, \dots, s_n)) = \sum_{j=1}^n \text{vb}_j(s) \text{vb}(s_j) + \sum_{j>n} \text{vb}_j(s).$$

Lemma 3. *Let $\gamma_j(t_1, \dots, t_{n_j}) \in F_{(\tau, \tau')}^*(X)$, $s_1, \dots, s_n \in W_\tau(X)$. Then we have*

$$\begin{aligned} & \text{vb}(R^n(\gamma_j(t_1, \dots, t_{n_j}), s_1, \dots, s_n)) = \\ & \sum_{k=1}^n \text{vb}_k(\gamma_j(t_1, \dots, t_{n_j})) \text{vb}(s_k) + \sum_{k>n} \text{vb}_k(\gamma_j(t_1, \dots, t_{n_j})). \end{aligned}$$

Proof. Assume that the equation is satisfied for $t_1, \dots, t_{n_j}$, then we have $\text{vb}(R^n(\gamma_j(t_1, \dots, t_{n_j}), s_1, \dots, s_n))$

$$\begin{aligned} &= \sum_{i=1}^{n_j} \left(\sum_{k=1}^n \text{vb}_k(t_i) \text{vb}(s_k) + \sum_{k>n} \text{vb}_k(t_i) \right) \\ &= \sum_{i=1}^{n_j} \left(\sum_{k=1}^n \text{vb}_k(t_i) \text{vb}(s_k) \right) + \sum_{i=1}^{n_j} \left(\sum_{k>n} \text{vb}_k(t_i) \right) \\ &= \sum_{k=1}^n \left(\sum_{i=1}^{n_j} \text{vb}_k(t_i) \text{vb}(s_k) \right) + \sum_{k>n} \left(\sum_{i=1}^{n_j} \text{vb}_k(t_i) \right) \\ &= \sum_{k=1}^n \left(\left(\sum_{i=1}^{n_j} \text{vb}_k(t_i) \right) \text{vb}(s_k) \right) + \sum_{k>n} \text{vb}_k(\gamma_j(t_1, \dots, t_{n_j})) \\ &= \sum_{k=1}^n \text{vb}_k(\gamma_j(t_1, \dots, t_{n_j})) \text{vb}(s_k) + \sum_{k>n} \text{vb}_k(\gamma_j(t_1, \dots, t_{n_j})). \end{aligned}$$

□

The total number of variables that appear in a pair (t, F) from the Cartesian product $W \times F_{(\tau, \tau')}^*(X)$, denoted by $\text{vb}((t, F))$, is defined by the equation: $\text{vb}((t, F)) = \text{vb}(t) + \text{vb}(F)$.

Consequently, the formula for counting the number of variables occurring in pairs from $W \times F_{(\tau, \tau')}^*(X)$ under the binary operation $+_n$ is now produced.

Lemma 4. *Let $(t, F) \in W \times F_{(\tau, \tau')}^*(X)$. Then*

$$\begin{aligned} & \text{vb}((t, F) +_n ((t', F'))) \\ &= \text{vb}(t) \left(\sum_{k=1}^n \text{vb}_k(t') + \sum_{k=1}^n \text{vb}_k(F') \right) + \left(\sum_{k>n} \text{vb}_k(t') + \sum_{k>n} \text{vb}_k(F') \right). \end{aligned}$$

Proof. The proof follows from Lemma 3. $\square$

Having these preparations at hand, we prove the following result, which reveals the order of elements in $(W \times F_{(\tau, \tau')}^*(X), +_n)$.

Theorem 4. *The following assertions hold:*

- (1) *every idempotent element in E_{+n} has order 1;*
- (2) *every 2-potent element in $2 - \text{potent}_{+n}$ has order 2;*
- (3) *the order of each element in $W \times F_{(\tau, \tau')}^*(X) \setminus (2 - \text{potent}_{+n} \cup E_{+n})$ is infinite.*

Proof. The statement (1) follows directly from Theorem 1. By Theorem 2, we obtain that the order of elements in $2 - \text{potent}_{+n}$ is 2. To show that the statement (3) holds, i.e., $W \times F_{(\tau, \tau')}^*(X) \setminus (2 - \text{potent}_{+n} \cup E_{+n})$ is infinite, we need to show that $\text{vb}((t, F)^{n+1}) > \text{vb}((t, F)^n)$ for every natural number $n \in \mathbb{N}$. It can be seen that $(t, F)^n = (t', F')$ for some $(t', F') \in W \times F_{(\tau, \tau')}^*(X) \setminus (2 - \text{potent}_{+n} \cup E_{+n})$. Thus, by Lemma 4 we have

$$\begin{aligned}
 \text{vb}((t, F)^{n+1}) &= \text{vb}((t, F) +_n (t, F)^n) \\
 &= \text{vb}((t, F) +_n (t', F')) \\
 &= \text{vb}((S^n(t', t, \dots, t), R^n(F', t, \dots, t))) \\
 &> \text{vb}((t', F')) \\
 &= \text{vb}((t, F)^n).
 \end{aligned}$$

Therefore, the order of element (t, F) where $(t, F) \in W \times F_{(\tau, \tau')}^*(X) \setminus (2 - \text{potent}_{+n} \cup E_{+n})$ is infinite. $\square$

3. Concluding remarks

The semigroup comprising of the Cartesian product between the set $W_\tau(X)$ of all terms of type τ and the set $F_{(\tau, \tau')}^*(X)$ of all relational formulas of type (τ, τ') and the associative binary operation $+_n$ where $n \in \mathbb{N}$ is the main structure studied in this paper. Characterizations of idempotent (with $+_n$) are mentioned. In fact, it can be seen that the set of all idempotent elements on $W \times F_{(\tau, \tau')}^*(X)$ is completely given. Conditions for which a pair $(t, F) \in W \times F_{(\tau, \tau')}^*(X)$ to be 2-potent are determined. We also describe the order of each element in that semigroup. However, the characterization of regular elements is more complicated and is not addressed in this paper. Actually, we say that a

pair (t, F) in $W \times F_{(\tau, \tau')}^*(X)$ is regular (with respect to $+_n$) if there exists a pair (s, Q) from the Cartesian product $W \times F_{(\tau, \tau')}^*(X)$ such that $(t, F) = (t, F) +_n (s, Q) +_n (t, F)$, which means that a term t equals $S^n(t, S^n(s, t, \dots, t), \dots, S^n(s, t, \dots, t))$ and a relational formula F is equal to $R^n(F, R^n(Q, t, \dots, t), \dots, R^n(Q, t, \dots, t))$. Besides, we denote Reg_{+_n} to be the set of all regular elements in the semigroup $W \times F_{(\tau, \tau')}^*(X)$ with the binary operation $+_n$. It is obvious that $E_{+_n} \subseteq \text{Reg}_{+_n}$. However, the opposite side is not generally valid. For this reason, the following questions arise! Try to characterize the set Reg_{+_n} of all regular elements. For this, we suggest that elements in $W \times F_{(\tau, \tau')}^*(X) \setminus E_{+_n}$ should be firstly considered. Moreover, left (right) regular elements, and various kinds of ideals in $(W \times F_{(\tau, \tau')}^*(X), +_n)$ are also interesting topics.

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