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RESEARCH ARTICLE

On the algebra of derivations of some Leibniz algebras

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Abstract. Let L be an algebra over a field F with the binary operations $+$ and $[-,-]$. Then L is called a left Leibniz algebra if it satisfies the left Leibniz identity $[[a,b],c]=[a,[b,c]]-[b,[a,c]]$ for all $a,b,c\in L$. We study algebras of derivations of some non–nilpotent Leibniz algebras of low dimensions.

Let V be a vector space over a field F. Denote by $End_F(V)$ the set of all linear transformations of L. Then $End_F(V)$ is an associative algebra by the operations $+$ and $\circ$. As usual, $End_F(V)$ is a Lie algebra by the operations $+$ and $[-,-]$ where $[f,g]=f\circ g-g\circ f$ for all $f,g\in End_F(V)$.

Now, let L be an algebra over a field F with the operations $+$ and $[-,-]$.

A linear transformation f of an algebra L is called a *derivation* if

$$f([a,b])=[f(a),b]+[a,f(b)]$$

for all $a,b\in L$.

Derivations play a very important role in studying the structure of many types of non-associative algebras. Such is, in particular, especially true for Lie and Leibniz algebras.

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Let L be an algebra over a field F with the binary operations $+$ and $[-,-]$. Then L is called a *left Leibniz algebra* if it satisfies the left Leibniz identity,

$$[[a,b],c] = [a,[b,c]] - [b,[a,c]],$$

for all $a,b,c \in L$. We will also use another form of this identity:

$$[a,[b,c]] = [[a,b],c] + [b,[a,c]].$$

Leibniz algebras first appeared in the paper of A. Blokh [2], but the term "Leibniz algebra" appears in the book of J.-L. Loday [11] and his article [12]. In [13], J.-L. Loday and T. Pirashvili conducted an in-depth study on Leibniz algebras' properties. The theory of Leibniz algebras has developed very intensely in many different directions. Some of the results of this theory were presented in the book [1]. Note that Lie algebras present a partial case of Leibniz algebras. Conversely, if L is a Leibniz algebra in which $[a,a] = 0$ for every element $a \in L$, then it is a Lie algebra. Thus, Lie algebras can be characterized as anticommutative Leibniz algebras.

Let $Der(L)$ be the subset of all derivations of a Leibniz algebra L. It can prove that $Der(L)$ is a subalgebra of the Lie algebra $End_F(L)$. $Der(L)$ is called the *algebra of derivations* of the Leibniz algebra L.

The influence on the structure of a Leibniz algebra of its algebra of derivations can be observed in the following result: If A is an ideal of a Leibniz algebra, then the factor-algebra of L by the annihilator of A is isomorphic to some subalgebra of $Der(A)$ [3, Proposition 3.2]. The structure of the algebra of derivations of finite-dimensional one-generator Leibniz algebras was described in the papers [7,15], and the one belonging to infinite-dimensional one-generator Leibniz algebras was delineated in the paper [10]. The question about the algebras of derivations of Leibniz algebras of small dimensions naturally arises. In contrast to Lie algebras, the situation with Leibniz algebras of dimension 3 is very diverse. The Leibniz algebras of dimension 3 have been described, and their most detailed description can be found in [4]. The papers [5, 8, 9] described the algebras of derivations of nilpotent Leibniz algebras of dimension 3. In the paper [6], the description of the algebras of derivations of some non-nilpotent Leibniz algebras of dimension 3 has been started. More concretely, it describes the algebra of derivations of non-nilpotent Leibniz algebras of dimension 3, which are not one-generator and have a Leibniz kernel of dimension 2. In this paper, we will continue the study of the algebras of derivations of non-nilpotent Leibniz algebras of dimension 3.

As usual, we will suppose that L is not a Lie algebra, so $Leib(L)$ is non-zero.

First, we will finish with a case where L is a non-nilpotent Leibniz algebra of dimension 3, having a Leibniz kernel of dimension 2. The last type of these algebras is the following.

Let L be a non-nilpotent Leibniz algebra, generated by an element a, and having a Leibniz kernel of dimension 2. Put $a_1 = a$, $b = [a_1, a_1]$. If we suppose that $[a_1, b] \in Fb$, then a subalgebra generated by an element a_1 coincides with $Fa_1 \oplus Fb$. In particular, it has dimension 2, and hence it is proper. Thus, we obtain that $d = [a_1, b] \notin Fb$. It follows that $Leib(L) = Fb \oplus Fd$. Further we have $[a_1, d] = \kappa_2 b + \kappa_3 d$. If $\kappa_2 = 0$, we obtain the specific case when a subspace Fd is an ideal. Since L is not nilpotent, $\kappa_3 \neq 0$. Put $c = \kappa_3 b - d$. Then

$$[a_1, c] = [a_1, \kappa_3 b - d] = \kappa_3[a_1, b] - [a_1, d] = \kappa_3 d - \kappa_3 d = 0.$$

Since $c \in Leib(L)$, $[c, a_1] = [c, b] = [c, d] = [b, c] = [d, c] = 0$. It follows that $c \in \zeta(L)$. Moreover, clearly $Fc = \zeta(L)$. The fact that $\kappa_3 \neq 0$ implies that $Leib(L) = Fc \oplus Fd$. Also we have $b = \kappa_3^{-1}(c + d)$, so that $[a_1, a_1] = \kappa_3^{-1}(c + d)$. Put $a_2 = \kappa_3^{-1}c$, $a_3 = \kappa_3^{-1}d$, then

$$[a_1, a_1] = a_2 + a_3, \ [a_1, a_2] = [a_2, a_1] = 0,$$
$$[a_1, a_3] = [a_1, \kappa_3^{-1}d] = \kappa_3^{-1}[a_1, d] = \kappa_3^{-1}\kappa_3 d = d = \kappa_3 a_3.$$

The fact that Fa_3 is an ideal and $Fa_2 = \zeta(L)$ implies that the factor-algebra L/Fa_3 is nilpotent. It follows that $Fa_3 = \gamma_3(L)$. Thus, we come to the following type of Leibniz algebras:

$$Lei_8(3, F) = Fa_1 \oplus Fa_2 \oplus Fa_3 \text{ where } [a_1, a_1] = a_2 + a_3,$$
$$[a_1, a_3] = \kappa a_3, \ 0 \neq \kappa \in F,$$
$$[a_1, a_2] = [a_2, a_1] = [a_2, a_2] = [a_2, a_3] = [a_3, a_1] = [a_3, a_2] = [a_3, a_3] = 0.$$

In other words, $Lei_8(3, F) = L$ is one-generator, has the Leibniz kernel $Leib(L) = Fa_2 \oplus Fa_3 = [L, L] = \zeta^{\text{left}}(L)$ such that $\zeta(L) = Fa_2$, $Fa_3 = \gamma_3(L)$, $\zeta^{\text{right}}(L) = \zeta(L)$.

Let again $\kappa_2 = 0$, $y = \alpha_2 b + \alpha_3 d$ and suppose that $[a_1, y] = \beta y$ for some scalar $0 \neq \beta \in F$. Then we have

$$\begin{aligned}\beta y = \beta(\alpha_2 b + \alpha_3 d) &= \beta\alpha_2 b + \beta\alpha_3 d = [a_1, y]\\ &= [a_1, \alpha_2 b + \alpha_3 d] = \alpha_2[a_1, b] + \alpha_3[a_1, d]\\ &= \alpha_2 d + \alpha_3\kappa_3 d = (\alpha_2 + \alpha_3\kappa_3)d,\end{aligned}$$

so that $\beta\alpha_2 = 0$ and $\beta\alpha_3 = \alpha_2 + \alpha_3\kappa_3$. Since $\beta \neq 0$, $\alpha_2 = 0$. It follows that $y \in Fd$ and $Fy = Fd$.

Suppose now that $\kappa_2 \neq 0$, let $y = \alpha_2 b + \alpha_3 d$ and suppose that $[a_1, y] = \beta y$ for some scalar $\beta \in F$. Then we have

$$\begin{aligned}\beta y &= \beta(\alpha_2 b + \alpha_3 d) = \beta\alpha_2 b + \beta\alpha_3 d = [a_1, y]\\ &= [a_1, \alpha_2 b + \alpha_3 d] = \alpha_2[a_1, b] + \alpha_3[a_1, d]\\ &= \alpha_2 d + \alpha_3(\kappa_2 b + \kappa_3 d) = \alpha_3\kappa_2 b + (\alpha_2 + \alpha_3\kappa_3)d,\end{aligned}$$

so that $\beta\alpha_2 = \alpha_3\kappa_2$ and $\beta\alpha_3 = \alpha_2 + \alpha_3\kappa_3$. If we suppose that $\beta = 0$, then taking into account the fact $\kappa_2 \neq 0$, we obtain that $\alpha_3 = 0$, and it implies that $\alpha_2 = 0$. Hence, if $\kappa_2 \neq 0$, then $\zeta(L) = \langle 0 \rangle$.

Suppose that $\beta \neq 0$. Then the assumption $\alpha_2 = 0$ implies that $\alpha_3 = 0$, and conversely, the assumption $\alpha_3 = 0$ implies that $\alpha_2 = 0$. Hence, we can suppose that $\alpha_2 \neq 0$ and $\alpha_3 \neq 0$. Put $\sigma = \alpha_2\alpha_3^{-1}$, then we obtain $\beta\sigma = \kappa_2$ and $\beta = \sigma + \kappa_3$. It follows that $\sigma^2 + \kappa_3\sigma - \kappa_2 = 0$. Hence, if the polynomial $X^2 + \kappa_3 X - \kappa_2$ has no roots in field F, $Leib(L)$ does not include the ideals of dimension 1. In other words, $Leib(L)$ is a minimal ideal of L. Put $a_2 = b$, $a_3 = d$. Thus, we come to the following type of Leibniz algebras:

$$\begin{gathered}Lei_9(3, F) = Fa_1 \oplus Fa_2 \oplus Fa_3 \text{ where } [a_1, a_1] = a_2,\ [a_1, a_2] = a_3,\\ [a_1, a_3] = \kappa_2 a_2 + \kappa_3 a_3,\ 0 \neq \kappa_2, \kappa_3 \in F,\\ [a_2, a_1] = [a_2, a_2] = [a_2, a_3] = [a_3, a_1] = [a_3, a_2] = [a_3, a_3] = 0\end{gathered}$$

and a polynomial $X^2 + \kappa_3 X - \kappa_2$ has no roots in a field F.

Note that if the polynomial $X^2 + \kappa_3 X - \kappa_2$ has roots in field F, $Leib(L)$ includes the ideals of dimension 1, and we come to the previous type of Leibniz algebras.

Check that in this way, we obtain a Leibniz algebra.

Let $L = Lei_9(3, F)$ and x, y, z be the arbitrary elements of L,

$$\begin{aligned}x &= \xi_1 a_1 + \xi_2 a_2 + \xi_3 a_3,\\ y &= \eta_1 a_1 + \eta_2 a_2 + \eta_3 a_3,\\ z &= \tau_1 a_1 + \tau_2 a_2 + \tau_3 a_3\end{aligned}$$

where $\xi_1, \xi_2, \xi_3, \eta_1, \eta_2, \eta_3, \tau_1, \tau_2, \tau_3$ are arbitrary scalars. We have

$$\begin{aligned}[x, y] &= [\xi_1 a_1 + \xi_2 a_2 + \xi_3 a_3, \eta_1 a_1 + \eta_2 a_2 + \eta_3 a_3] \\ &= \xi_1\eta_1[a_1, a_1] + \xi_1\eta_2[a_1, a_2] + \xi_1\eta_3[a_1, a_3] \\ &= \xi_1\eta_1 a_2 + \xi_1\eta_2 a_3 + \xi_1\eta_3(\kappa_2 a_2 + \kappa_3 a_3) \\ &= (\xi_1\eta_1 + \xi_1\eta_3\kappa_2)a_2 + (\xi_1\eta_2 + \xi_1\eta_3\kappa_3)a_3, \\ [x, z] &= (\xi_1\tau_1 + \xi_1\tau_3\kappa_2)a_2 + (\xi_1\tau_2 + \xi_1\tau_3\kappa_3)a_3, \\ [y, z] &= (\eta_1\tau_1 + \eta_1\tau_3\kappa_2)a_2 + (\eta_1\tau_2 + \eta_1\tau_3\kappa_3)a_3.\end{aligned}$$

Therefore,

$$\begin{aligned}[[x, y], z] &= 0, \\ [x, [y, z]] &= [\xi_1 a_1 + \xi_2 a_2 + \xi_3 a_3, (\eta_1\tau_1 + \eta_1\tau_3\kappa_2)a_2 + (\eta_1\tau_2 + \eta_1\tau_3\kappa_3)a_3] \\ &= [\xi_1 a_1, (\eta_1\tau_1 + \eta_1\tau_3\kappa_2)a_2] + [\xi_1 a_1, (\eta_1\tau_2 + \eta_1\tau_3\kappa_3)a_3] \\ &= \xi_1(\eta_1\tau_1 + \eta_1\tau_3\kappa_2)[a_1, a_2] + \xi_1(\eta_1\tau_2 + \eta_1\tau_3\kappa_3)[a_1, a_3] \\ &= \xi_1(\eta_1\tau_1 + \eta_1\tau_3\kappa_2)a_3 + \xi_1(\eta_1\tau_2 + \eta_1\tau_3\kappa_3)(\kappa_2 a_2 + \kappa_3 a_3) \\ &= (\xi_1\eta_1\tau_2\kappa_2 + \xi_1\eta_1\tau_3\kappa_3\kappa_2)a_2 \\ &+ (\xi_1\eta_1\tau_1 + \xi_1\eta_1\tau_3\kappa_2 + \xi_1\eta_1\tau_2\kappa_3 + \xi_1\eta_1\tau_3\kappa_3^2)a_3, \\ [y, [x, z]] &= [\eta_1 a_1 + \eta_2 a_2 + \eta_3 a_3, (\xi_1\tau_1 + \xi_1\tau_3\kappa_2)a_2 + (\xi_1\tau_2 + \xi_1\tau_3\kappa_3)a_3] \\ &= [\eta_1 a_1, (\xi_1\tau_1 + \xi_1\tau_3\kappa_2)a_2] + [\eta_1 a_1, (\xi_1\tau_2 + \xi_1\tau_3\kappa_3)a_3] \\ &= \eta_1(\xi_1\tau_1 + \xi_1\tau_3\kappa_2)[a_1, a_2] + \eta_1(\xi_1\tau_2 + \xi_1\tau_3\kappa_3)[a_1, a_3] \\ &= \eta_1(\xi_1\tau_1 + \xi_1\tau_3\kappa_2)a_3 + \eta_1(\xi_1\tau_2 + \xi_1\tau_3\kappa_3)(\kappa_2 a_2 + \kappa_3 a_3) \\ &= (\eta_1\xi_1\tau_2\kappa_2 + \eta_1\xi_1\tau_3\kappa_3\kappa_2)a_2 \\ &+ (\eta_1\xi_1\tau_1 + \eta_1\xi_1\tau_3\kappa_2 + \eta_1\xi_1\tau_2\kappa_3 + \eta_1\xi_1\tau_3\kappa_3^2)a_3.\end{aligned}$$

Thus, we obtain $[[x, y], z] = [x, [y, z]] - [y, [x, z]]$. Hence, $Lei_9(3, F)$ is a Leibniz algebra.

We begin with some general properties of an algebra of derivations of a Leibniz algebra. Here, we show some basic elementary properties of derivations that have been proven in the paper [10]. First, let us recall some definitions.

Every Leibniz algebra L has a specific ideal. Denote by $Leib(L)$ the subspace generated by the elements $[a, a]$, $a \in L$. It is possible to prove that $Leib(L)$ is an ideal of L. The ideal $Leib(L)$ is called the *Leibniz kernel* of algebra L. By the definition, factor-algebra $L/Leib(L)$ is a Lie algebra. Conversely, if K is an ideal of L, such that L/K is a Lie algebra, then K includes the Leibniz kernel.

Let L be a Leibniz algebra. Define the *lower central series* of L,

$$L = \gamma_1(L) \geq \gamma_2(L) \geq \dots \gamma_\alpha(L) \geq \gamma_{\alpha+1}(L) \geq \dots \gamma_\delta(L),$$

by the following rule: $\gamma_1(L) = L$, $\gamma_2(L) = [L, L]$, recursively, $\gamma_{\alpha+1}(L) = [L, \gamma_\alpha(L)]$ for every ordinal α, and $\gamma_\lambda(L) = \bigcap_{\mu<\lambda} \gamma_\mu(L)$ for every limit ordinal λ. The last term $\gamma_\delta(L) = \gamma_\infty(L)$ is called the *lower hypocenter* of L. We have: $\gamma_\delta(L) = [L, \gamma_\delta(L)]$.

As usual, we say that a Leibniz algebra L is called *nilpotent* if a positive integer k exists, such that $\gamma_k(L) = \langle 0 \rangle$. More precisely, L is said to be *nilpotent of nilpotency class* c if $\gamma_{c+1}(L) = \langle 0 \rangle$ but $\gamma_c(L) \neq \langle 0 \rangle$.

The *left* (respectively *right*) *center* $\zeta^{\text{left}}(L)$ (respectively $\zeta^{\text{right}}(L)$) of a Leibniz algebra L is defined by the rule below:

$$\zeta^{\text{left}}(L) = \{x \in L|\ [x, y] = 0 \text{ for each element } y \in L\}$$

(respectively

$$\zeta^{\text{right}}(L) = \{x \in L|\ [y, x] = 0 \text{ for each element } y \in L\}).$$

It is not hard to prove that the left center of L is an ideal, but this is not true for the right center. Moreover, $Leib(L) \leq \zeta^{\text{left}}(L)$, so that $L/\zeta^{\text{left}}(L)$ is a Lie algebra. The right center is a subalgebra of L; the left and right centers are generally different; they may even have different dimensions (see [3]).

The center of L is defined by the rule below:

$$\zeta(L) = \{x \in L|\ [x, y] = 0 = [y, x] \text{ for each element } y \in L\}.$$

The center is an ideal of L. Note that if K is an ideal of L, then the center of K is an ideal of L [14, Lemma 4].

Lemma 1. *Let L be a Leibniz algebra over a field F and f be a derivation of L. Then $f(\zeta^{\text{left}}(L)) \leq \zeta^{\text{left}}(L)$, $f(\zeta^{\text{right}}(L)) \leq \zeta^{\text{right}}(L)$ and $f(\zeta(L)) \leq \zeta(L)$.*

Corollary 1. *Let L be a Leibniz algebra over a field F and f be a derivation of L. Then $f(\zeta_\alpha(L)) \leq \zeta_\alpha(L)$ for every ordinal α.*

Lemma 2. *Let L be a Leibniz algebra over a field F and f be a derivation of L. Then $f([L, L]) \leq [L, L]$.*

Corollary 2. *Let L be a Leibniz algebra over a field F and f be a derivation of L. Then $f(\gamma_\alpha(L)) \leq \gamma_\alpha(L)$ for every ordinal α.*

Proof. Lemma 2 shows that $f(\gamma_2(L)) \leq \gamma_2(L)$. Let $\alpha > 2$ and suppose that we have already proved that $f(\gamma_\beta(L)) \leq \gamma_\beta(L)$ for all ordinals $\beta \leq \alpha$. Suppose first that α is a not limit ordinal, $\alpha = \mu + 1$ for some ordinal μ. Let x be an arbitrary element of $\gamma_\alpha(L)$, then

$$x = \sigma_1[a_1, b_1] + \ldots + \sigma_n[a_n, b_n]$$

where $a_1, \ldots, a_n \in L$, $b_1, \ldots, b_n \in \gamma_\mu(L)$, $\sigma_1, \ldots, \sigma_n \in F$.

Now we obtain

$$\begin{aligned} f(x) &= f(\sigma_1[a_1, b_1] + \ldots + \sigma_n[a_n, b_n]) \\ &= \sigma_1 f([a_1, b_1]) + \ldots + \sigma_n f([a_n, b_n]) \\ &= \sigma_1[f(a_1), b_1] + \ldots + \sigma_n[f(a_n), b_n] \\ &+ \sigma_1[a_1, f(b_1)] + \ldots + \sigma_n[a_n, f(b_n)]. \end{aligned}$$

By $b_j \in \gamma_\mu(L)$, we have $[f(a_j), b_j] \in [L, \gamma_\mu(L)] = \gamma_{\mu+1}(L) = \gamma_\alpha(L)$, $1 \leq j \leq n$. Since $\mu < \alpha$, $f(b_j) \in \gamma_\mu(L)$ by the induction hypothesis, so that $[a_j, f(b_j)] \in [L, \gamma_\mu(L)] = \gamma_{\mu+1}(L) = \gamma_\alpha(L)$, $1 \leq j \leq n$. Hence $f(x) \in \gamma_\alpha(L)$.

Suppose now that α is a limit ordinal. Then $\gamma_\alpha(L) = \bigcap_{\tau<\alpha} \gamma_\tau(L)$. It follows that $x \in \gamma_\tau(L)$ for all ordinal $\tau < \alpha$. Then, by induction hypothesis, $f(x) \in \gamma_\tau(L)$ for all ordinal $\tau < \alpha$ and therefore $f(x) \in \bigcap_{\tau<\alpha} \gamma_\tau(L) = \gamma_\alpha(L)$. □

Denote by Ξ the classic monomorphism of $End(L)$ in $M_3(F)$ (i.e., the mapping, assigning to each endomorphism its matrix concerning the basis $\{a_1, a_2, a_3\}$).

Theorem 1. *Let D be an algebra of derivations of the Leibniz algebra $Lei_8(3, F)$. Then D is isomorphic to a Lie subalgebra of $M_3(F)$ consisting of the matrices of the following form:*

$$\begin{pmatrix} 0 & 0 & 0 \\ \alpha & 0 & 0 \\ \beta & 0 & \kappa\beta \end{pmatrix},$$

$\alpha, \beta \in F$. Furthermore, D is abelian and isomorphic to the direct sum of two copies of the additive group of field F.

Proof. Let $L = Lei_8(3, F)$ and $f \in Der(L)$. By Lemma 1, $f(\zeta(L)) \leq \zeta(L) = Fa_2$, and by Corollary 2, $f(Fa_3) = f(\gamma_3(L)) \leq \gamma_3(L)$. So that

$$\begin{aligned} f(a_1) &= \alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3, \\ f(a_2) &= \beta a_2, \\ f(a_3) &= \gamma a_3, \end{aligned}$$

$\alpha_1, \alpha_2, \alpha_3, \beta, \gamma \in F$. Then

$$\begin{aligned} f(a_3) &= f(\kappa^{-1}[a_1, a_3]) = \kappa^{-1} f([a_1, a_3]) \\ &= \kappa^{-1}([f(a_1), a_3] + [a_1, f(a_3)]) \\ &= \kappa^{-1}([\alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3, a_3] + [a_1, \gamma a_3]) \\ &= \kappa^{-1}(\alpha_1[a_1, a_3] + \alpha_2[a_2, a_3] + \alpha_3[a_3, a_3] + \gamma\kappa a_3) \\ &= \kappa^{-1}(\alpha_1 \kappa a_3 + \gamma\kappa a_3) = (\alpha_1 + \gamma)a_3; \\ f([a_1, a_1]) &= [f(a_1), a_1] + [a_1, f(a_1)] \\ &= [\alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3, a_1] + [a_1, \alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3] \\ &= \alpha_1[a_1, a_1] + \alpha_1[a_1, a_1] + \alpha_2[a_1, a_2] + \alpha_3[a_1, a_3] \\ &= 2\alpha_1(a_2 + a_3) + \kappa\alpha_3 a_3 = 2\alpha_1 a_2 + (2\alpha_1 + \kappa\alpha_3)a_3, \\ f([a_1, a_1]) &= f(a_2 + a_3) = f(a_2) + f(a_3) = \beta a_2 + \gamma a_3. \end{aligned}$$

Then we obtain

$$(\alpha_1 + \gamma)a_3 = \gamma a_3, \ 2\alpha_1 a_2 + (2\alpha_1 + \kappa\alpha_3)a_3 = \beta a_2 + \gamma a_3,$$

so that

$$\alpha_1 + \gamma = \gamma, \ 2\alpha_1 = \beta, \ 2\alpha_1 + \kappa\alpha_3 = \gamma.$$

It follows that $\alpha_1 = 0$, $\beta = 0$, $\kappa\alpha_3 = \gamma$. Hence, $\Xi(f)$ is the following matrix:

$$\begin{pmatrix} 0 & 0 & 0 \\ \alpha_2 & 0 & 0 \\ \alpha_3 & 0 & \kappa\alpha_3 \end{pmatrix},$$

$\alpha_2, \alpha_3, \kappa \in F$.

Conversely, let x, y be arbitrary elements of L,

$$\begin{aligned} x &= \xi_1 a_1 + \xi_2 a_2 + \xi_3 a_3, \\ y &= \eta_1 a_1 + \eta_2 a_2 + \eta_3 a_3 \end{aligned}$$

where $\xi_1, \xi_2, \xi_3, \eta_1, \eta_2, \eta_3$ are arbitrary scalars. Then

$$\begin{aligned}
[x, y] &= [\xi_1 a_1 + \xi_2 a_2 + \xi_3 a_3, \eta_1 a_1 + \eta_2 a_2 + \eta_3 a_3] \\
&= \xi_1\eta_1[a_1, a_1] + \xi_1\eta_2[a_1, a_2] + \xi_1\eta_3[a_1, a_3] \\
&= \xi_1\eta_1(a_2 + a_3) + \xi_1\eta_3\kappa a_3 \\
&= \xi_1\eta_1 a_2 + (\xi_1\eta_1 + \kappa\xi_1\eta_3)a_3, \\
f(x) &= f(\xi_1 a_1 + \xi_2 a_2 + \xi_3 a_3) = \xi_1 f(a_1) + \xi_2 f(a_2) + \xi_3 f(a_3) \\
&= \xi_1\alpha_2 a_2 + \xi_1\alpha_3 a_3 + \xi_3\kappa\alpha_3 a_3 \\
&= \xi_1\alpha_2 a_2 + (\xi_1\alpha_3 + \xi_3\kappa\alpha_3)a_3, \\
f(y) &= \eta_1\alpha_2 a_2 + (\eta_1\alpha_3 + \eta_3\kappa\alpha_3)a_3, \\
f([x, y]) &= f(\xi_1\eta_1 a_2 + (\xi_1\eta_1 + \kappa\xi_1\eta_3)a_3) \\
&= \xi_1\eta_1 f(a_2) + (\xi_1\eta_1 + \kappa\xi_1\eta_3)f(a_3) \\
&= \kappa\alpha_3(\xi_1\eta_1 + \kappa\xi_1\eta_3)a_3 = \kappa\alpha_3\xi_1(\eta_1 + \kappa\eta_3)a_3.
\end{aligned}$$

Therefore

$$\begin{aligned}
[f(x), y] + [x, f(y)] &= [\xi_1\alpha_2 a_2 + (\xi_1\alpha_3 + \xi_3\kappa\alpha_3)a_3, \eta_1 a_1 + \eta_2 a_2 + \eta_3 a_3] \\
&\quad + [\xi_1 a_1 + \xi_2 a_2 + \xi_3 a_3, \eta_1\alpha_2 a_2 + (\eta_1\alpha_3 + \eta_3\kappa\alpha_3)a_3] \\
&= \xi_1(\eta_1\alpha_3 + \eta_3\kappa\alpha_3)[a_1, a_3] = \kappa\xi_1(\eta_1\alpha_3 + \eta_3\kappa\alpha_3)a_3 \\
&= \kappa\xi_1\alpha_3(\eta_1 + \eta_3\kappa)a_3,
\end{aligned}$$

so that $f([x, y]) = [f(x), y] + [x, f(y)]$.

Denote by Φ the mapping of a vector space $F \oplus F$ in $\Xi(L)$ defined by the rule:

$$(\alpha, \beta) \to \begin{pmatrix} 0 & 0 & 0 \\ \alpha & 0 & 0 \\ \beta & 0 & \kappa\beta \end{pmatrix},$$

$\alpha, \beta, \kappa \in F$. Clearly

$$\Phi((\alpha, \beta) + (\lambda, \mu)) = \Phi((\alpha + \lambda, \beta + \mu)) = \Phi(\alpha, \beta) + \Phi(\lambda, \mu)$$

and

$$\Phi(\sigma(\alpha, \beta)) = \Phi(\sigma\alpha, \sigma\beta) = \sigma\Phi(\alpha, \beta).$$

Furthermore, the equality

$$\begin{pmatrix} 0 & 0 & 0 \\ \alpha & 0 & 0 \\ \beta & 0 & \kappa\beta \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ \lambda & 0 & 0 \\ \mu & 0 & \kappa\mu \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \kappa\beta\mu & 0 & \kappa^2\beta\mu \end{pmatrix}$$

shows that a Lie algebra $\Xi(L)$ is abelian. Thus D is isomorphic to an abelian Lie algebra $F \oplus F$. □

Lemma 3. *Let L be a finite-dimensional Leibniz algebra over a field F, $\{a_1, \dots, a_n\}$ be the basis of L. If f is a linear transformation of L such that $f([a_j, a_t]) = [f(a_j), a_t] + [a_j, f(a_t)]$ for all j, t, $1 \le j, t \le n$, then f is a derivation of L.*

Proof. Let x, y be the arbitrary elements of L, then $x = \sum_{1\le j\le n} \lambda_j a_j$, $y = \sum_{1\le t\le n} \mu_t a_t$. We have

$$[x, y] = \left[\sum_{1\le j\le n} \lambda_j a_j, \sum_{1\le t\le n} \mu_t a_t\right] = \sum_{\substack{1\le j\le n,\\ 1\le t\le n}} \lambda_j \mu_t [a_j, a_t].$$

Then

$$f([x, y]) = f\left(\sum_{\substack{1\le j\le n,\\ 1\le t\le n}} \lambda_j \mu_t [a_j, a_t]\right) = \sum_{\substack{1\le j\le n,\\ 1\le t\le n}} \lambda_j \mu_t f([a_j, a_t])$$
$$= \sum_{\substack{1\le j\le n,\\ 1\le t\le n}} \lambda_j \mu_t ([f(a_j), a_t] + [a_j, f(a_t)])$$

and

$$[f(x), y] + [x, f(y)]$$
$$= \left[f\left(\sum_{1\le j\le n} \lambda_j a_j\right), \sum_{1\le t\le n} \mu_t a_t\right] + \left[\sum_{1\le j\le n} \lambda_j a_j, f\left(\sum_{1\le t\le n} \mu_t a_t\right)\right]$$
$$= \left[\sum_{1\le j\le n} \lambda_j f(a_j), \sum_{1\le t\le n} \mu_t a_t\right] + \left[\sum_{1\le j\le n} \lambda_j a_j, \sum_{1\le t\le n} \mu_t f(a_t)\right]$$
$$= \sum_{\substack{1\le j\le n,\\ 1\le t\le n}} \lambda_j \mu_t [f(a_j), a_t] + \sum_{\substack{1\le j\le n,\\ 1\le t\le n}} \lambda_j \mu_t [a_j, f(a_t)],$$

so that $f([x, y]) = [f(x), y] + [x, f(y)]$. Hence f is a derivation of L. □

Lemma 4. *Let S be a subset of a Lie algebra $M_3(F)$ of matrices consisting of matrices having a form*

$$\begin{pmatrix} \alpha_1 & 0 & 0 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{pmatrix},$$

$\alpha_1, \alpha_2, \alpha_3, \beta_2, \beta_3, \gamma_2, \gamma_3 \in F$. *Then S is a Lie subalgebra of $M_3(F)$, and the mapping*

$$\theta : \begin{pmatrix} \alpha_1 & 0 & 0 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{pmatrix} \to \begin{pmatrix} \beta_2 & \gamma_2 \\ \beta_3 & \gamma_3 \end{pmatrix}$$

is an epimorphism of S on $M_2(F)$.

Proof. Indeed, an equality

$$\begin{pmatrix} \alpha_1 & 0 & 0 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 & 0 \\ \lambda_2 & \mu_2 & \sigma_2 \\ \lambda_3 & \mu_3 & \sigma_3 \end{pmatrix}$$
$$= \begin{pmatrix} \alpha_1\lambda_1 & 0 & 0 \\ \alpha_2\lambda_1 + \beta_2\lambda_2 + \gamma_2\lambda_3 & \beta_2\mu_2 + \gamma_2\mu_3 & \beta_2\sigma_2 + \gamma_2\sigma_3 \\ \alpha_3\lambda_1 + \beta_3\lambda_2 + \gamma_3\lambda_3 & \beta_3\mu_2 + \gamma_3\mu_3 & \beta_3\sigma_2 + \gamma_3\sigma_3 \end{pmatrix}$$

shows that a subset S is closed by multiplication. It follows that $[x, y] \in S$ for every matrices $x, y \in S$. Hence S is a Lie subalgebra of $M_3(F)$. Finally, the equality

$$\begin{pmatrix} \beta_2 & \gamma_2 \\ \beta_3 & \gamma_3 \end{pmatrix} \begin{pmatrix} \mu_2 & \sigma_2 \\ \mu_3 & \sigma_3 \end{pmatrix} = \begin{pmatrix} \beta_2\mu_2 + \gamma_2\mu_3 & \beta_2\sigma_2 + \gamma_2\sigma_3 \\ \beta_3\mu_2 + \gamma_3\mu_3 & \beta_3\sigma_2 + \gamma_3\sigma_3 \end{pmatrix}$$

shows that a mapping θ is an epimorphism. □

Lemma 5. *Let F be a field of characteristic 2, κ be a fixed non-zero element of F, and let S be a subset of a Lie algebra $M_2(F)$ of matrices consisting of matrices having a form*

$$\begin{pmatrix} \beta_1 & \kappa\beta_2 \\ \beta_2 & \beta_3 \end{pmatrix},$$

$\beta_1, \beta_2, \beta_3 \in F$. *Then S is a Lie subalgebra of $M_2(F)$, moreover, $S = Z \oplus L$ where Z is the center of $M_2(F)$ (the subset of all scalar matrices) and L is a non-abelian Lie subalgebra of dimension 2.*

Proof. Indeed, let

$$X = \begin{pmatrix} \beta_1 & \kappa\beta_2 \\ \beta_2 & \beta_3 \end{pmatrix}, \ Y = \begin{pmatrix} \lambda_1 & \kappa\lambda_2 \\ \lambda_2 & \lambda_3 \end{pmatrix}.$$

We have

$$\begin{aligned}
XY &= \begin{pmatrix} \beta_1 & \kappa\beta_2 \\ \beta_2 & \beta_3 \end{pmatrix} \begin{pmatrix} \lambda_1 & \kappa\lambda_2 \\ \lambda_2 & \lambda_3 \end{pmatrix} \\
&= \begin{pmatrix} \beta_1\lambda_1 + \kappa\beta_2\lambda_2 & \kappa\beta_1\lambda_2 + \kappa\beta_2\lambda_3 \\ \beta_2\lambda_1 + \beta_3\lambda_2 & \kappa\beta_2\lambda_2 + \beta_3\lambda_3 \end{pmatrix}, \\
YX &= \begin{pmatrix} \lambda_1 & \kappa\lambda_2 \\ \lambda_2 & \lambda_3 \end{pmatrix} \begin{pmatrix} \beta_1 & \kappa\beta_2 \\ \beta_2 & \beta_3 \end{pmatrix} \\
&= \begin{pmatrix} \lambda_1\beta_1 + \kappa\lambda_2\beta_2 & \kappa\lambda_1\beta_2 + \kappa\lambda_2\beta_3 \\ \lambda_2\beta_1 + \lambda_3\beta_2 & \kappa\lambda_2\beta_2 + \lambda_3\beta_3 \end{pmatrix}, \\
[X,Y] &= \begin{pmatrix} \beta_1\lambda_1 + \kappa\beta_2\lambda_2 & \kappa\beta_1\lambda_2 + \kappa\beta_2\lambda_3 \\ \beta_2\lambda_1 + \beta_3\lambda_2 & \kappa\beta_2\lambda_2 + \beta_3\lambda_3 \end{pmatrix} \\
&+ \begin{pmatrix} \lambda_1\beta_1 + \kappa\lambda_2\beta_2 & \kappa\lambda_1\beta_2 + \kappa\lambda_2\beta_3 \\ \lambda_2\beta_1 + \lambda_3\beta_2 & \kappa\lambda_2\beta_2 + \lambda_3\beta_3 \end{pmatrix} \\
&= \begin{pmatrix} \beta_1\lambda_1 + \kappa\beta_2\lambda_2 + \lambda_1\beta_1 + \kappa\lambda_2\beta_2 & \kappa\beta_1\lambda_2 + \kappa\beta_2\lambda_3 + \kappa\lambda_1\beta_2 + \kappa\lambda_2\beta_3 \\ \beta_2\lambda_1 + \beta_3\lambda_2 + \lambda_2\beta_1 + \lambda_3\beta_2 & \kappa\beta_2\lambda_2 + \beta_3\lambda_3 + \kappa\lambda_2\beta_2 + \lambda_3\beta_3 \end{pmatrix} \\
&= \begin{pmatrix} 0 & \kappa(\beta_1\lambda_2 + \beta_2\lambda_3 + \lambda_1\beta_2 + \lambda_2\beta_3) \\ \beta_2\lambda_1 + \beta_3\lambda_2 + \lambda_2\beta_1 + \lambda_3\beta_2 & 0 \end{pmatrix} \in S.
\end{aligned}$$

It follows that S is a subalgebra of a Lie algebra $M_2(F)$.

For every matrix $X \in S$, we have decomposition

$$\begin{aligned}
X &= \begin{pmatrix} \beta_1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \beta_3 \end{pmatrix} + \begin{pmatrix} 0 & \kappa\beta_2 \\ \beta_2 & 0 \end{pmatrix} \\
&= \beta_1 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \beta_3 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \beta_2 \begin{pmatrix} 0 & \kappa \\ 1 & 0 \end{pmatrix}.
\end{aligned}$$

Put

$$U = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},\ V = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},\ W = \begin{pmatrix} 0 & \kappa \\ 1 & 0 \end{pmatrix}.$$

We have

$$\begin{aligned}
UW &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & \kappa \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \kappa \\ 0 & 0 \end{pmatrix}, \\
WU &= \begin{pmatrix} 0 & \kappa \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \\
[U,W] &= \begin{pmatrix} 0 & \kappa \\ 1 & 0 \end{pmatrix} = W,
\end{aligned}$$

$$VW = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & \kappa \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$
$$WV = \begin{pmatrix} 0 & \kappa \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & \kappa \\ 0 & 0 \end{pmatrix},$$
$$[V, W] = \begin{pmatrix} 0 & \kappa \\ 1 & 0 \end{pmatrix} = W.$$

Furthermore

$$U + V = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in S.$$

Then S includes the center Z of a Lie algebra $M_2(F)$. Moreover, $S = Z \oplus L$ where L is a Lie subalgebra generated by U, W. In other words, L is a non-abelian Lie algebra of dimension 2. □

Theorem 2. *Let D be an algebra of derivations of the Leibniz algebra $Lei_9(3, F)$. If $char(F) \neq 2$, $\kappa_3 = 1$, then D is isomorphic to a Lie subalgebra of $M_3(F)$ consisting of the matrices having the following form:*

$$\begin{pmatrix} 0 & 0 & 0 \\ \alpha_2 & \alpha_3\kappa_2 & \alpha_2\kappa_2 + \alpha_3\kappa_2 \\ \alpha_3 & \alpha_2 + \alpha_3 & \alpha_3\kappa_2 + \alpha_2 + \alpha_3 \end{pmatrix},$$

$\alpha_2, \alpha_3 \in F$.

If $char(F) \neq 2$, $\kappa_3 \neq 1$, then D is isomorphic to a one-dimensional Lie subalgebra of $M_3(F)$, generated by the matrix

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & -\kappa_3 & \kappa_2 - \kappa_3^2 \\ -\kappa_2^{-1}\kappa_3 & 1 - \kappa_2^{-1}\kappa_3^2 & 2\kappa_3 - \kappa_2^{-1}\kappa_3^3 \end{pmatrix}.$$

If $char(F) = 2$, then D is isomorphic to a Lie subalgebra of $M_3(F)$ consisting of the matrices having the following form:

$$\begin{pmatrix} (\kappa_3 - 1)(\alpha_3\kappa_2 + \alpha_2\kappa_3) & 0 & 0 \\ \alpha_2 & \alpha_2\kappa_3 & \alpha_2\kappa_2 + \alpha_3\kappa_3\kappa_2 \\ \alpha_3 & \alpha_2 + \alpha_3\kappa_3 & \alpha_3\kappa_2\kappa_3 + \alpha_2\kappa_3^2 + \alpha_3\kappa_3^2 \end{pmatrix},$$

$\alpha_2, \alpha_3 \in F$.

If $\kappa_2 = \kappa_3^2$, then D is a abelian Lie algebra of dimension 2. In this case, a polynomial $X^2 + \kappa_3 X + \kappa_3^2$ has no roots in a field F.

If $\kappa_2 \neq \kappa_3^2$, D is a non-abelian Lie algebra of dimension 2.

Proof. Let $L = Lei_9(3, F)$ and $f \in Der(L)$. Let $f(a_1) = \alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3$. Then

$$\begin{aligned}
f(a_2) &= f([a_1, a_1]) = [f(a_1), a_1] + [a_1, f(a_1)] \\
&= [\alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3, a_1] + [a_1, \alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3] \\
&= \alpha_1 [a_1, a_1] + \alpha_1 [a_1, a_1] + \alpha_2 [a_1, a_2] + \alpha_3 [a_1, a_3] \\
&= 2\alpha_1 a_2 + \alpha_2 a_3 + \alpha_3 (\kappa_2 a_2 + \kappa_3 a_3) \\
&= (2\alpha_1 + \alpha_3 \kappa_2) a_2 + (\alpha_2 + \alpha_3 \kappa_3) a_3, \\
f(a_3) &= f([a_1, a_2]) = [f(a_1), a_2] + [a_1, f(a_2)] \\
&= [\alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3, a_2] + [a_1, (2\alpha_1 + \alpha_3 \kappa_2) a_2 + (\alpha_2 + \alpha_3 \kappa_3) a_3] \\
&= \alpha_1 [a_1, a_2] + (2\alpha_1 + \alpha_3 \kappa_2)[a_1, a_2] + (\alpha_2 + \alpha_3 \kappa_3)[a_1, a_3] \\
&= \alpha_1 a_3 + (2\alpha_1 + \alpha_3 \kappa_2) a_3 + (\alpha_2 + \alpha_3 \kappa_3)(\kappa_2 a_2 + \kappa_3 a_3) \\
&= (\alpha_2 \kappa_2 + \alpha_3 \kappa_3 \kappa_2) a_2 + (\alpha_1 + 2\alpha_1 + \alpha_3 \kappa_2 + \alpha_2 \kappa_3 + \alpha_3 \kappa_3^2) a_3.
\end{aligned}$$

Moreover, by $[a_1, a_3] = \kappa_2 a_2 + \kappa_3 a_3$ we obtain

$$\begin{aligned}
f([a_1, a_3]) &= [f(a_1), a_3] + [a_1, f(a_3)] \\
&= [\alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3, a_3] + [a_1, (\alpha_2 \kappa_2 + \alpha_3 \kappa_3 \kappa_2) a_2 \\
&+ (\alpha_1 + 2\alpha_1 + \alpha_3 \kappa_2 + \alpha_2 \kappa_3 + \alpha_3 \kappa_3^2) a_3] \\
&= \alpha_1 [a_1, a_3] + (\alpha_2 \kappa_2 + \alpha_3 \kappa_3 \kappa_2)[a_1, a_2] \\
&+ (\alpha_1 + 2\alpha_1 + \alpha_3 \kappa_2 + \alpha_2 \kappa_3 + \alpha_3 \kappa_3^2)[a_1, a_3] \\
&= \alpha_1 (\kappa_2 a_2 + \kappa_3 a_3) + (\alpha_2 \kappa_2 + \alpha_3 \kappa_3 \kappa_2) a_3 \\
&+ (\alpha_1 + 2\alpha_1 + \alpha_3 \kappa_2 + \alpha_2 \kappa_3 + \alpha_3 \kappa_3^2)(\kappa_2 a_2 + \kappa_3 a_3) \\
&= (\alpha_1 \kappa_2 + \alpha_1 \kappa_2 + 2\alpha_1 \kappa_2 + \alpha_3 \kappa_2^2 + \alpha_2 \kappa_3 \kappa_2 + \alpha_3 \kappa_3^2 \kappa_2) a_2 \\
&+ (\alpha_1 \kappa_3 + \alpha_2 \kappa_2 + \alpha_3 \kappa_3 \kappa_2 + \alpha_1 \kappa_3 + 2\alpha_1 \kappa_3 + \alpha_3 \kappa_2 \kappa_3 \\
&+ \alpha_2 \kappa_3^2 + \alpha_3 \kappa_3^3) a_3.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
f([a_1, a_3]) &= f(\kappa_2 a_2 + \kappa_3 a_3) = \kappa_2 f(a_2) + \kappa_3 f(a_3) \\
&= \kappa_2 ((2\alpha_1 + \alpha_3 \kappa_2) a_2 + (\alpha_2 + \alpha_3 \kappa_3) a_3) \\
&+ \kappa_3 ((\alpha_2 \kappa_2 + \alpha_3 \kappa_3 \kappa_2) a_2 + (3\alpha_1 + \alpha_3 \kappa_2 + \alpha_2 \kappa_3 + \alpha_3 \kappa_3^2) a_3) \\
&= (\kappa_2 (2\alpha_1 + \alpha_3 \kappa_2) + \kappa_3 (\alpha_2 \kappa_2 + \alpha_3 \kappa_3 \kappa_2)) a_2 \\
&+ (\kappa_2 (\alpha_2 + \alpha_3 \kappa_3) + (3\alpha_1 + \alpha_3 \kappa_2 + \alpha_2 \kappa_3 + \alpha_3 \kappa_3^2)) a_3 \\
&= (2\alpha_1 \kappa_2 + \alpha_3 \kappa_2^2 + \kappa_3 \alpha_2 \kappa_2 + \alpha_3 \kappa_3^2 \kappa_2) a_2 \\
&+ (\kappa_2 \alpha_2 + \kappa_2 \alpha_3 \kappa_3 + 3\alpha_1 + \alpha_3 \kappa_2 + \alpha_2 \kappa_3 + \alpha_3 \kappa_3^2) a_3.
\end{aligned}$$

It follows that

$$\begin{gathered}2\alpha_1\kappa_2+\alpha_3\kappa_2^2+\kappa_3\alpha_2\kappa_2+\alpha_3\kappa_3^2\kappa_2\\=\alpha_1\kappa_2+\alpha_1\kappa_2+2\alpha_1\kappa_2+\alpha_3\kappa_2^2+\alpha_2\kappa_3\kappa_2+\alpha_3\kappa_3^2\kappa_2\end{gathered}$$

and

$$\begin{gathered}\kappa_2\alpha_2+\kappa_2\alpha_3\kappa_3+3\alpha_1+\alpha_3\kappa_2+\alpha_2\kappa_3+\alpha_3\kappa_3^2\\=\alpha_1\kappa_3+\alpha_2\kappa_2+\alpha_3\kappa_3\kappa_2+\alpha_1\kappa_3+2\alpha_1\kappa_3+\alpha_3\kappa_2\kappa_3+\alpha_2\kappa_3^2+\alpha_3\kappa_3^3.\end{gathered}$$

Then we obtain $2\alpha_1\kappa_2=0$ and $3\alpha_1+\alpha_3\kappa_2+\alpha_2\kappa_3=4\alpha_1\kappa_3+\alpha_3\kappa_2\kappa_3+\alpha_2\kappa_3^2$. Since $\kappa_2\neq 0$, $2\alpha_1=0$, and we come to

$$\alpha_1+\alpha_3\kappa_2+\alpha_2\kappa_3=\kappa_3(\alpha_3\kappa_2+\alpha_2\kappa_3)\quad\text{or}\quad\alpha_1=(\kappa_3-1)(\alpha_3\kappa_2+\alpha_2\kappa_3).$$

As we can see, the following two situations appear: $\alpha_1=0$ and $\alpha_1\neq 0$, and then $char(F)=2$.

Suppose first that $\alpha_1=0$, then $0=(\kappa_3-1)(\alpha_3\kappa_2+\alpha_2\kappa_3)$. We obtained two subcases: $\kappa_3=1$ or $\kappa_3\neq 1$ and $\alpha_3\kappa_2+\alpha_2\kappa_3=0$. In the first case, $\Xi(f)$ is the following matrix:

$$\begin{pmatrix}0&0&0\\\alpha_2&\alpha_3\kappa_2&\alpha_2\kappa_2+\alpha_3\kappa_2\\\alpha_3&\alpha_2+\alpha_3&\alpha_3\kappa_2+\alpha_2+\alpha_3\end{pmatrix},$$

$\alpha_2,\alpha_3\in F$.

In the second case, $\alpha_3=-\kappa_2^{-1}\kappa_3\alpha_2$ and $\Xi(f)$ is the following matrix:

$$\begin{pmatrix}0&0&0\\\alpha_2&-\alpha_2\kappa_3&\alpha_2\kappa_2-\alpha_2\kappa_3^2\\-\kappa_2^{-1}\kappa_3\alpha_2&\alpha_2-\kappa_2^{-1}\kappa_3^2\alpha_2&2\alpha_2\kappa_3-\kappa_2^{-1}\alpha_2\kappa_3^3\end{pmatrix},$$

$\alpha_2,\alpha_3\in F$.

Suppose now that $\alpha_1\neq 0$ and $char(F)=2$. Then we come to the equalities $\alpha_1+\alpha_3\kappa_2+\alpha_2\kappa_3=\kappa_3(\alpha_3\kappa_2+\alpha_2\kappa_3)$ or $\alpha_1=(\kappa_3-1)(\alpha_3\kappa_2+\alpha_2\kappa_3)$. Here

$$\begin{aligned}&\alpha_1+\alpha_3\kappa_2+\alpha_2\kappa_3+\alpha_3\kappa_3^2\\&=(\kappa_3-1)(\alpha_3\kappa_2+\alpha_2\kappa_3)+\alpha_3\kappa_2+\alpha_2\kappa_3+\alpha_3\kappa_3^2\\&=\alpha_3\kappa_2\kappa_3+\alpha_2\kappa_3^2-\alpha_3\kappa_2-\alpha_2\kappa_3+\alpha_3\kappa_2+\alpha_2\kappa_3+\alpha_3\kappa_3^2\\&=\alpha_3\kappa_2\kappa_3+\alpha_2\kappa_3^2+\alpha_3\kappa_3^2,\end{aligned}$$

so that $\Xi(f)$ is the following matrix:

$$\begin{pmatrix} (\kappa_3-1)(\alpha_3\kappa_2+\alpha_2\kappa_3) & 0 & 0 \\ \alpha_2 & \alpha_2\kappa_3 & \alpha_2\kappa_2+\alpha_3\kappa_3\kappa_2 \\ \alpha_3 & \alpha_2+\alpha_3\kappa_3 & \alpha_3\kappa_2\kappa_3+\alpha_2\kappa_3^2+\alpha_3\kappa_3^3 \end{pmatrix},$$

$\alpha_2, \alpha_3 \in F$.

Conversely, let f be the linear transformation of L, defined by the above matrices. We found the elements $f([a_1, a_j])$ from the equations

$$f([a_1, a_j]) = [f(a_1), a_j] + [a_1, f(a_j)],\ j \in \{1, 2, 3\}.$$

For commutators $[a_j, a_k]$, $j \in \{2, 3\}$, we have $[a_j, a_k] = 0$. By Lemma 2 $f(a_j) \in Leib(L)$, and it follows that $[f(a_j), x] = 0$ for each element $x \in L$. Then we obtain

$$f([a_j, a_k]) = f(0) = 0,\ [f(a_j), a_k] + [a_j, f(a_k)] = 0 + 0 = 0,$$

so that $f([a_j, a_k]) = [f(a_j), a_k] + [a_j, f(a_k)]$, $j, k \in \{1, 2, 3\}$. Lemma 3 implies that f is a derivation of L.

Consider now the structure of D in more detail. Assume first that $\alpha_1 = 0$ and $\kappa_3 = 1$. As we have seen above, in this case, $\Xi(D)$ consists of the following matrices:

$$\begin{pmatrix} 0 & 0 & 0 \\ \alpha_2 & \alpha_3\kappa_2 & \alpha_2\kappa_2+\alpha_3\kappa_2 \\ \alpha_3 & \alpha_2+\alpha_3 & \alpha_3\kappa_2+\alpha_2+\alpha_3 \end{pmatrix},$$

$\alpha_2, \alpha_3 \in F$.

Using Lemma 4 we obtain that the mapping

$$\vartheta : \begin{pmatrix} 0 & 0 & 0 \\ \alpha_2 & \alpha_3\kappa_2 & \alpha_2\kappa_2+\alpha_3\kappa_2 \\ \alpha_3 & \alpha_2+\alpha_3 & \alpha_3\kappa_2+\alpha_2+\alpha_3 \end{pmatrix} \to \begin{pmatrix} \alpha_3\kappa_2 & \alpha_2\kappa_2+\alpha_3\kappa_2 \\ \alpha_2+\alpha_3 & \alpha_3\kappa_2+\alpha_2+\alpha_3 \end{pmatrix}.$$

is a homomorphism.

We can see that $Ker(\vartheta)$ is zero, so in this case $\Xi(D)$ is isomorphic to the Lie subalgebra of $M_2(F)$, consisting of the matrices, having the following form

$$\begin{pmatrix} \alpha_3\kappa_2 & \alpha_2\kappa_2+\alpha_3\kappa_2 \\ \alpha_2+\alpha_3 & \alpha_3\kappa_2+\alpha_2+\alpha_3 \end{pmatrix}.$$

We have

$$XY = \begin{pmatrix} \alpha_3\kappa_2 & \alpha_2\kappa_2 + \alpha_3\kappa_2 \\ \alpha_2 + \alpha_3 & \alpha_3\kappa_2 + \alpha_2 + \alpha_3 \end{pmatrix} \begin{pmatrix} \sigma_3\kappa_2 & \sigma_2\kappa_2 + \sigma_3\kappa_2 \\ \sigma_2 + \sigma_3 & \sigma_3\kappa_2 + \sigma_2 + \sigma_3 \end{pmatrix}$$
$$= \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix},$$

where

$$x_{11} = \alpha_3\sigma_3\kappa_2^2 + (\alpha_2 + \alpha_3)(\sigma_2 + \sigma_3)\kappa_2,$$
$$x_{12} = \alpha_3(\sigma_2 + \sigma_3)\kappa_2^2 + (\alpha_2\kappa_2 + \alpha_3\kappa_2)(\sigma_3\kappa_2 + \sigma_2 + \sigma_3),$$
$$x_{21} = (\alpha_2 + \alpha_3)\sigma_3\kappa_2 + (\alpha_3\kappa_2 + \alpha_2 + \alpha_3)(\sigma_2 + \sigma_3),$$
$$x_{22} = (\alpha_2 + \alpha_3)(\sigma_2 + \sigma_3)\kappa_2 + (\alpha_3\kappa_2 + \alpha_2 + \alpha_3)(\sigma_3\kappa_2 + \sigma_2 + \sigma_3),$$

and

$$YX = \begin{pmatrix} \sigma_3\kappa_2 & \sigma_2\kappa_2 + \sigma_3\kappa_2 \\ \sigma_2 + \sigma_3 & \sigma_3\kappa_2 + \sigma_2 + \sigma_3 \end{pmatrix} \begin{pmatrix} \alpha_3\kappa_2 & \alpha_2\kappa_2 + \alpha_3\kappa_2 \\ \alpha_2 + \alpha_3 & \alpha_3\kappa_2 + \alpha_2 + \alpha_3 \end{pmatrix}$$
$$= \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix},$$

where

$$y_{11} = \alpha_3\sigma_3\kappa_2^2 + (\alpha_2 + \alpha_3)(\sigma_2 + \sigma_3)\kappa_2,$$
$$y_{12} = \sigma_3(\alpha_2 + \alpha_3)\kappa_2^2 + (\sigma_2\kappa_2 + \sigma_3\kappa_2)(\alpha_3\kappa_2 + \alpha_2 + \alpha_3),$$
$$y_{21} = (\sigma_2 + \sigma_3)\alpha_3\kappa_2 + (\sigma_3\kappa_2 + \sigma_2 + \sigma_3)(\alpha_2 + \alpha_3),$$
$$y_{22} = (\sigma_2 + \sigma_3)(\alpha_2 + \alpha_3)\kappa_2 + (\sigma_3\kappa_2 + \sigma_2 + \sigma_3)(\alpha_3\kappa_2 + \alpha_2 + \alpha_3).$$

We have

$$\alpha_3(\sigma_2 + \sigma_3)\kappa_2^2 + (\alpha_2\kappa_2 + \alpha_3\kappa_2)(\sigma_3\kappa_2 + \sigma_2 + \sigma_3)$$
$$= \alpha_3\sigma_2\kappa_2^2 + \alpha_3\sigma_3\kappa_2^2 + \alpha_2\sigma_3\kappa_2^2 + \alpha_2\kappa_2\sigma_2 + \alpha_2\kappa_2\sigma_3 + \alpha_3\sigma_3\kappa_2^2$$
$$+\alpha_3\kappa_2\sigma_2 + \alpha_3\kappa_2\sigma_3;$$
$$\sigma_3(\alpha_2 + \alpha_3)\kappa_2^2 + (\sigma_2\kappa_2 + \sigma_3\kappa_2)(\alpha_3\kappa_2 + \alpha_2 + \alpha_3)$$
$$= \sigma_3\alpha_2\kappa_2^2 + \sigma_3\alpha_3\kappa_2^2 + \sigma_2\alpha_3\kappa_2^2 + \sigma_2\kappa_2\alpha_2 + \sigma_2\kappa_2\alpha_3 + \sigma_3\alpha_3\kappa_2^2$$
$$+\sigma_3\kappa_2\alpha_2 + \sigma_3\kappa_2\alpha_3,$$

and

$$(\alpha_2 + \alpha_3)\sigma_3\kappa_2 + (\alpha_3\kappa_2 + \alpha_2 + \alpha_3)(\sigma_2 + \sigma_3)$$
$$= \alpha_2\sigma_3\kappa_2 + \alpha_3\sigma_3\kappa_2 + \alpha_3\kappa_2\sigma_2 + \alpha_2\sigma_2 + \alpha_3\sigma_2 + \alpha_3\kappa_2\sigma_3 + \alpha_2\sigma_3 + \alpha_3\sigma_3;$$
$$(\sigma_2 + \sigma_3)\alpha_3\kappa_2 + (\sigma_3\kappa_2 + \sigma_2 + \sigma_3)(\alpha_2 + \alpha_3)$$
$$= \sigma_2\alpha_3\kappa_2 + \sigma_3\alpha_3\kappa_2 + \sigma_3\kappa_2\alpha_2 + \sigma_2\alpha_2 + \sigma_3\alpha_2 + \sigma_3\kappa_2\alpha_3 + \sigma_2\alpha_3 + \sigma_3\alpha_3.$$

Thus, we can see that $XY = YX$, so $\Xi(D)$ is abelian. It is not hard to see that this algebra has dimension 3.

Suppose now that $\alpha_1 = 0$ and $\kappa_3 \neq 1$. By equality $(\kappa_3 - 1)(\alpha_3\kappa_2 + \alpha_2\kappa_3) = \alpha_1 = 0$, we obtain that $\alpha_3\kappa_2 + \alpha_2\kappa_3 = 0$. As we have seen above, in this case, $\Xi(D)$ consists of the following matrix:

$$\begin{pmatrix} 0 & 0 & 0 \\ \alpha_2 & -\alpha_2\kappa_3 & \alpha_2\kappa_2 - \alpha_2\kappa_3^2 \\ -\kappa_2^{-1}\kappa_3\alpha_2 & \alpha_2 - \kappa_2^{-1}\kappa_3^2\alpha_2 & 2\alpha_2\kappa_3 - \kappa_2^{-1}\alpha_2\kappa_3^3 \end{pmatrix}$$
$$= \alpha_2 \begin{pmatrix} 0 & 0 & 0 \\ 1 & -\kappa_3 & \kappa_2 - \kappa_3^2 \\ -\kappa_2^{-1}\kappa_3 & 1 - \kappa_2^{-1}\kappa_3^2 & 2\kappa_3 - \kappa_2^{-1}\kappa_3^3 \end{pmatrix},$$

$\alpha_2 \in F$. Thus, we can see that $\Xi(D)$ is a one-dimensional Lie algebra generated by the matrix

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & -\kappa_3 & \kappa_2 - \kappa_3^2 \\ -\kappa_2^{-1}\kappa_3 & 1 - \kappa_2^{-1}\kappa_3^2 & 2\kappa_3 - \kappa_2^{-1}\kappa_3^3 \end{pmatrix}.$$

Suppose now that $\alpha_1 \neq 0$ and $char(F) = 2$. As we have seen above, in this case, $\Xi(f)$ consists of the following matrices

$$\begin{pmatrix} (\kappa_3 - 1)(\alpha_3\kappa_2 + \alpha_2\kappa_3) & 0 & 0 \\ \alpha_2 & \alpha_2\kappa_3 & \alpha_2\kappa_2 + \alpha_3\kappa_3\kappa_2 \\ \alpha_3 & \alpha_2 + \alpha_3\kappa_3 & \alpha_3\kappa_2\kappa_3 + \alpha_2\kappa_3^2 + \alpha_3\kappa_3^2 \end{pmatrix},$$

$\alpha_2, \alpha_3 \in F$. Using Lemma 4 again we obtain that the mapping

$$\vartheta : \begin{pmatrix} (\kappa_3 - 1)(\alpha_3\kappa_2 + \alpha_2\kappa_3) & 0 & 0 \\ \alpha_2 & \alpha_2\kappa_3 & \alpha_2\kappa_2 + \alpha_3\kappa_3\kappa_2 \\ \alpha_3 & \alpha_2 + \alpha_3\kappa_3 & \alpha_3\kappa_2\kappa_3 + \alpha_2\kappa_3^2 + \alpha_3\kappa_3^2 \end{pmatrix}$$
$$\to \begin{pmatrix} \alpha_2\kappa_3 & \alpha_2\kappa_2 + \alpha_3\kappa_3\kappa_2 \\ \alpha_2 + \alpha_3\kappa_3 & \alpha_3\kappa_2\kappa_3 + \alpha_2\kappa_3^2 + \alpha_3\kappa_3^2 \end{pmatrix}$$

is a homomorphism. We can see that $Ker(\vartheta)$ is zero, so in this case $\Xi(D)$ is the Lie subalgebra of $M_2(F)$, consisting of the matrices, having the following form

$$\begin{pmatrix} \alpha_2\kappa_3 & \alpha_2\kappa_2 + \alpha_3\kappa_3\kappa_2 \\ \alpha_2 + \alpha_3\kappa_3 & \alpha_3\kappa_2\kappa_3 + \alpha_2\kappa_3^2 + \alpha_3\kappa_3^3 \end{pmatrix},$$

$\alpha_2, \alpha_3 \in F$.

Put

$$\begin{aligned}\beta_{11} &= \alpha_2\kappa_3,\ \beta_{21} = \alpha_2 + \alpha_3\kappa_3,\\ \beta_{12} &= \kappa_2(\alpha_2 + \alpha_3\kappa_3) = \kappa_2\beta_{21},\\ \beta_{22} &= \alpha_3\kappa_2\kappa_3 + \alpha_2\kappa_3^2 + \alpha_3\kappa_3^2.\end{aligned}$$

Let S be a Lie subset of a Lie algebra $M_2(F)$ of matrices having a form

$$\begin{pmatrix} \mu_1 & \kappa_2\mu_2 \\ \mu_2 & \mu_3 \end{pmatrix},$$

$\mu_1, \mu_2, \mu_3 \in F$. By Lemma 5 S is a Lie subalgebra of $M_2(F)$, moreover, $S = Z \oplus L$ where Z is the center of $M_2(F)$ and L is a non-abelian Lie subalgebra of dimension 2. We can see that $\vartheta(\Xi(D))$ is a subalgebra of S. In particular, it follows that

$$dim_F(D) = dim_F(\Xi(D)) = dim_F(\vartheta(\Xi(D))) \leq 3.$$

We have

$$\begin{pmatrix} \alpha_2\kappa_3 & \alpha_2\kappa_2 + \alpha_3\kappa_3\kappa_2 \\ \alpha_2 + \alpha_3\kappa_3 & \alpha_3\kappa_2\kappa_3 + \alpha_2\kappa_3^2 + \alpha_3\kappa_3^2 \end{pmatrix} \begin{pmatrix} \mu_2\kappa_3 & \mu_2\kappa_2 + \mu_3\kappa_3\kappa_2 \\ \mu_2 + \mu_3\kappa_3 & \mu_3\kappa_2\kappa_3 + \mu_2\kappa_3^2 + \mu_3\kappa_3^2 \end{pmatrix}$$
$$= \begin{pmatrix} \nu_{11} & \nu_{12} \\ \nu_{21} & \nu_{22} \end{pmatrix},$$

where

$$\begin{aligned}\nu_{11} &= \alpha_2\kappa_3\mu_2\kappa_3 + (\alpha_2\kappa_2 + \alpha_3\kappa_3\kappa_2)(\mu_2 + \mu_3\kappa_3)\\ &= \alpha_2\mu_2\kappa_3^2 + \alpha_2\mu_2\kappa_2 + \alpha_3\mu_2\kappa_2\kappa_3 + \alpha_2\mu_3\kappa_2\kappa_3 + \alpha_3\mu_3\kappa_2\kappa_3^2,\\ \nu_{12} &= \alpha_2\kappa_3(\mu_2\kappa_2 + \mu_3\kappa_3\kappa_2)\\ &+ (\alpha_2\kappa_2 + \alpha_3\kappa_2\kappa_3)(\mu_3\kappa_2\kappa_3 + \mu_2\kappa_3^2 + \mu_3\kappa_3^2)\\ &= \alpha_2\mu_2\kappa_2\kappa_3 + \alpha_2\mu_3\kappa_2\kappa_3^2 + \alpha_2\mu_3\kappa_2^2\kappa_3 + \alpha_2\mu_2\kappa_2\kappa_3^2\\ &+ \alpha_2\mu_3\kappa_2\kappa_3^2 + \alpha_3\mu_3\kappa_2^2\kappa_3^2 + \alpha_3\mu_2\kappa_2\kappa_3^3 + \alpha_3\mu_3\kappa_2\kappa_3^3\\ &= \alpha_2\mu_2\kappa_2\kappa_3 + \alpha_2\mu_3\kappa_2^2\kappa_3 + \alpha_2\mu_2\kappa_2\kappa_3^2\\ &+ \alpha_3\mu_3\kappa_2^2\kappa_3^2 + \alpha_3\mu_2\kappa_2\kappa_3^3 + \alpha_3\mu_3\kappa_2\kappa_3^3\\ &= \kappa_2(\alpha_2\mu_2\kappa_3 + \alpha_2\mu_3\kappa_2\kappa_3 + \alpha_2\mu_2\kappa_3^2\\ &+ \alpha_3\mu_3\kappa_2\kappa_3^2 + \alpha_3\mu_2\kappa_3^3 + \alpha_3\mu_3\kappa_3^3),\end{aligned}$$

and

$$
\begin{aligned}
\nu_{21} &= (\alpha_2+\alpha_3\kappa_2)\mu_2\kappa_3 + (\alpha_3\kappa_2\kappa_3+\alpha_2\kappa_3^2+\alpha_3\kappa_3^2)(\mu_2+\mu_3\kappa_3)\\
&= \alpha_2\mu_2\kappa_3 + \alpha_3\mu_2\kappa_3^2 + \alpha_3\mu_2\kappa_2\kappa_3 + \alpha_2\mu_2\kappa_3^2\\
&+ \alpha_3\mu_2\kappa_3^2 + \alpha_3\mu_3\kappa_2\kappa_3^2 + \alpha_2\mu_3\kappa_3^3 + \alpha_3\mu_3\kappa_3^3\\
&= \alpha_2\mu_2\kappa_3 + \alpha_3\mu_2\kappa_2\kappa_3 + \alpha_2\mu_2\kappa_3^2\\
&+ \alpha_3\mu_3\kappa_2\kappa_3^2 + \alpha_2\mu_3\kappa_3^3 + \alpha_3\mu_3\kappa_3^3,\\
\nu_{22} &= (\alpha_2+\alpha_3\kappa_3)(\mu_2\kappa_2+\mu_3\kappa_3\kappa_2)\\
&+ (\alpha_3\kappa_2\kappa_3+\alpha_2\kappa_3^2+\alpha_3\kappa_3^2)(\mu_3\kappa_2\kappa_3+\mu_2\kappa_3^2+\mu_3\kappa_3^2)\\
&= \alpha_2\mu_2\kappa_2 + \alpha_2\mu_3\kappa_2\kappa_3 + \alpha_3\mu_2\kappa_2\kappa_3 + \alpha_3\mu_3\kappa_2\kappa_3^2\\
&+ \alpha_3\mu_3\kappa_2^2\kappa_3^2 + \alpha_3\mu_2\kappa_2\kappa_3^3 + \alpha_3\mu_3\kappa_2\kappa_3^3 + \alpha_2\mu_3\kappa_2\kappa_3^3\\
&+ \alpha_2\mu_2\kappa_3^4 + \alpha_2\mu_3\kappa_3^4 + \alpha_3\mu_3\kappa_2\kappa_3^3 + \alpha_3\mu_2\kappa_3^4 + \alpha_3\mu_3\kappa_3^4.
\end{aligned}
$$

Furthermore

$$
\begin{pmatrix} \mu_2\kappa_3 & \mu_2\kappa_2+\mu_3\kappa_3\kappa_2\\ \mu_2+\mu_3\kappa_3 & \mu_3\kappa_2\kappa_3+\mu_2\kappa_3^2+\mu_3\kappa_3^2 \end{pmatrix}
\begin{pmatrix} \alpha_2\kappa_3 & \alpha_2\kappa_2+\alpha_3\kappa_3\kappa_2\\ \alpha_2+\alpha_3\kappa_3 & \alpha_3\kappa_2\kappa_3+\alpha_2\kappa_3^2+\alpha_3\kappa_3^2 \end{pmatrix}
= \begin{pmatrix} \tau_{11} & \tau_{12}\\ \tau_{21} & \tau_{22} \end{pmatrix},
$$

where

$$
\begin{aligned}
\tau_{11} &= \mu_2\alpha_2\kappa_3^2 + \mu_2\alpha_2\kappa_2 + \mu_3\alpha_2\kappa_2\kappa_3 + \mu_2\alpha_3\kappa_2\kappa_3 + \mu_3\alpha_3\kappa_2\kappa_3^2,\\
\tau_{12} &= \kappa_2(\mu_2\alpha_2\kappa_3 + \mu_2\alpha_3\kappa_2\kappa_3 + \mu_2\alpha_2\kappa_3^2 + \mu_3\alpha_3\kappa_2\kappa_3^2\\
&+ \mu_3\alpha_2\kappa_3^3 + \mu_3\alpha_3\kappa_3^3),\\
\tau_{21} &= \mu_2\alpha_2\kappa_3 + \mu_3\alpha_2\kappa_2\kappa_3 + \mu_2\alpha_2\kappa_3^2 + \mu_3\alpha_3\kappa_2\kappa_3^2\\
&+ \mu_2\alpha_3\kappa_3^3 + \mu_3\alpha_3\kappa_3^3,\\
\tau_{22} &= \mu_2\alpha_2\kappa_2 + \mu_2\alpha_3\kappa_2\kappa_3 + \mu_3\alpha_2\kappa_2\kappa_3 + \mu_3\alpha_3\kappa_2\kappa_3^2\\
&+ \mu_3\alpha_3\kappa_2^2\kappa_3^2 + \mu_3\alpha_2\kappa_2\kappa_3^3 + \mu_3\alpha_3\kappa_2\kappa_3^3 + \mu_2\alpha_3\kappa_2\kappa_3^3\\
&+ \mu_2\alpha_2\kappa_3^4 + \mu_2\alpha_3\kappa_3^4 + \mu_3\alpha_3\kappa_2\kappa_3^3 + \mu_3\alpha_2\kappa_3^4 + \mu_3\alpha_3\kappa_3^4.
\end{aligned}
$$

Now we obtain a commutator of these two matrices:

$$
\begin{aligned}
\nu_{11}+\tau_{11} &= \alpha_2\mu_2\kappa_3^2 + \alpha_2\mu_2\kappa_2 + \alpha_3\mu_2\kappa_2\kappa_3 + \alpha_2\mu_3\kappa_2\kappa_3\\
&+ \alpha_3\mu_3\kappa_2\kappa_3^2 + \mu_2\alpha_2\kappa_3^2 + \mu_2\alpha_2\kappa_2 + \mu_3\alpha_2\kappa_2\kappa_3\\
&+ \mu_2\alpha_3\kappa_2\kappa_3 + \mu_3\alpha_3\kappa_2\kappa_3^2\\
&= 0,
\end{aligned}
$$

$$\begin{aligned}
\nu_{12} + \tau_{12} &= \kappa_2(\alpha_2\mu_2\kappa_3 + \alpha_2\mu_3\kappa_2\kappa_3 + \alpha_2\mu_2\kappa_3^2 \\
&+ \alpha_3\mu_3\kappa_2\kappa_3^2 + \alpha_3\mu_2\kappa_3^3 + \alpha_3\mu_3\kappa_3^3) \\
&+ \kappa_2(\mu_2\alpha_2\kappa_3 + \mu_2\alpha_3\kappa_2\kappa_3 + \mu_2\alpha_2\kappa_3^2 \\
&+ \mu_3\alpha_3\kappa_2\kappa_3^2 + \mu_3\alpha_2\kappa_3^3 + \mu_3\alpha_3\kappa_3^3) \\
&= \kappa_2(\alpha_2\mu_3\kappa_2\kappa_3 + \alpha_3\mu_2\kappa_3^3 + \mu_2\alpha_3\kappa_2\kappa_3 + \mu_3\alpha_2\kappa_3^3), \\
\nu_{21} + \tau_{21} &= \alpha_2\mu_2\kappa_3 + \alpha_3\mu_2\kappa_2\kappa_3 + \alpha_2\mu_2\kappa_3^2 \\
&+ \alpha_3\mu_3\kappa_2\kappa_3^2 + \alpha_2\mu_3\kappa_3^3 + \alpha_3\mu_3\kappa_3^3 \\
&+ \mu_2\alpha_2\kappa_3 + \mu_3\alpha_2\kappa_2\kappa_3 + \mu_2\alpha_2\kappa_3^2 \\
&+ \mu_3\alpha_3\kappa_2\kappa_3^2 + \mu_2\alpha_3\kappa_3^3 + \mu_3\alpha_3\kappa_3^3 \\
&= \alpha_3\mu_2\kappa_2\kappa_3 + \alpha_2\mu_3\kappa_3^3 + \mu_3\alpha_2\kappa_2\kappa_3 + \mu_2\alpha_3\kappa_3^3, \\
\nu_{22} + \tau_{22} &= \alpha_2\mu_2\kappa_2 + \alpha_2\mu_3\kappa_2\kappa_3 + \alpha_3\mu_2\kappa_2\kappa_3 + \alpha_3\mu_3\kappa_2\kappa_3^2 \\
&+ \alpha_3\mu_3\kappa_2^2\kappa_3^2 + \alpha_3\mu_2\kappa_2\kappa_3^3 + \alpha_3\mu_3\kappa_2\kappa_3^3 + \alpha_2\mu_3\kappa_2\kappa_3^3 \\
&+ \alpha_2\mu_2\kappa_3^4 + \alpha_2\mu_3\kappa_3^4 + \alpha_3\mu_3\kappa_2\kappa_3^3 + \alpha_3\mu_2\kappa_3^4 + \alpha_3\mu_3\kappa_3^4 \\
&+ \mu_2\alpha_2\kappa_2 + \mu_2\alpha_3\kappa_2\kappa_3 + \mu_3\alpha_2\kappa_2\kappa_3 + \mu_3\alpha_3\kappa_2\kappa_3^2 \\
&+ \mu_3\alpha_3\kappa_2^2\kappa_3^2 + \mu_3\alpha_2\kappa_2\kappa_3^3 + \mu_3\alpha_3\kappa_2\kappa_3^3 + \mu_2\alpha_3\kappa_2\kappa_3^3 \\
&+ \mu_2\alpha_2\kappa_3^4 + \mu_2\alpha_3\kappa_3^4 + \mu_3\alpha_3\kappa_2\kappa_3^3 + \mu_3\alpha_2\kappa_3^4 + \mu_3\alpha_3\kappa_3^4 \\
&= 0.
\end{aligned}$$

Thus we can see that if $\alpha_3\mu_2\kappa_2\kappa_3 + \alpha_2\mu_3\kappa_3^3 + \mu_3\alpha_2\kappa_2\kappa_3 + \mu_2\alpha_3\kappa_3^3 = 0$, then $\vartheta(\Xi(D))$ is abelian. If $\kappa_2 = \kappa_3^2$ then

$$\begin{gathered}
\alpha_3\mu_2\kappa_2\kappa_3 + \alpha_2\mu_3\kappa_3^3 + \mu_3\alpha_2\kappa_2\kappa_3 + \mu_2\alpha_3\kappa_3^3 \\
= \alpha_3\mu_2\kappa_3^3 + \alpha_2\mu_3\kappa_3^3 + \mu_3\alpha_2\kappa_3^3 + \mu_2\alpha_3\kappa_3^3 \\
= 2\alpha_3\mu_2\kappa_3^3 + 2\alpha_2\mu_3\kappa_3^3 = 0,
\end{gathered}$$

so that in this case $\vartheta(\Xi(D))$ is abelian.

Suppose that $\vartheta(\Xi(D)) \cap Z \neq \langle 0 \rangle$ and let

$$\begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix} \in \vartheta(\Xi(D)) \cap Z.$$

Then we obtain $\alpha_2\kappa_3 = \sigma$, $\alpha_2 + \alpha_3\kappa_3 = 0$, $\alpha_3\kappa_2\kappa_3 + \alpha_2\kappa_3^2 + \alpha_3\kappa_3^2 = \sigma$.

$$\begin{pmatrix} \alpha_2\kappa_3 & \alpha_2\kappa_2 + \alpha_3\kappa_3\kappa_2 \\ \alpha_2 + \alpha_3\kappa_3 & \alpha_3\kappa_2\kappa_3 + \alpha_2\kappa_3^2 + \alpha_3\kappa_3^2 \end{pmatrix} = \begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix}$$

or $\alpha_2\kappa_3 = \sigma$, $\alpha_2 + \alpha_3\kappa_3 = 0$, $\alpha_3\kappa_2\kappa_3 + \alpha_2\kappa_3^2 + \alpha_3\kappa_3^2 = \sigma$. It follows that $\alpha_2 = \sigma\kappa_3^{-1}$, $\alpha_3 = \sigma\kappa_3^{-2}$. But in this case

$$\begin{aligned}\alpha_3\kappa_2\kappa_3 + \alpha_2\kappa_3^2 + \alpha_3\kappa_3^2 &= \sigma\kappa_3^{-2}\kappa_2\kappa_3 + \sigma\kappa_3^{-1}\kappa_3^2 + \sigma\kappa_3^{-2}\kappa_3^2 \\ &= \sigma\kappa_3^{-1}\kappa_2 + \sigma\kappa_3 + \sigma = \sigma\kappa_3^{-1}(\kappa_2 + \kappa_3^2 + \kappa_3).\end{aligned}$$

It follows that $\kappa_3^{-1}(\kappa_2 + \kappa_3^2 + \kappa_3) = 1$, and we obtain that in this case $\kappa_2 = \kappa_3^2$. Hence, if $\kappa_2 = \kappa_3^2$, then $\vartheta(\Xi(D))$ is abelian. Moreover, it includes the center of a Lie algebra $M_2(F)$. Since S is not abelian, we obtain that $dim_F(\vartheta(\Xi(D))) = 2$. A polynomial

$$X^2 + \kappa_3 X - \kappa_2 = X^2 + \kappa_3 X + \kappa_3^2$$

has no roots in a field F.

Suppose now that $\kappa_2 \neq \kappa_3^2$. As we have seen above, then

$$\vartheta(\Xi(D)) \cap Z = \langle 0 \rangle$$

and we obtain that

$$\vartheta(\Xi(D)) \cong \vartheta(\Xi(D))/(\vartheta(\Xi(D)) \cap Z) \cong (\vartheta(\Xi(D)) + Z)/Z \leq S/Z \cong L.$$

In particular, $dim_F(\vartheta(\Xi(D))) \leq 2$. Let

$$B = \begin{pmatrix} \kappa_3 & \kappa_2 \\ 1 & \kappa_3^2 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & \kappa_3\kappa_2 \\ \kappa_3 & \kappa_2\kappa_3 + \kappa_3^2 \end{pmatrix},$$

then $B, C \in \vartheta(\Xi(D))$.

Suppose that λ, μ be the elements of a field F such that $\lambda B + \mu C = O$. We have

$$\begin{aligned}\lambda B + \mu C &= \lambda \begin{pmatrix} \kappa_3 & \kappa_2 \\ 1 & \kappa_3^2 \end{pmatrix} + \mu \begin{pmatrix} 0 & \kappa_3\kappa_2 \\ \kappa_3 & \kappa_2\kappa_3 + \kappa_3^2 \end{pmatrix} \\ &= \begin{pmatrix} \lambda\kappa_3 & \lambda\kappa_3 + \mu\kappa_3\kappa_2 \\ \lambda + \mu\kappa_3 & \lambda\kappa_3^2 + \mu\kappa_2\kappa_3 + \mu\kappa_3^2 \end{pmatrix}.\end{aligned}$$

It follows that $\lambda\kappa_3 = 0$, $\lambda + \mu\kappa_3 = 0$, $\lambda\kappa_3^2 + \mu\kappa_2\kappa_3 + \mu\kappa_3^2 = 0$. Since $\kappa_3 \neq 0$, we obtain that $\lambda = 0$, $\mu = 0$. Hence, the matrices B and C are linearly independent. It follows that $dim_F(\vartheta(\Xi(D))) = 2$. By above proved we obtain that $\vartheta(\Xi(D)) \cong L$, so that

$$D \cong \Xi(D) \cong \vartheta(\Xi(D))$$

is a non-abelian Lie algebra of dimension 2. □

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