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Multiplicative Jordan triple (θ, ϕ)-derivations of rings and standard operator algebras

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Abstract. In this paper we study the additivity and the characterization of multiplicative Jordan triple (θ, ϕ)-derivations of rings. As a consequence, we show that multiplicative Jordan triple (θ, ϕ)-derivations of standard operator algebras are (θ, ϕ)-derivations.

Introduction

The study of the additivity of multiplicative derivation type maps on rings have attracted the attention of many algebraists. The first result about the additivity of multiplicative derivation of a ring (a concept based on the notion of a derivation) was given by Daif [3]. Lu [9] studied the additivity of multiplicative Jordan triple derivations of rings (also called of Jordan semitriple derivable maps), a concept based on the notion of a Jordan triple derivation, and Jing and Lu [5] generalized the results of [9] to a larger class of rings. Motivated by these facts, in this paper we present the notion of a multiplicative Jordan triple (θ, φ)-derivation, a concept based on the notion of a Jordan triple (θ, φ)-derivation, presented by Liu and Shiue [8], and we present a study on its additivity. As an application we prove that multiplicative Jordan triple (θ, φ)-derivations of standard operator algebras are (θ, φ)-derivations.

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1. Multiplicative Jordan triple (θ, ϕ)-derivations of rings

Let $\mathscr{R}$ be a ring and $\theta, \phi : \mathscr{R} \to \mathscr{R}$ endomorphisms of $\mathscr{R}$. A map $\delta : \mathscr{R} \to \mathscr{R}$ is called a *multiplicative* (θ, ϕ)*-derivation of* $\mathscr{R}$ if the following condition is satisfied:

$$\delta(ab) = \delta(a)\theta(b) + \phi(a)\delta(b),$$

for all elements $a, b \in \mathscr{R}$. An additive multiplicative (θ, ϕ)-derivation of $\mathscr{R}$ is called a (θ, ϕ)*-derivation of* $\mathscr{R}$.

A map $\delta : \mathscr{R} \to \mathscr{R}$ is called a *multiplicative Jordan triple* (θ, ϕ)*-derivation of* $\mathscr{R}$ if the following condition is satisfied:

$$\delta(aba) = \delta(a)\theta(b)\theta(a) + \phi(a)\delta(b)\theta(a) + \phi(a)\phi(b)\delta(a),$$

for all elements $a, b \in \mathscr{R}$. An additive multiplicative Jordan triple (θ, ϕ)-derivation of $\mathscr{R}$ is called a *Jordan triple* (θ, ϕ)*-derivation of* $\mathscr{R}$.

Our main result in this section reads as follows.

Theorem 1. *Let* $\mathscr{R}$ *be a ring containing a non-trivial idempotent* e_1, $\mathscr{R} = \bigoplus_{i,j=1,2} \mathscr{R}_{ij}$ *the Peirce decomposition of* $\mathscr{R}$, *relative to* e_1, *and* $\theta, \phi : \mathscr{R} \to \mathscr{R}$ *endomorphisms of* $\mathscr{R}$. *Suppose further that for* $t \in \mathscr{R}$ *the following properties hold:*

(♣) $\phi(x_{ij})t\theta(x_{ij}) = 0$, *for all* $x_{ij} \in \mathscr{R}_{ij}$ $(i, j = 1, 2)$, *implies* $t = 0$,

(♠) $\phi(x_{ij})t\theta(x_{ij}) = 0$, *for all* $x_{ij} \in \mathscr{R}_{ij}$ $(i \neq j; i, j = 1, 2)$, *implies* $t = 0$, *provided that the conditions are satisfied:*

$$\phi(x_{ji})t\theta(x_{ii}) = 0 \text{ and } \phi(x_{ii})t\theta(x_{ij}) = 0,$$

for all elements $x_{ii} \in \mathscr{R}_{ii}$, $x_{ij} \in \mathscr{R}_{ij}$ *and* $x_{ji} \in \mathscr{R}_{ji}$ $(i \neq j; i, j = 1, 2)$.

Then every multiplicative Jordan triple (θ, ϕ)*-derivation of* $\mathscr{R}$ *is additive.*

In addition, if $\mathscr{R}$ *is a* 2*-torsion free semiprime ring and* $\theta, \phi : \mathscr{R} \to \mathscr{R}$ *are automorphisms of* $\mathscr{R}$, *then every multiplicative Jordan triple* (θ, ϕ)*-derivation of* $\mathscr{R}$ *is a* (θ, ϕ)*-derivation of* $\mathscr{R}$.

To prove the Theorem 1, let $\delta : \mathscr{R} \to \mathscr{R}$ be a multiplicative Jordan triple (θ, ϕ)-derivation of $\mathscr{R}$.

Based on the techniques used by Jing and Lu [5], we shall organize the proof of Theorem 1 in a series of lemmas. The following lemma will be used throughout this paper, whose proof is elementary and therefore omitted.

Lemma 1. $\delta(0) = 0$.

The following well known result will be used throughout this paper: Let e_1 be a non-trivial idempotent of $\mathscr{R}$ and formally set $e_2 = 1_{\mathscr{R}} - e_1$ ($\mathscr{R}$ need not have an identity element $1_{\mathscr{R}}$). Then $\mathscr{R}$ has a Peirce decomposition $\mathscr{R} = \mathscr{R}_{11} \oplus \mathscr{R}_{12} \oplus \mathscr{R}_{21} \oplus \mathscr{R}_{22}$, relative to e_1, where $\mathscr{R}_{ij} = e_i\mathscr{R}e_j$ $(i, j = 1, 2)$, satisfying the following multiplicative relations: $\mathscr{R}_{ij}\mathscr{R}_{kl} \subseteq \delta_{jk}\mathscr{R}_{il}$, where δ_{jk} is the *Kronecker delta function*. More details about the Peirce decomposition and its properties, can be found in references [4] and [10].

Lemma 2. *For every elements* $a_{11} \in \mathscr{R}_{11}$, $a_{12} \in \mathscr{R}_{12}$, $a_{21} \in \mathscr{R}_{21}$ *and* $a_{22} \in \mathscr{R}_{22}$ *we have:* $\delta(a_{11} + a_{12} + a_{21} + a_{22}) = \delta(a_{11}) + \delta(a_{12}) + \delta(a_{21}) + \delta(a_{22})$.

Proof. By definition of the map δ we get

$$\begin{aligned} &\delta(x_{ij}(a_{11} + a_{12} + a_{21} + a_{22})x_{ij}) - \delta(x_{ij}a_{11}x_{ij}) - \delta(x_{ij}a_{12}x_{ij}) \\ &\qquad -\delta(x_{ij}a_{21}x_{ij}) - \delta(x_{ij}a_{22}x_{ij}) \\ &= \phi(x_{ij})(\delta(a_{11} + a_{12} + a_{21} + a_{22}) - \delta(a_{11}) - \delta(a_{12}) - \delta(a_{21}) - \delta(a_{22}))\theta(x_{ij}), \end{aligned}$$

for all $x_{ij} \in \mathscr{R}_{ij}$ $(i, j = 1, 2)$. Thus,

$$\phi(x_{ij})(\delta(a_{11} + a_{12} + a_{21} + a_{22}) - \delta(a_{11}) - \delta(a_{12}) - \delta(a_{21}) - \delta(a_{22}))\theta(x_{ij}) = 0,$$

for all $x_{ij} \in \mathscr{R}_{ij}$ $(i, j = 1, 2)$, which leads to $\delta(a_{11} + a_{12} + a_{21} + a_{22}) - \delta(a_{11}) - \delta(a_{12}) - \delta(a_{21}) - \delta(a_{22}) = 0$, by property (♣). □

Lemma 3. *For every elements* $a_{12}, b_{12} \in \mathscr{R}_{12}$, $a_{21}, b_{21} \in \mathscr{R}_{21}$ *and* $t_{22} \in \mathscr{R}_{22}$ *we have: (i)* $\delta(a_{12} + b_{12}t_{22}) = \delta(a_{12}) + \delta(b_{12}t_{22})$ *and (ii)* $\delta(a_{21} + t_{22}b_{21}) = \delta(a_{21}) + \delta(t_{22}b_{21})$.

Proof. First, note that the following identity holds:

$$e_1 + a_{12} + b_{12}t_{22} = (e_1 + a_{12} + t_{22})(e_1 + b_{12})(e_1 + a_{12} + t_{22}),$$

for all elements a_{12}, $b_{12} \in \mathscr{R}_{12}$ and $t_{22} \in \mathscr{R}_{22}$. Hence, by Lemma 2 we have

$$\begin{aligned} &\delta(e_1) + \delta(a_{12} + a_{12}t_{22}) \\ &= \delta(e_1 + a_{12} + a_{12}t_{22}) \\ &= \delta((e_1 + a_{12} + t_{22})(e_1 + b_{12})(e_1 + a_{12} + t_{22})) \\ &= \delta(e_1 + a_{12} + t_{22})\theta(e_1 + b_{12})\theta(e_1 + a_{12} + t_{22}) \end{aligned}$$

$$
\begin{aligned}
&+ \phi(e_1 + a_{12} + t_{22})\delta(e_1 + b_{12})\theta(e_1 + a_{12} + t_{22}) \\
&+ \phi(e_1 + a_{12} + t_{22})\phi(e_1 + b_{12})\delta(e_1 + a_{12} + t_{22}) \\
&= \delta(e_1 + a_{12} + t_{22})(\theta(e_1) + \theta(b_{12}))\theta(e_1 + a_{12} + t_{22}) \\
&+ \phi(e_1 + a_{12} + t_{22})(\delta(e_1) + \delta(b_{12}))\theta(e_1 + a_{12} + t_{22}) \\
&+ \phi(e_1 + a_{12} + t_{22})(\phi(e_1) + \phi(b_{12}))\delta(e_1 + a_{12} + t_{22}) \\
&= \delta(e_1 + a_{12} + t_{22})\theta(e_1)\theta(e_1 + a_{12} + t_{22}) \\
&+ \phi(e_1 + a_{12} + t_{22})\delta(e_1)\theta(e_1 + a_{12} + t_{22}) \\
&+ \phi(e_1 + a_{12} + t_{22})\phi(e_1)\delta(e_1 + a_{12} + t_{22}) \\
&+ \delta(e_1 + a_{12} + t_{22})\theta(b_{12})\theta(e_1 + a_{12} + t_{22}) \\
&+ \phi(e_1 + a_{12} + t_{22})\delta(b_{12})\theta(e_1 + a_{12} + t_{22}) \\
&+ \phi(e_1 + a_{12} + t_{22})\phi(b_{12})\delta(e_1 + a_{12} + t_{22}) \\
&= \delta((e_1 + a_{12} + t_{22})e_1(e_1 + a_{12} + t_{22})) \\
&+ \delta((e_1 + a_{12} + t_{22})b_{12}(e_1 + a_{12} + t_{22})) \\
&= \delta(e_1 + a_{12}) + \delta(b_{12}t_{22}) \\
&= \delta(e_1) + \delta(a_{12}) + \delta(b_{12}t_{22}).
\end{aligned}
$$

This immediately leads to $\delta(a_{12} + a_{12}t_{22}) = \delta(a_{12}) + \delta(a_{12}t_{22})$. Similarly, we prove that $\delta(a_{21} + t_{22}a_{21}) = \delta(a_{21}) + \delta(t_{22}a_{21})$ from the identity:

$$e_1 + a_{21} + t_{22}b_{21} = (e_1 + a_{21} + t_{22})(e_1 + b_{21})(e_1 + a_{21} + t_{22}),$$

for all elements $a_{21}, b_{21} \in \mathscr{R}_{21}$ and $t_{22} \in \mathscr{R}_{22}$. $\square$

Lemma 4. *For every elements $a_{12}, b_{12} \in \mathscr{R}_{12}$ and $a_{21}, b_{21} \in \mathscr{R}_{21}$ we have: (i) $\delta(a_{12} + b_{12}) = \delta(a_{12}) + \delta(b_{12})$ and (ii) $\delta(a_{21} + b_{21}) = \delta(a_{21}) + \delta(b_{21})$.*

Proof. On account of the definition of the map δ we have

$$
\begin{aligned}
&\delta(x_{ij}(a_{12} + b_{12})x_{ij}) - \delta(x_{ij}a_{12}x_{ij}) - \delta(x_{ij}b_{12}x_{ij}) \\
&\quad = \phi(x_{ij})(\delta(a_{12} + b_{12}) - \delta(a_{12}) - \delta(b_{12}))\theta(x_{ij}),
\end{aligned}
$$

for all $x_{ij} \in \mathscr{R}_{ij}$ $(i, j = 1, 2)$. Then by Lemma 3(ii) we see that

$$\phi(x_{ij})(\delta(a_{12} + b_{12}) - \delta(a_{12}) - \delta(b_{12}))\theta(x_{ij}) = 0,$$

for all $x_{ij} \in \mathscr{R}_{ij}$ $(i, j = 1, 2)$, which leads to $\delta(a_{12}+b_{12})-\delta(a_{12})-\delta(b_{12}) = 0$.

Similarly, we prove the case (ii). $\square$

Lemma 5. *For every elements $a_{11}, b_{11} \in \mathscr{R}_{11}$ and $a_{22}, b_{22} \in \mathscr{R}_{22}$ we have: (i) $\delta(a_{11} + b_{11}) = \delta(a_{11}) + \delta(b_{11})$ and (ii) $\delta(a_{22} + b_{22}) = \delta(a_{22}) + \delta(b_{22})$.*

Proof. By definition of the map δ we get

$$\delta(x_{ij}(a_{11}+b_{11})x_{ij}) - \delta(x_{ij}a_{11}x_{ij}) - \delta(x_{ij}b_{11}x_{ij}) = \phi(x_{ij})(\delta(a_{11}+b_{11}) - \delta(a_{11}) - \delta(b_{11}))\theta(x_{ij}),$$

for all $x_{ij} \in \mathscr{R}_{ij}$ $(i \neq j; i,j = 1,2)$. Thus

$$\phi(x_{ij})(\delta(a_{11}+b_{11}) - \delta(a_{11}) - \delta(b_{11}))\theta(x_{ij}) = 0, \tag{1}$$

for all $x_{ij} \in \mathscr{R}_{ij}$ $(i \neq j; i,j = 1,2)$. Next, we compute

$$\begin{aligned} &\delta(x_{ii}(a_{11}+b_{11})x_{ii}) + \delta(x_{ji}(a_{11}+b_{11})x_{ii}) \\ &= \delta((x_{ii}+x_{ji})(a_{11}+b_{11})(x_{ii}+x_{ji})) \\ &= \delta(x_{ii}+x_{ji})\theta(a_{11}+b_{11})\theta(x_{ii}+x_{ji}) \\ &+ \phi(x_{ii}+x_{ji})\delta(a_{11}+b_{11})\theta(x_{ii}+x_{ji}) \\ &+ \phi(x_{ii}+x_{ji})\phi(a_{11}+b_{11})\delta(x_{ii}+x_{ji}), \end{aligned} \tag{2}$$

$$\begin{aligned} \delta(x_{ii}a_{11}x_{ii}) + \delta(x_{ji}a_{11}x_{ii}) &= \delta((x_{ii}+x_{ji})a_{11}(x_{ii}+x_{ji})) \\ = \delta(x_{ii}+x_{ji})\theta(a_{11})\theta(x_{ii}+x_{ji}) &+ \phi(x_{ii}+x_{ji})\delta(a_{11})\theta(x_{ii}+x_{ji}) \\ &+ \phi(x_{ii}+x_{ji})\phi(a_{11})\delta(x_{ii}+x_{ji}) \end{aligned} \tag{3}$$

and

$$\begin{aligned} \delta(x_{ii}b_{11}x_{ii}) + \delta(x_{ji}b_{11}x_{ii}) &= \delta((x_{ii}+x_{ji})b_{11}(x_{ii}+x_{ji})) \\ = \delta(x_{ii}+x_{ji})\theta(b_{11})\theta(x_{ii}+x_{ji}) &+ \phi(x_{ii}+x_{ji})\delta(b_{11})\theta(x_{ii}+x_{ji}) \\ &+ \phi(x_{ii}+x_{ji})\phi(b_{11})\delta(x_{ii}+x_{ji}), \end{aligned} \tag{4}$$

for all elements $x_{ii} \in \mathscr{R}_{ii}$ and $x_{ji} \in \mathscr{R}_{ji}$ $(i \neq j; i,j = 1,2)$. Adding (3) and (4) and subtracting this sum from (2), we see that

$$\begin{aligned} &\delta(x_{ii}(a_{11}+b_{11})x_{ii}) - \delta(x_{ii}a_{11}x_{ii}) - \delta(x_{ii}b_{11}x_{ii}) \\ &= \phi(x_{ii}+x_{ji})(\delta(a_{11}+b_{11}) - \delta(a_{11}) - \delta(b_{11}))\theta(x_{ii}+x_{ji}), \end{aligned} \tag{5}$$

by Lemma 4. Since

$$\begin{aligned} &\delta(x_{ii}(a_{11}+b_{11})x_{ii}) - \delta(x_{ii}a_{11}x_{ii}) - \delta(x_{ii}b_{11}x_{ii}) \\ &= \phi(x_{ii})(\delta(a_{11}+b_{11}) - \delta(a_{11}) - \delta(b_{11}))\theta(x_{ii}), \end{aligned} \tag{6}$$

then by (5) and (6) we get

$$\phi(x_{ii}+x_{ji})(\delta(a_{11}+b_{11}) - \delta(a_{11}) - \delta(b_{11}))\theta(x_{ii}+x_{ji})$$

$$= \phi(x_{ii})(\delta(a_{11}+b_{11})-\delta(a_{11})-\delta(b_{11}))\theta(x_{ii})$$

and by (1) we obtain

$$\phi(x_{ii})(\delta(a_{11}+b_{11})-\delta(a_{11})-\delta(b_{11}))\theta(x_{ji}) + \phi(x_{ji})(\delta(a_{11}+b_{11})-\delta(a_{11})-\delta(b_{11}))\theta(x_{ii}) = 0. \tag{7}$$

Taking into account that ϕ is an endomorphism of $\mathscr{R}$, then multiplying (1) on the left by $\phi(e_1)$, we obtain that

$$\phi(x_{ji})(\delta(a_{11}+b_{11})-\delta(a_{11})-\delta(b_{11}))\theta(x_{ii}) = 0, \tag{8}$$

for all elements $x_{ii} \in \mathscr{R}_{ii}$ and $x_{ji} \in \mathscr{R}_{ji}$ $(i \neq j; i,j = 1,2)$. By a similar argument, we can show that

$$\phi(x_{ii})(\delta(a_{11}+b_{11})-\delta(a_{11})-\delta(b_{11}))\theta(x_{ij}) = 0, \tag{9}$$

for all elements $x_{ii} \in \mathscr{R}_{11}$ and $x_{ij} \in \mathscr{R}_{ij}$ $(i \neq j; i,j = 1,2)$. It follows from the identities (1), (8), (9) and property (♠) that $\delta(a_{11}+b_{11}) - \delta(a_{11}) - \delta(b_{11}) = 0$.

Similarly, we prove the case (ii). □

Lemma 6. *δ is an additive map.*

Proof. The result follows directly from the Lemmas 2, 4, and 5. □

For concluding the proof of the Theorem 1 we will suppose in addition that $\mathscr{R}$ is a 2-torsion free semiprime ring and that $\theta,\phi : \mathscr{R} \to \mathscr{R}$ are automorphisms of $\mathscr{R}$.

Lemma 7. *δ is a (θ,ϕ)-derivation of $\mathscr{R}$.*

Proof. The result follows directly from the [8, Theorem 1]. □

2. Multiplicative Jordan triple (θ,ϕ)-derivations of standard operator algebras

Let $\mathscr{X}$ be a Banach space. We denote by $\mathscr{B}(\mathscr{X})$ the algebra of all bounded linear operators on $\mathscr{X}$ and $\mathscr{F}(\mathscr{X})$ the ideal of all bounded finite rank operators in $\mathscr{B}(\mathscr{X})$. A subalgebra $\mathscr{A}$ of $\mathscr{B}(\mathscr{X})$ is called *prime* if $a\mathscr{A}b = 0$ implies $a = 0$ or $b = 0$. A subalgebra $\mathscr{A}$ of $\mathscr{B}(\mathscr{X})$ is called a *standard operator algebra* if $\mathscr{A}$ contain $\mathscr{F}(\mathscr{X})$. It is clear that $\mathscr{B}(\mathscr{X})$ is a standard operator algebra.

The main result described in the previous section allows to establish the following theorem.

Theorem 2. *Let $\mathscr{X}$ be a Banach space with* $\dim \mathscr{X} \geq 2$, $\mathscr{A} \subset \mathscr{B}(\mathscr{X})$ *a standard operator algebra on $\mathscr{X}$ and $\theta, \phi : \mathscr{A} \to \mathscr{A}$ automorphisms of $\mathscr{A}$. Then the following statement holds: every multiplicative Jordan triple (θ, ϕ)-derivation of $\mathscr{A}$ is a (θ, ϕ)-derivation of $\mathscr{A}$.*

Proof. First of all, it is well known that $\mathscr{B}(\mathscr{X})$ is a prime ring and $\mathscr{A}$ contains a non-trivial idempotent e_1. Write $e_2 = 1_{\mathscr{B}(\mathscr{X})} - e_1$, where $1_{\mathscr{B}(\mathscr{X})}$ is a multiplicative identity of $\mathscr{B}(\mathscr{X})$. Also, $\mathscr{A}$ is dense in $\mathscr{B}(\mathscr{X})$ under the strong operator topology and, by [1, Corollary 7.3.], there exist linear invertible operators of $\mathscr{B}(\mathscr{X})$, $x, y : \mathscr{X} \to \mathscr{X}$, such that $\phi(a) = xax^{-1}$ and $\theta(a) = yay^{-1}$, for any element $a \in \mathscr{A}$.

Let $\mathscr{B}(\mathscr{X}) = \bigoplus_{i,j=1,2} \mathscr{B}(\mathscr{X})_{ij}$ and $\mathscr{A} = \bigoplus_{i,j=1,2} \mathscr{A}_{ij}$ be the Peirce decompositions of $\mathscr{B}(\mathscr{X})$ and $\mathscr{A}$, relative to e_1, respectively, and for every element $t \in \mathscr{A}$ let $x^{-1}ty = (x^{-1}ty)_{11} + (x^{-1}ty)_{12} + (x^{-1}ty)_{21} + (x^{-1}ty)_{22}$ be the Peirce decomposition of $x^{-1}ty$, relative to e_1.

If $\phi(r_{ij})t\theta(r_{ij}) = 0$, for all $r_{ij} \in \mathscr{A}_{ij}$ $(i, j = 1, 2)$, then $xr_{ij}x^{-1}tyr_{ij}y^{-1} = 0$ which shows that $r_{ij}x^{-1}tyr_{ij} = 0$. As $\mathscr{A}$ is dense in $\mathscr{B}(\mathscr{X})$, then we get that $r_{ij}x^{-1}tyr_{ij} = 0$, for all $r_{ij} \in \mathscr{B}(\mathscr{X})_{ij}$ $(i, j = 1, 2)$. Hence, by [7, Lemma 2(i)], we obtain $(x^{-1}ty)_{ji} = 0$ $(i, j = 1, 2)$. It follows that $x^{-1}ty = 0$ which leads to $t = 0$. This shows that the property (♣) is satisfied. Now, if $\phi(r_{ij})t\theta(r_{ij}) = 0$, for all $r_{ij} \in \mathscr{A}_{ij}$ $(i \neq j; i, j = 1, 2)$, then $(x^{-1}ty)_{ji} = 0$ $(i \neq j; i, j = 1, 2)$, by a similar reasoning to the previous case. On the other hand, the hypotheses that $\phi(r_{ji})t\theta(r_{ii}) = 0$ and $\phi(r_{ii})t\theta(r_{ij}) = 0$, for all elements $r_{ii} \in \mathscr{A}_{ii}$, $r_{ij} \in \mathscr{A}_{ij}$ and $r_{ji} \in \mathscr{A}_{ji}$ $(i \neq j; i, j = 1, 2)$, imply that $xr_{ji}x^{-1}tyr_{ii}y^{-1} = 0$ and $xr_{ii}x^{-1}tyr_{ij}y^{-1} = 0$, respectively, which lead to identities $r_{ji}x^{-1}tyr_{ii} = 0$ and $r_{ii}x^{-1}tyr_{ij} = 0$. As $\mathscr{A}$ is dense in $\mathscr{B}(\mathscr{X})$, then we can deduce that $r_{ji}x^{-1}tyr_{ii} = 0$ and $r_{ii}x^{-1}tyr_{ij} = 0$, for all elements $r_{ii} \in \mathscr{B}(\mathscr{X})_{ii}$, $r_{ij} \in \mathscr{B}(\mathscr{X})_{ij}$ and $r_{ji} \in \mathscr{B}(\mathscr{X})_{ji}$ $(i \neq j; i, j = 1, 2)$, respectively. This implies that $(x^{-1}ty)_{ii} = 0$, in view of the primeness of $\mathscr{B}(\mathscr{X})$. The last two results combined show that $x^{-1}ty = 0$ which results that $t = 0$. This shows that the property (♠) is also satisfied. Therefore, by Theorem 1, d is an additive map and the stated result follows. □

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