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# Flip graphs of coloured triangulations of convex polygons

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Abstract. A triangulation of a polygon is a subdivision of it into triangles, using diagonals between its vertices. Two different triangulations of a polygon can be related by a sequence of flips: a flip replaces a diagonal by the unique other diagonal in the quadrilateral it defines. In this paper, we study coloured triangulations and coloured flips. In this more general situation, it is no longer true that any two triangulations can be linked by a sequence of (coloured) flips. In this paper, we study the connected components of the coloured flip graphs of triangulations. The motivation for this is a result of Gravier and Payan proving that the Four-Colour Theorem is equivalent to a property of the flip graph of 2-coloured triangulations: that any two triangulations can be 2-coloured in such a way that they belong to the same connected component of the 2-coloured flip graph.

## 1. Introduction

A triangulation of a polygon is a subdivision of it into triangles, using diagonals between its vertices. Two different triangulations of a polygon can be related by a sequence of flips: a flip replaces a diagonal by the unique other diagonal in the quadrilateral it defines. In this paper,

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we study $m$-coloured triangulations and $m$-coloured flips: we allocate $n$ colours to the triangles and flip diagonals only if the two triangles incident with it have the same colour, say $i$. The flip assigns the colour $i+1$ (reducing modulo $n$) to the two introduced triangles. When using colours, it is no longer true that any two triangulations can be linked by a sequence of (coloured) flips. In this paper, we study the connected components of the coloured flip graphs of triangulations. The motivation for this is a result of Gravier and Payan proving that the Four-Colour Theorem is equivalent to the property that for any two triangulations of a convex polygon, one can find a 2-colouring of their triangles in such a way that they belong to the same connected component of the 2-coloured flip graph. This problem is naturally very hard to solve and still wide open. Our contribution sheds light on the size of the connected components under coloured flips and on the shapes of the components.

Flip graphs of triangulated surfaces are well known and have been studied a lot, in particular the polygon case is one of the many instances of the Catalan combinatorics. Different generalisations appear in various contexts. Here, we only mention two directions. Edge colouring, their orbits under flipping and the diameters of the flip graphs have been studied in [3] and [11]. Triangulated surfaces and flips also appear in the context of cluster algebras and cluster categories: any (monochromatic) triangulation of a so-called marked surface gives rise to a cluster in the cluster algebra of the surface and any two clusters are linked by a sequence of flips. A different notion of coloured flips also appears in cluster combinatorics, in the context of higher cluster categories, see [4] and [13].

This article is structured as follows: Section 2 contains the background on triangulated polygons and introduces coloured triangulations. Then it explains the link between coloured triangulations and the Four-Colour theorem. In Section 3, we study the size and structure of the connected components of coloured flip graph. Section 4 contains further observations and a conjecture about the possible occurrence of a triangle in a component of the 2-coloured flip graph: we expect that a triangle cannot appear with two different colours in a single component. In an appendix, we provide a translation of the results we use from [7], we describe the connected components of the 2-coloured flip graphs for small polygons and compute the size of their components.

## 2. Background

Here we recall the notions of triangulations of convex polygons. We write $P_n$ to denote a convex polygon with $n$ vertices.

**Definition 2.1** (Triangulation)**.** A *triangulation* of $P_n$ is a subdivision of the polygon into triangles, using pairwise non-crossing diagonals.

Boundary segments are not considered to be diagonals. Note that any triangulation of $P_n$ decomposes the polygon into $n-2$ triangles, using $n-3$ diagonals.

**Example 2.2.** A triangulation given by $n-3$ diagonals incident with a common vertex will be called a *fan triangulation*. An example of a fan triangulation of a hexagon is Figure 1.

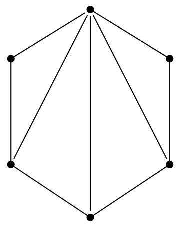


Figure 1: A fan triangulation of a hexagon.

The following result is well-known. We include a proof for convenience. The strategy of the proof is illustrated for $n = 8$ in Figure 2.

**Lemma 2.3.** *The number of triangulations of a convex $(n+2)$-gon is given by the $n$-th Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$.*

*Proof.* The proof can be done using an inductive argument. One checks that the claim is true for $n = 1$. Choose an edge $E$, and consider the triangle it is a part of. In an $(n+2)$-gon, there are $n$ other options for the third vertex of this triangle. All of these reduce the problem to one or two smaller cases, as to the left and right of this triangle, there are smaller polygons of size $m-1$ and $n+4-m$ respectively, for $m = 3, \dots, n+2$. (For $m = 3$, there is only a polygon of size $n+1$ on the right of the triangle, for $m = n+2$, there is only a polygon of size $n+1$ on the left of the triangle.) We count the number of triangulations of these two subpolygons and let $m$ run. This gives the total number of triangulations as $C_{n-1} + C_1C_{n-2} + ... + C_{n-2}C_1 + C_{n-1}$, which is a well-known recursive formula for the Catalan numbers. □

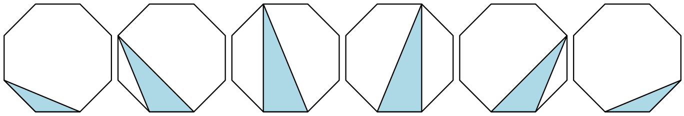


Figure 2: Every triangulation of the octagon falls into one of 6 types.

Triangulations of a polygon are related by the so-called flip:

**Definition 2.4.** Let $t$ be a diagonal in the triangulation $T$ of $P_n$. The union of the two triangles containing $t$ is a quadrilateral. We obtain a new triangulation $T'$ by replacing the diagonal $t$ with the other diagonal of that quadrilateral. This local move is called a *flip*.

Any two triangulation of $P_n$ can be linked by a sequence of flips. This is true for more general surfaces and has been proven by several authors independently and for different set-ups. See for example [5,8,12] or [10] for triangulations of planar point sets. In the convex polygon case, it can be shown using the procedure of [14] that consists in flipping two triangulations into the same fan triangulation.

**Definition 2.5.** The *flip graph* of $P_n$ is the graph whose vertices are triangulations of the polygon, and two vertices $T_1, T_2$ are connected by an edge if and only if there exists a (single) flip linking $T_1$ with $T_2$.

## 2.1. Coloured triangulations, coloured flips

In this article, we are interested in a generalisation of triangulations: we equip triangulations with a set of colours and define a new flip operation for them.

Let $m \geq 1$ and let $C = \{1, \ldots, m\}$ be a set of $m$ different colours. If $T$ is a triangulation of a polygon, we write $F(T)$ for the set of its triangles (or faces). We write $S_m$ for the symmetric group on $m$ letters.

**Definition 2.6** (Colouring)**.** Let $T$ be a triangulation of a convex polygon. By a *colouring* of $T$ we mean an assignment of colours from $1, \ldots, m$ for every triangle of $T$.

**Definition 2.7** (Coloured flip)**.** Let $C = \{1, \ldots, m\}$ be a set of colours and $\sigma \in S_m$ be a permutation. Let $T$ be a triangulation of a convex polygon with each triangle a colour in $C$. Let $t \in T$ be a diagonal incident with two triangles of the same colour $i$. Then the $\sigma$-*flip* of $T$ at $t$ is defined as follows:

1. Replace $t$ by the flip of $t'$ in the underlying uncoloured triangulation.
2. Change the colours of the two triangles incident with $t'$ to the colour $\sigma(i)$.

If the permutation $\sigma$ is a single cycle of the form $(1, 2, \ldots, m)$ (i.e. $i \mapsto i+1$), we call a $\sigma$-flip simply an *$m$-coloured flip.*

**Definition 2.8.** Let $P$ be a convex polygon and let $C = \{1, \ldots, m\}$ be a set of colours, let $\sigma \in S_m$ be a permutation. The *coloured flip graph* of $P$ *with colours $C$ and permutation $\sigma$* or the *$\sigma$-flip graph of $P$* is the graph whose vertices are the coloured triangulations of $P_n$ and whose edges correspond to $\sigma$-flips. We will often just call it the *flip graph* of the polygon.

The coloured triangulations are also counted in terms of Catalan numbers. We note that whenever no two adjacent triangles have the same colour, no edge can be flipped and we have an isolated vertex in the flip graph.

**Lemma 2.9.** *Consider a convex $n+2$-gon $P_{n+2}$ and a set $C$ of $m$ colours.*

*(i) There are $C_n m^n$ coloured triangulations of $P_{n+2}$;*

*(ii) There are $C_n m(m-1)^{n-1}$ triangulations of $P_{n+2}$ where none of the diagonals can be flipped.*

*Proof.* (i) Any triangulation of $P_{n+2}$ has $n$ triangles, so there are $m^n$ different ways to colour a triangulation. The claim then follows from Lemma 2.3.

(ii) We consider the dual graph $G_T$ to a given triangulation $T$ of $P_{n+2}$: it has as vertices the triangles in $T$ and an edge between the two vertices of adjacent triangles. This graph is known to be a tree, it has $n$ vertices and at least two leaves. We start by colouring one leaf with one of the $m$ colours and then proceed to colour adjacent vertices. Since $G$ is a tree, for any new vertex we want to colour, there are $m-1$ options. Hence the factor $(m-1)^{n-1}$. The claim then follows with Lemma 2.3. □

## 2.2. Triangulations of polygons and the Four-Colour Theorem

The *Four-Colour Theorem* is one of the most famous mathematical problems in history. It concerns the question whether four colours are enough to colour any map drawn in the plane. In 1977, Appel, Haken and Koch established that four colours are enough (see [1] and [2]):

**Theorem 2.10** (Four-Colour Theorem)**.** *Any map on $\mathbb{R}^2$ can be coloured using four colours such that any two regions sharing an edge are of different colours.*

This result was proved with the assistance of computers. So far, there is no abstract proof of this theorem. The main motivation of this project is the search for an alternative approach to its proof. In 1999, Eliahou formulated the following conjecture about signed flips (2-coloured flips) [6, §1]:

**Conjecture 1.** *Let $T_1, T_2$ be two arbitrary triangulations of a convex polygon $P$. Let $C = \{1, 2\}$ be two colours. Then there exist colourings of $T_1$ and of $T_2$ such that there is a sequence of 2-coloured flips between the two coloured triangulations.*

Eliahou showed that the signed flip conjecture implies the Four-Colour Theorem. Gravier and Payan later proved that the two are equivalent [7], completing the earlier work of Eliahou and of Krychkov (see [9], a reprint of the original preprint from 1992):

**Theorem 2.11** ([7])**.** *Given any two triangulations of a convex polygon, it is possible to find a 2-colouring of each triangulation such that they can be transformed into one another by a sequence of 2-coloured flips if and only if the Four-Colour Theorem holds.*

So in order to give an abstract approach to the Four-Colour Theorem, it is enough to give an abstract prove of Conjecture 1. We will recall the proof of Theorem 2.11 in Section 1 of the Appendix.

## 3. Connected components of coloured flip graphs

We will now show how to reduce the study of coloured flip graphs to permutations which are a single cycle (Lemma 3.1) and the study of its components to 1 or 2 colours (Lemma 3.3).

**Lemma 3.1.** *Every coloured flip graph is the Cartesian product of coloured flip graphs associated to cyclic permutations.*

*Proof.* Any permutation can be written as a product of disjoint cycles, say $\sigma = \sigma_s \dots \sigma_2\sigma_1$ (where the $\sigma_i \in S_m$ are single cycle permutations). Let $m_i$ be the size of $\sigma_i$. Let $T$ be a triangulation of a polygon and equip it with an $m$-colouring.

For $i \in \{1, 2, \ldots, s\}$ let $D_i$ be the union of all triangles of $T$ which are coloured with a colour appearing in $\sigma_i$. Each $D_i$ is a possibly disconnected union of triangles of $T$. Note that if none of the colours of $\sigma_i$ appear in the $m$-colouring of $T$, then $D_i$ is empty. See Figure 3 for an illustration with $s = 3$. Each cycle $\sigma_i$ only acts on $D_i$, leaving the rest of the coloured triangulation fixed. Let $G_i$ be the flip graph of the $m_i$-coloured region $D_i$. Then the $m$-coloured flip graph of the polygon is a cartesian product of the $G_i$. □

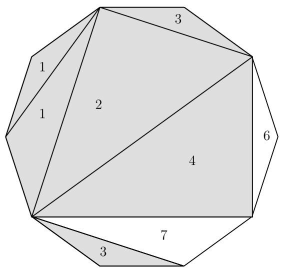



Figure 3: A 7-coloured triangulated decagon with two regions $D_1$ (shaded) and $D_3$ (white) for $\sigma$ as in Example 3.2.

**Example 3.2.** Consider the triangulation of the decagon of Figure 3. Let $\sigma = (1,2,3,4)(5)(6,7) = \sigma_1\sigma_2\sigma_3$ be the product of three disjoint cycles. $D_1$ is the union of six triangles, shaded in grey in the figure. $D_2$ is empty and $D_3$ is the union of two triangles (in white in the figure). The coloured flip graph is the product of the flip graph of $D_1$ on the four colours $\{1,2,3,4\}$, the trivial flip graph of $D_2$ and of the flip graph of $D_3$ on the two colours $\{6,7\}$.

From now on we will concentrate on the case where $\sigma$ is a single cycle of length $m$. As a consequence of Lemma 3.1 in order to understand the flip graph of coloured triangulations, it is enough to understand the case from now on we will assume that $\sigma$ is a single cycle of length $m$. The next result shows that we can reduce the study of flip graphs to the case with one or two colours.

**Lemma 3.3.** *Let $m \geq 1$ and let $\sigma$ be a single cycle of length $m$.*

*(1) If $m$ is odd, then any two triangulations can be $m$-coloured in such a way that they are linked by a sequence of $m$-coloured flips.*

*(2) If $m$ is even, two triangulations can be $T_1$ and $T_2$ can be $m$-coloured in such a way that they are linked by an $m$-coloured flip sequence if and*

*only if they can be 2-coloured in such a way that they are linked by a 2-coloured flip sequence.*

*Proof.* Let $T_1$ and $T_2$ be two triangulations of the same polygon. Let $\sigma = (1, 2, \dots, m)$ be the single cycle permutation changing colour $i$ to colour $i+1$. The idea is to use an uncoloured flip sequence from $T_1$ to $T_2$ and to replace it with an appropriate coloured sequence.

(1) Let $m$ be odd. Let $\underline{\nu} = \nu_s \cdots \nu_1$ be a sequence of (uncoloured) flips such that $T_2 = \underline{\nu}(T_1)$. For $i = 1, \dots, s$ let $d_i$ be the diagonal of the polygon which will be flipped under $\nu_i$ (the diagonal $d_i$ is a diagonal of the triangulation $\nu_{i-1} \cdots \nu_1(T_1)$). The diagonal $d_i$ determines a quadrilateral $D_i$ in $\nu_{i-1} \cdots \nu_1(T_1)$. We write $d_i'$ to denote the other diagonal of $D_i$.

We colour all triangles of $T_1$ with colour 1 and do $m$ coloured flips on $D_1$ (switching between $d_1$ and $d_1'$ and back while doing so). By doing so, we replace the first flip $\nu_1$ by the sequence $\mu_{d_1} \dots \mu_{d_1'} \mu_{d_1}$ with $m$ terms. The resulting triangulation again has colour 1 on all triangles but the diagonal $d_1$ got replaced with $d_1'$. Then we iterate, replacing each uncoloured flip $\nu_i = \nu_{d_i}$ by a sequence $\mu_{d_i} \dots \mu_{d_i'} \mu_{d_i}$ of length $m$ of coloured flips. The result of this longer sequence of $m$-coloured flips is a coloured version of $T_2$.

(2) Let $m$ be even. Assume first that there are 2-colourings of $T_1$ and $T_2$ such that there is a 2-coloured flip sequence $\underline{\nu} = \nu_s \dots \nu_1$ from $T_1$ to $T_2$, on the two colours $\{1, 2\}$. We construct a coloured flip sequence for $\sigma = (1, 2, \dots, m)$ from $\underline{\nu}$. As in part (1) of the proof, we denote by $d_i$ the diagonal of the polygon which is flipped under $\nu_i$ and we let $D_i$ be the quadrilateral of the polygon determined by $d_i$ (it is a quadrilateral in the triangulation $\nu_{i-1} \cdots \nu_1(T_1)$). We modify the sequence $\underline{\nu}$ by replacing $\nu_i$ if needed: If the colour in $D_i$ is 1, we keep $\nu_i$ and view it as part of a $\sigma$-flip for the cyclic permutation $\sigma = (1, 2, \dots, m)$. If the colour is 2, we replace $\nu_i$ by the sequence $\mu_{d_i} \dots \mu_{d_i'} \mu_{d_i}$ with $m-1$ terms. In both cases, the effect is to replace the diagonal $d_i$ by $d_i'$ in $D_i$ and the resulting coloured triangulation is the same as $\nu_i \cdots \nu_1(T_1)$. We can iterate the procedure and replace (or keep) $\nu_{i+1}$ with the same rule. After going through all flips of $\underline{\nu}$, we obtain a coloured flip sequence from $T_1$ to $T_2$ for $m$ colours, where both $T_1$ and $T_2$ are coloured as for the original 2-coloured sequence.

Assume now that there exists a 4-colouring for $T_1$ and $T_2$ and a 4-coloured sequence $\underline{\nu} = \nu_s \cdots \nu_1$ linking them. We replace every even-numbered colour in $T_1, T_2$ by 2 and any odd-numbered colour by 1. If $d_i$ is the diagonal flipped under $\nu_i$, we define $\mu_i$ to be the same flip, but for

$\sigma = (1,2)$. This provides a 2-coloured flip sequence for the 2-coloured versions of $T_1$ and $T_2$. □

The strategy to go from a 2-coloured flip sequence to a 4-coloured flip sequence is illustrated in Figure 4.

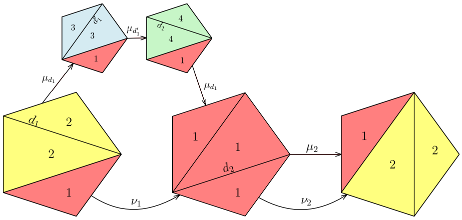


Figure 4: A flip sequence $\underline{\nu}$ with two colours, translated into a sequence with four colours (with $\sigma = (1,2,3,4) =$ (red, yellow, light blue, green))

**Example 3.4.** We will again consider an example of coloured hexagon $P_6$ triangulations, with 2 colours and where $\sigma$ switches the two colours. In Figure 5, we list all types of connected components of the associated coloured flip graph.

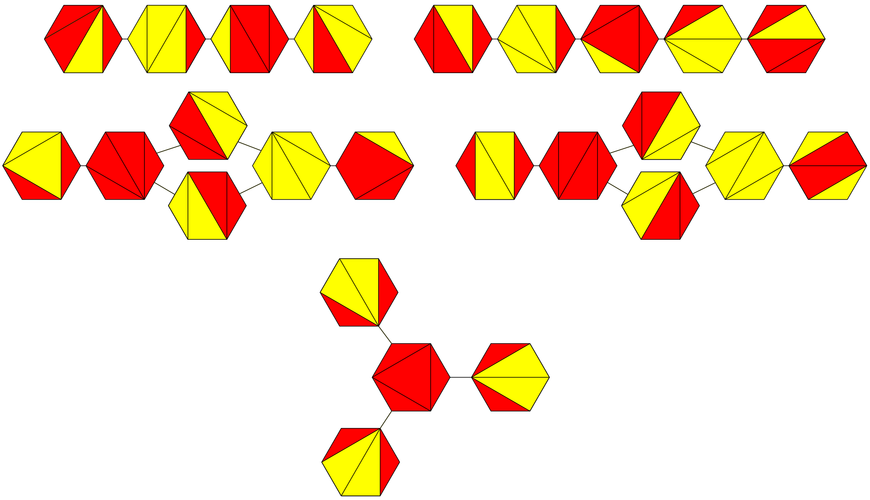


Figure 5: Connected components of the 2-coloured flip graph of $P_6$ up to isometry, excluding the isolated points.

From now on, we restrict to 2-coloured flips, i.e. to the case $m = 2$. We will often choose $C = \{1, -1\}$ and indicate these colours by $+, -$ in the examples. We often use yellow and red as the two colours.

In the statement of the theorem below, we use the notion of a flip sequence: a *2-coloured flip sequence* $\underline{\mu} = \mu_2 \cdots \mu_1$ (with $s > 0$) is a sequence of 2-coloured flips, applied successively.

**Theorem 3.5.** *Let $G$ be the coloured flip graph of $P_{n+2}$. Then every connected component of $G$ is either an isolated point or is of size $\geq n$. Moreover, if $T$ is a triangulation in a non-trivial connected component and $t$ a diagonal of $T$, then either $t$ can be 2-coloured flipped or there exists a 2-coloured flip sequence $\underline{\mu} = \mu_2 \cdots \mu_1$ (where $s \geq 1$) such that $t \in \underline{\mu}(T)$ and such that $t$ can be colour-flipped in $\underline{\mu}(T)$.*

*Proof.* Let $T$ be a triangulation of $P_{n+2}$, with a colouring. We assume that there is at least one diagonal which can be 2-colour-flipped. We mark the diagonals of $T$ with a blue or a red dot: a diagonal $t$ is marked blue if it can be flipped after some (possibly empty, if it can be flipped immediately) sequence of coloured flips. Diagonals of $T$ which can never be flipped are marked with a red point. By assumption, at least one diagonal is marked with a blue dot. If there exists a diagonal with a red dot, find a triangle with a red and a blue dot on two of its diagonals. Such a triangle always exists (see Remark 3.6). Call these two diagonals $B$ and $R$. The two triangles incident with $R$ must be of different colours, since otherwise, we can flip it immediately. We now execute a (coloured) flip sequence in order to flip the edge $B$. At some point in this sequence (or at the end of the sequence), the colour of one of the triangles incident with edge $R$ has changed colour, and at this point, the edge $R$ can be flipped.

This is a contradiction, hence there can be no red dots and every edge can be flipped eventually. Hence there are at least $n$ vertices in this connected component of the coloured flip graph. □

*Remark* 3.6. Consider a triangulation $T$ where at least one diagonal can be 2-colour flipped. We mark the diagonals of $T$ by a blue dot if the diagonal can eventually be colour-flipped and with a red dot otherwise (as in the proof of Theorem 3.5). Then there exists a triangle with a red and blue dot on two of its sides: We consider the line graph $L(T)$ of $T$: it has vertices for the diagonals of $T$ and edges between vertices $t, t'$ whenever there is a triangle containing $t$ and $t'$. We equip this graph with the two colours. Since the graph $L(T)$ is connected, there has to be an edge between a red and a blue node.

The following result shows that connected components of size $n$ do exist in the coloured flip graph of $P_{n+2}$.

**Example 3.7.** Consider a fan triangulation $T$, with alternatingly coloured triangles apart from at one end. See Figure 6 for an illustration of such a fan triangulation of a decagon. Then the connected component containing $T$ is a single line with $n$ vertices: in every step, only a single 2-coloured flip can be made. Under this, the two triangles of the same colour move from one side of the fan to the other end of the fan.

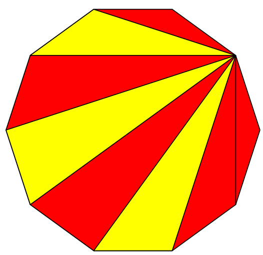


Figure 6: A coloured fan triangulation of a decagon.

We recall the notion of a weighting on the vertices of a triangulated polygon with a 2-colouring from [7].

If we have a coloured triangulation and if $\triangle$ is a triangle of $T$, we denote its colour by $s(\triangle)$. Recall that $s(\triangle) \in \{-1, +1\}$.

**Definition 3.8.** A *weighting* of the polygon $P$ is given by a choice of a triangulation $T$ and a function $p$ assigning to each vertex of $P$ an element of $\{-1, 0, 1\}$ such that there is a 2-colouring of $T$ where for every vertex $x$ of $P$, we have $p(x) = \sum_{x \in \triangle} s(\triangle) \mod 3$ (the sum is taken over all triangles incident with the vertex $x$).

Two weightings of coloured triangulated quadrilaterals are shown in Figure 8. The statement that flips do not change the weighting already appears in [7, Text and Figure 3 on page 819]. We include a proof for completeness.

**Lemma 3.9.** *Any two 2-coloured triangulations which are in the same connected component of the flip graph have the same weighting.*

*Proof.* Let $p$ be a weighting of a triangulated polygon with 2-colouring. If $x$ is a vertex of the quadrilateral where the flip happens, the flip either changes two triangles with $+1$ to one triangle with $-1$ (or vice versa) or two triangles with $-1$ to one triangle with $+1$ (or vice versa). In all cases, $p(x)$ remains the same. (See Figure 8). □

However, there exist 2-colourings which are not flip equivalent but have the same weighting, see Figure 7 for an example.

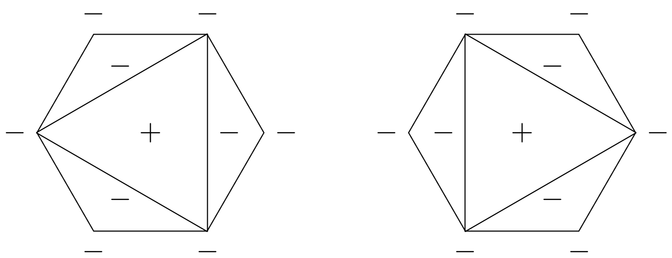


Figure 7: Two triangulations which are not flip equivalent but have the same weighting

The following statement is mentioned in [7]. We include a proof for completeness.

**Theorem 3.10.** *Given a weighting of a triangulation, there is at most one way to colour it to match the weighting.*

*Proof.* Since any triangulation must contain a triangle with 2 of its sides being sides of the polygon, there is a vertex which is only contained in one triangle. At this vertex, if the value is zero then there is no such colouring, and if it is $-1$ or $+1$ then this determines the colour of the triangle. Consider removing this triangle, and subtracting off the value it contributes to the neighbouring triangles to give a valuation and triangulation for a $(n-1)$-gon, repeat until we either reach a contradiction or have completely coloured the shape. Hence if the colouring exists, it must be unique (see Figure 8). □

**Theorem 3.11.** *Any cycle in the coloured flip graph is of even length.*

*Proof.* Let $T$ be a 2-coloured triangulation, let $X$ be the number of triangles marked $+$ in this triangulation. After every flip, $X$ either increases or decreases by 2, hence every flip changes $X$ by $\pm 2$. Therefore, if we

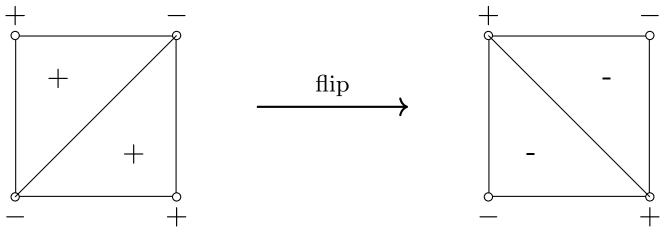



Figure 8: Example of quadrilateral valuation

reach $T$ again after a sequence of coloured flips, this sequence has to have even length, since the number of triangles marked with a + will be equal to $X$ again. □

## 3.1. Structure of the flip graph

In this section, we show properties of the flip graph. In particular, we prove the existence of hypercubes in the flip graph.

**Definition 3.12.** Let $t, t'$ be two diagonals in a triangulation of a convex polygon. If two quadrilaterals have disjoint interiors, we say that the quadrilaterals are *disjoint*. In this case, we say that the flips of $t$ and of $t'$ are *independent*.

A triangulated convex $n + 2$-gon has $n$ triangles and since two independent flips require two disjoint quadrilaterals, there can be at most $\lfloor \frac{n}{2} \rfloor$ independent flips in any triangulation of $P_{n+2}$.

**Example 3.13.** Let $T$ be a fan triangulation of $P_{n+2}$ where each face is assigned the same colour (see Figure 9 for an example). There are $\lfloor \frac{n}{2} \rfloor$ independent flips in this case: We start by choosing the first two triangles on the left in $T$. The we continue, choosing quadrilaterals close to the previous one, but independent. We end up with $\lfloor \frac{n}{2} \rfloor$ disjoint quadrilaterals (and possibly one left over triangle). So this coloured triangulation reaches the upper bound of possible independent flips.

**Proposition 3.14.** *Let $T$ be a 2-coloured triangulation of a convex polygon. Let $G$ be the connected component of the flip graph containing $T$. Assume that there are $k > 1$ diagonals in $T$ which can be 2-coloured flipped and whose quadrilaterals are pairwise disjoint. Then $G$ contains a $k$-dimensional hypercube.*

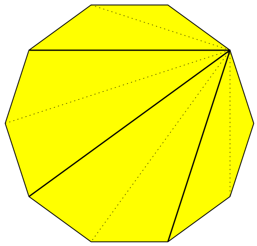


Figure 9: A flip graph of the monochromatic $n+2$-gon, $n \geq 8$ contains 4 disjointquadrilaterals.

*Proof.* Denote that $k$ diagonals of $T$ which can be flipped independently by $1, 2, \ldots, k$. For any $i \neq j$, $1 \leq i, j \leq k$, the flips $\mu_i$ and $\mu_j$ commute. We consider all the triangulations which can be reached from $T$ by arbitrary 2-coloured flips of these $k$ diagonals. In the subgraph of the flip graph they define, each of them has degree $k$. So they form a subgraph isomorphic to a $k$-dimensional hypercube as claimed. □

**Corollary 3.15.** *For $n \geq 8$, the 2-coloured flip graph of $P_{n+2}$ contains a connected component which is not planar.*

*Proof.* Consider the fan triangulation where every triangle is coloured with the same colour. Let $G$ be the connected component of the coloured flip graph which contains this coloured fan triangulation. Since $n \geq 8$, there is a sub-polygon with the same structure as the fan decagon (see figure 8). Hence there are at least four quadrilaterals which can be flipped independently, given by the thick lines. Therefore, $G$ contains a 4-dimensional hypercube by Proposition 3.14. Denote this by $Q_4$. Since $Q_4$ has the complete bipartite graph $K_{3,3}$ as a subgraph, and the latter is not planar, $G$ cannot be planar. □

*Notation.* We consider two $k$-dimensional hypercubes in a connected component of the flip graph to be *distinct* if they are disjoint or if their intersection is a union of hypercubes of smaller dimension.

**Lemma 3.16.** *Suppose $T$ that is a fan triangulation of a convex $(n+2)$-gon where all triangles have the same colour. Let $G$ be the connected component of the coloured flip graph that contains $T$. Then*

*(i) if $n$ is even, then $G$ contains a $\frac{n}{2}$-dimensional hypercube, meeting at least one disjoint*[1] *$(\frac{n}{2}-1)$-dimensional hypercube;*

[1] apart from the common vertex with coloured triangulation $T$.

*(ii) if $n$ is odd, then $G$ contains $\frac{n+1}{2}$ hypercubes of dimension $\frac{n-1}{2}$, meeting at the vertex with coloured triangulation $T$.*

*Proof.* We label the vertices of $P_{n+2}$ so that the fan is based at vertex 1 and that there are two "boundary triangles" with vertices $1, 2, 3$ and $n+1, n+2, 1$.

(i) When we assume $n$ to be even, the maximum number of independent coloured flips is $\frac{n}{2}$. We reach this if we group pairs of adjacent triangles into quadrilaterals starting at the boundary, i.e. with the quadrilateral on the vertices $1, 2, 3, 4$ and continuing with adjacent triangles, see Example 3.13. If instead we group the pairs of adjacent triangles of $T$ starting the quadrilateral on the vertices $1, 3, 4, 5$, we only get $\frac{n}{2} - 1$ disjoint quadrilaterals and find a hypercube based on $\frac{n}{2} - 1$ independent flips. These two hypercubes meet at the monochromatic fan triangulation.

(ii) If $n$ is odd, we "ignore" one of the two triangles of $T$ and pair all other triangles to form quadrilaterals, in the same way as in part (i). The triangles we pick for this have vertices $1, 2\ell, 2\ell+1$ where $1 \leq \ell \leq \frac{n+1}{2}$. The remaining triangles form $\frac{n-1}{2}$ disjoint quadrilaterals and so we get $\frac{n-1}{2}$ independent flips in each case, giving $\frac{n-1}{2}$ hypercubes of dimension $\frac{n-1}{2}$, meeting at the coloured triangulation $T$ as their common vertex.

□

Note that a version of Lemma 3.16 can be proved for more general triangulations: the number of disjoint quadrilaterals in an arbitrary 2-coloured triangulations gives a lower bound on the dimension of maximal dimensional hypercubes it contains.

To get the minimal number of independent flips in a monochromatic triangulation, one considers a triangulation with as many "ear triangles" as possible (an ear triangle is a triangle formed by two boundary segments and one diagonal). This way, the number of independent quadrilaterals stays as small as possible. We choose a triangulation built with nested sequence of ear triangles: first draw the diagonals $(1,3)$, $(3,5)$, etc., ending with $(n-1, n+1)$. Then the diagonals $(1,5)$, $(5,9)$, etc. There are $\lfloor \frac{n+2}{2} \rfloor$ ear triangles in such a triangulation. Around half of them will not get matched when forming independent quadrilaterals, depending on the factors of $n$. A lower bound on the dimension of the largest hypercube in a monochromatic triangulation of this shape is $\lfloor \frac{n+1}{3} \rfloor$.

**Example 3.17.** We illustrate Lemma 3.16 by showing for the monochromatic fan triangulation of $P_n$ with $6 \le n \le 9$ in Figures 10, 11, 12 and 13. In each figure from left to right, the first graph is the original triangulation, and then are the possible hypercubes of different dimensions, and the last one is the combination of all these hypercubes. We number the diagonals in $T$, and the numbers on edges of the $k$-dimensional cube represents a flip of that diagonal. The black points represent the fan triangulation.

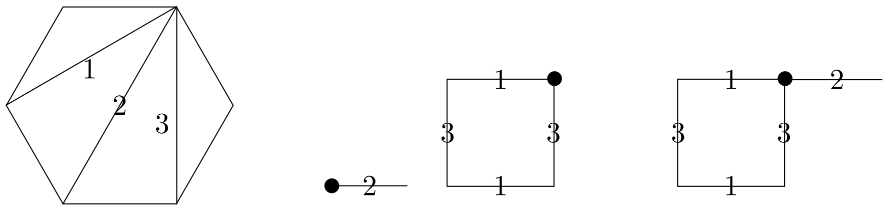



Figure 10: The connected component with the monochromatic fan triangulation of the hexagon contains two shown 1-dimensional and 2-dimensional cubes.

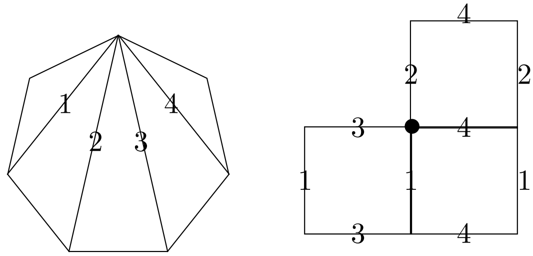



Figure 11: The connected component with the monochromatic fan triangulation of the heptagon with three 2-dimensional cubes.

## 4. Observations and a conjecture

We conclude this paper by a number of observations and a conjecture. Let $P_{n+2}$ be a convex $n+2$-gon.

**Observation 1.** For $n \le 4$ any connected component of the 2-coloured flip graphs of $P_{n+2}$ is either a tree, or obtained from adding leaves onto a 4-cycle. See Appendix 2. For $n > 4$, this is not true anymore. An example is a component for $n = 7$ in Figure 14.

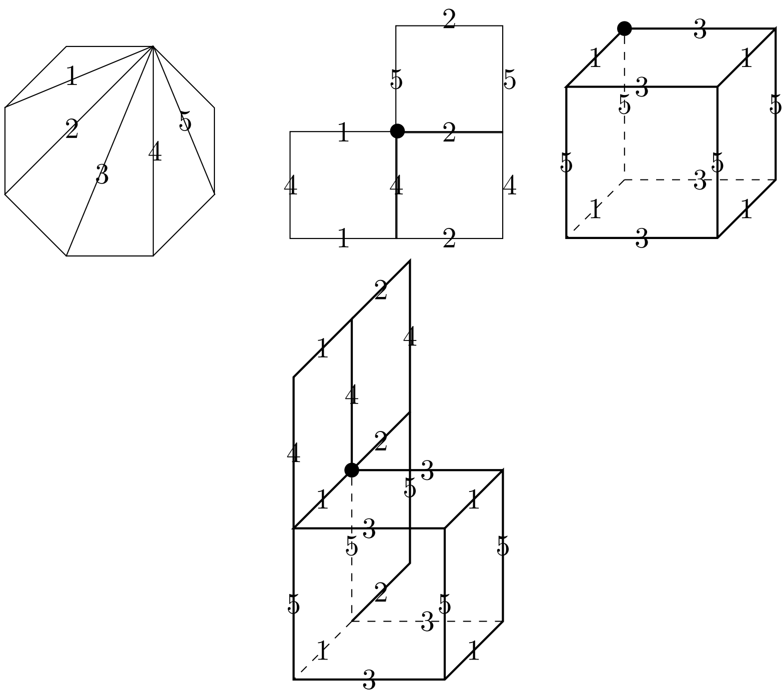



Figure 12: The connected component with the monochromatic fan triangulation of the octagon has a 3-dimensional cube.

**Observation 2.** There are connected components of the flip graph which do not have any leaves, see e.g. Figure 14 for $n = 7$ or Appendix 2 in the case of $n = 6$.

In the examples we considered, no two triangulations in a connected component contained two triangles with the same vertices but with different colours. See for example Figure 14 for an illustration. We suspect that this could be true in general:

**Conjecture 2.** *In a connected component of the 2-coloured flip graph, a triangle cannot appear in the same position but with different colours.*

**APPENDIX**

## 1. Proof of Theorem 2.11

We recall the statement of Theorem 2.11 from the Introduction: the Four-Colour Theorem holds if and only if for any two triangulations of a convex

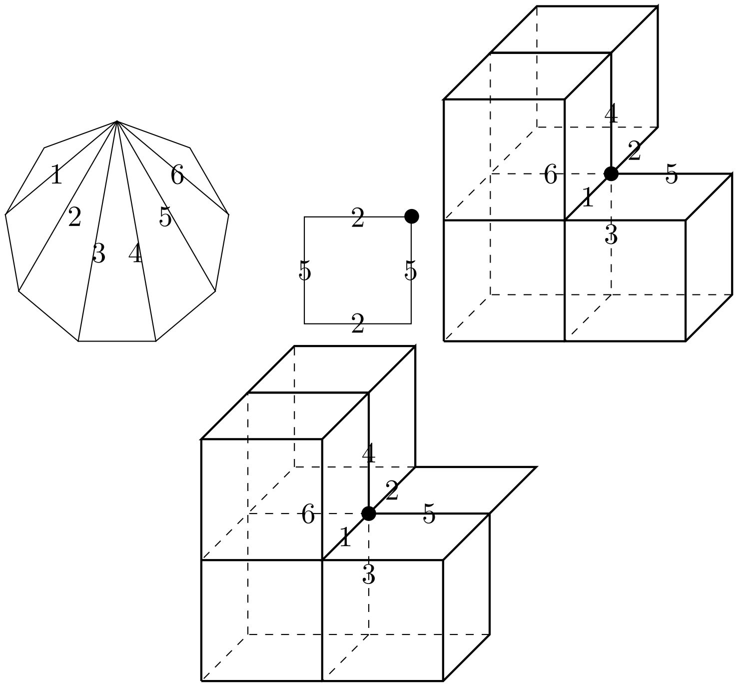



Figure 13: The connected component with the monochromatic fan triangulation of nonagon contains four 3-dimensional cubes.

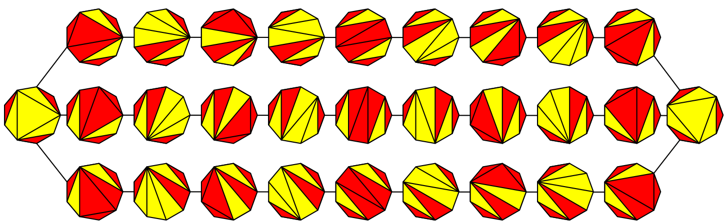


Figure 14: A connected component of the flip graph for coloured nonagon triangulations which contains no fans, and has minimum cycle size 20.

polygon, one can 2-colour them in a way that there exists a sequence of 2-coloured flips linking the two triangulations. This result by Gravier and Payan motivates the notion of coloured mutation. The work of Gravier and Payan has appeared in French in 2002. For the convenience of the reader, we illustrate their reasoning in this section. We first recall the notions needed. In this section, we will use 'signed triangulations' to refer to 2-coloured triangulations in order to distinguishing from the notion of a colour in the 4-colour theorem.

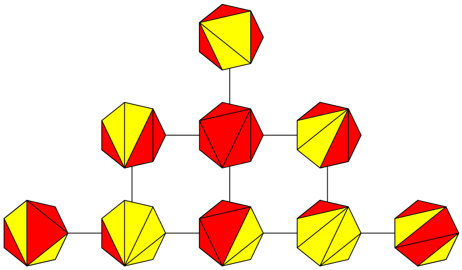


Figure 15: A connected component for the heptagon, which has three leaves of two different triangulation types.

**Definition 1.1.** Let $P$ be a convex polygon. We introduce the following definitions:

- Let $T$ be a triangulation of $P$. We write $\mathcal{D}(T)$ for the set of all diagonals of $T$ and $\mathcal{F}(T)$ for its faces (the triangles).
- A *sub-polygon* $S \subset P$ is a polygon whose vertices are a subset of those of $P$, and which respects the cyclic order of the vertices of $P$.
- $S - x$ denotes the sub-polygon induced by all vertices except $x$.
- A *signed triangulation* of $P$ is a 2-coloured triangulation $T$ of the polygon, i.e. a pair $\{T, s\}$, where $s : \mathcal{F}(T) \to \{+1, -1\}$ is a 2-colouring of the triangles of $T$. Let $\overline{s}$ be the signed triangulation obtained from $s$ by changing all signs. We write $(T, s)$ to denote the class $\{\{T, s\}, \{T, \overline{s}\}\}$.
- A *signed flip* is a 2-coloured flip of a diagonal of a signed triangulation.
- We recall the definition of a weighting of $P$ (Definition 3.8) and introduce a notation suitable with the other terms of this section: The pair $\{T, p\}$ where $p : V(T) \to \{-1, 0, +1\}$ is a function on the vertices of $T$ (or of $P$) is called a *weighting* of $T$ is there exists a 2-colouring $s$ of $T$ such that for every vertex $x$ of $T$ we have $p(x) = \sum_{t \in \mathcal{F}(T)} s(t) \mod 3$. Similarly as before, if $\{T, p\}$ is a weighting and

$s$ a 2-colouring giving rise to it, we write $\{T, \overline{p}\}$ for the weighting associated to $\overline{s}$. We use $(T, p)$ to denote the valuation $p$ of $T$, up to exchanging $s$ with $\overline{s}$.

- A *valuation* of $T$ is a pair $(T, v)$, where $v : \mathcal{D}(T) \to \{0, 1\}$ assigns 0 or 1 to every diagonal of $T$.
- A *colouring* of $T$ is a pair $(T, col)$, where $col$ is a 4-proper colouring of the vertices of $T$ (i.e., no two vertices adjacent under $T$ share the same colour). We will often use the letters $a, b, c, d$ to indicate the four colours of a colouring. We only consider colourings up to permutation of colours.

Let $\{T, s\}$ be a signed triangulation. The signs determine a weighting of $T$ by definition. There is a natural way to associate a valuation $(T, v)$ to any signed triangulation $\{T, s\}$ if a diagonal is incident with two triangles of the same sign, its valuation is set to be 0. Otherwise, its valuation is set to be 1. By definition, this procedure associates the same valuation $v$ to $\{T, \overline{s}\}$. So we can naturally assign a valuation $(T, v)$ to $(T, s)$.

**Example 1.2.** See Figure 16 for an example of a signed triangulation $T$ of heptagon, with associated weighting (on the left), valuation (in the middle) and with a colouring for $T$ (on the right).

Notice that "signed triangulations, weighting, valuation and colouring" are equivalent notions, up to taking the opposite signs/weights:

1. $(T, s) \equiv (T, p)$. Weightings and 2-colourings are equivalent by definition.
2. $(T, s) \equiv (T, v)$. Any signed triangulation $(T, s)$ gives a valuation $(T, v)$ as we have explained above (for any diagonal $xy \in T$, $v(xy) = 0$ if and only if the two triangles adjacent to $xy$ have the same sign). Conversely, any valuation $(T, v)$ gives rise to two signed triangulations $\{T, s\}$ and $\{T, \overline{s}\}$.
3. $(T, v) \equiv (T, col)$. Given a valuation $(T, v)$, we construct a 4-colouring $col$ of the vertices of $P$ compatible with $T$, denoted by $col(T, v)$: Choose a vertex of degree 2 in $T$. Such a vertex lies in a triangle which has two boundary edges (every triangulation has at least two such triangles). We colour the three vertices of this triangles in three different colours. We proceed as follows: for any quadrilateral

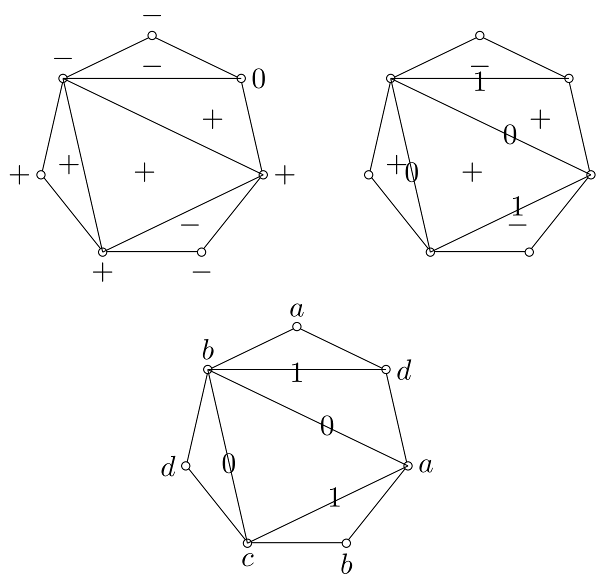



Figure 16: A signed triangulation of heptagon with weighting, valuation, colouring.

with vertices $xyzt$, formed by two adjacent triangles sharing the common diagonal $yt$, we colour $x, z$ in the same colour if and only if the diagonal $yt$ is valued 1 under $v$. Starting with the above triangle, we thus obtain a colouring of $T$ with (up to) four colours. The colouring $col(T, v)$ is unique up to permutation of the colours.

Reciprocally, starting from $(T, col)$, we get a valuation of $T$ by setting a diagonal of any quadrilateral to be 0 if and only if the four vertices of the quadrilateral this diagonal determines are all coloured differently.

By the above, it makes sense to write $(T, \varepsilon)$ where $\varepsilon$ is in $\{s, p, v, col\}$ as these are all equivalent.

*Remark* 1.3. Let $T$ be a triangulation of a polygon. We comment on the effect of a flip on the notions weighting, valuation and colouring. See Figure 17 for an illustration.

- Any flippable diagonal has valuation 0. If we flip it, the new diagonal also has valuation 0 while the diagonals bounding the corresponding quadrilateral change their valuation.

- The weighting of the vertices remains unchanged under flips.
- Any colouring for $T$ is still a colouring for the new triangulation.

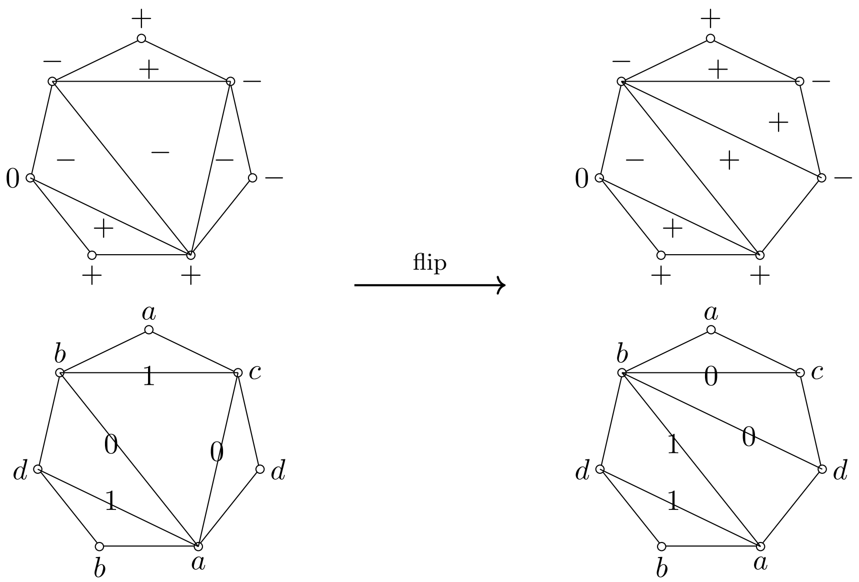



Figure 17: The effect of a signed flip on weighting, valuation, colouring.

Note that a 3-colour colouring of a triangulation corresponds to the case where each diagonal has value 1, and such signed triangulation is called *alternating*. Alternating signed triangulations are isolated vertices in the flip exchange graph and so they are not of interest for us.

**Definition 1.4.** Let $(T, \varepsilon)$ and $(T', \varepsilon')$ be two signed triangulations of the same polygon. We write $(T, \varepsilon) \sim (T', \varepsilon')$ if there exists a sequence of 2-coloured flips from $(T, \varepsilon)$ to $(T', \varepsilon')$. This sequence may be empty (i.e. we allow $T = T'$ with $\varepsilon = \varepsilon'$). One can check that $\sim$ is an equivalence relation, we denote the class of $(T, \varepsilon)$ by $[T, \varepsilon]$.

Now we are ready to prove Theorem 2.11 which we reformulate as follows:

**Theorem 1.5.** *Let $(T, v) \neq (T', v')$ be signed triangulations of $P$. Then $(T, v) \sim (T', v')$ if and only if $col(T, v) = col(T', v')$ and it uses 4 colours.*

*Proof of $\Longrightarrow$ of Theorem 1.5.* Using Remark 1.3 one can see that a coloured flip does not change the colouring of the vertices. Iterating, we get that $(T, v) \sim (T', v')$ implies $col(T, v) = col(T', v')$. Since we assumed that the two triangulations are different, the sequence of signed flips needed to go from $(T, v)$ to $(T', v')$ is not empty, i.e. the flip graph is not a single point and there is at least one diagonal valued with 0. Hence $col(T, v)$ uses four colours. □

To prove the converse of the theorem, we first show three lemmas. We have to study the vertices of $P$ and their neighbours. In a triangulated polygon any vertex of $P$ has neighbours on the boundary and potentially neighbours through diagonals of the triangulation. When dealing with the former, we refer to them as neighbours along the boundary (or on the boundary).

**Lemma 1.6.** *Let $(T, \varepsilon)$ be a signed triangulation of a polygon $P$ and $x$ a vertex of $P$. Assume that the two neighbours of $x$ along the boundary are the only two neighbours of $x$ with the same colour. Then $x$ has 3 or 4 neighbours, and $p(x) = 0$.*

*Proof.* Clearly, $x$ cannot have only 2 neighbours in this case as in that case, these would belong to a common triangle with $x$.

Suppose for contradiction that the vertex $x$ has at least five neighbours. Then the two neighbours on the polygon are not the only two neighbours of the same colour in $T$: we can only colour three neighbours of $x$ with distinct colours (different from the colour of $x$). And we would have at least three vertices of the same colour or another pair of neighbours with the same colour. Hence $x$ cannot have more than 4 neighbours.

In case $x$ has three neighbours, these four vertices span a quadrilateral (with $x$) and the diagonal ending at $x$ has value 1 as the other end must be of a different colour. In particular, the two triangles incident with $x$ have opposite sign and $x$ has weight 0.

In case $x$ has four neighbours, the two neighbours which are linked to $x$ by diagonals must be of two different colours which are also different from the colour of $x$. In particular, both these diagonals have value 0. Therefore, $x$ is incident with three triangles of the same sign and the weight $p(x)$ is 0 (mod 3). □

**Lemma 1.7.** *Let $(T, \varepsilon)$ be a signed triangulation of $P$ and $x$ a vertex of $P$. If $x$ has no two neighbours of the same colour, then $x$ has 2 or 3 neighbours and the weight $p(x)$ of $x$ is not 0.*

*Proof.* It is clear that $x$ can only have 2 or 3 neighbours as if there are more, there would be at least two of them with the same colour. In case $x$ has only two neighours, it is incident with only one triangle and so $p(x)$ is 1 or 2 (mod 3).

So assume that $x$ has three neighbours. In the quadrilateral spanned by $x$ and its three neighbours, $T$ has a diagonal connecting $x$ with the fourth vertex, say $y$. The vertices $x$ and $y$ have to be of different colour and so all four vertices of this quadrilateral are of different colours. Hence the diagonal $xy$ has value 0. So the two triangles at $x$ are of the same sign and the weight $p(x)$ is in $\{1, 2\} \mod 3$. □

**Lemma 1.8.** *Let $(T, \varepsilon)$ be a signed triangulation of a polygon $P$. Let $x$ be a vertex of $P$. Then $p(x) = 0$ if and only if its two neighbours on $P$ have the same colour.*

*Proof.* It is enough to consider the full subgraph of the triangulated polygon induced by $x$ (it consists of $x$, of all vertices connected with $x$ and of all boundary edges and diagonals connecting them). The idea is to use induction on the degree of the vertex $x$.

(1) If $x$ has no two neighbours of the same colour, then $x$ has degree 2 or 3 and $p(x) \neq 0$ by Lemma 1.7.

(2) If the two neighbours of $x$ on the polygon are the only neighbours of $x$ with the same colour, then $x$ has degree 3 or 4 and $p(x) = 0$ by Lemma 1.6.

With (1) and (2) we have covered all cases where $x$ has degree 2 or 3 (in degree 3, if there are vertices of the same colour among the neighbours of $x$, they have to be on the boundary, for a colouring to be valid).

So the result holds for vertices $x$ of degree $\leq 3$.

(3) It remains to check the general situation. Let $y_1, y_2, \dots, y_r$ be the neighbours of $x$, with $y_1$ and $y_r$ being along the boundary and where $r \geq 4$. See left hand picture of Figure 18.

Since $r \geq 4$, there are vertices among the $y_i$ of the same colour. Pick $y_i$, $y_j$, $i < j - 1$ of the same colour such that there are no two vertices of the same colour among $y_{i+1}, \dots, y_{j-1}$. Consider the triangulated subpolygon on the vertices $x, y_i, y_{i+1}, \dots, y_j$. Using the same argument as in Lemma 1.6, we find that either $j = i + 2$ or $j = i + 3$ and that the triangles incident with $x$ and that $p(x) = 0$ in this subpolygon (there are either two triangles of opposite signs or three triangles of the same sign).

We then identify $y_i$ with $y_j$, getting a new polygon $P'$, reducing the degree of $x$ in it, see right hand side of Figure 18. So in the polygon

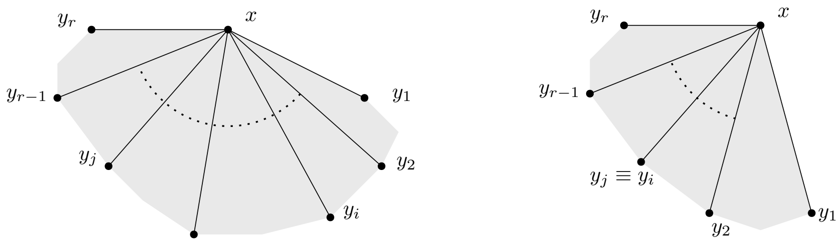



Figure 18: The neighbourhood of $x$ in $P$ and in $P'$

$P'$, the weight of $x$ is 0 if and only if the two neighbours $y_1$ and $y_r$ on the boundary have the same colour. Since the region between $y_i$ and $y_j$ contributes by 0 to the weight, the claim holds. □

*Proof of* $\Longleftarrow$ *of Theorem 1.5.* Assume that there exists a polygon $P$ and two triangulations $(T, v)$ and $(T'v')$ of $P$ which provide a counterexample. Let $P$ be minimal with this property. The polygon $P$ has at least 5 vertices (one can check that the theorem is true for 4 vertices). So $col(T, v) = col(T', v')$, this colouring uses all four colours, and there is no sequence of signed flips between these two signed triangulations. Among the vertices of $T$ of degree 2 we choose a vertex $x$ with the property that $T - x$ (the triangulated polygon without the triangle at $x$) is still coloured with four colours. Such a vertex always exists as $P$ has at least 5 vertices and among them at least two vertices of degree 2. At least one of them satisfies this condition (if one removes a degree 2 vertex $y$ and the remaining colouring only uses 3 colours, one replaces $y$ by another degree 2 vertex in $T$). Since all four colours are present in $T - x$, there exists a diagonal with valuation 0.

If there exists a signed triangulation $T''$ in the equivalence class $[T', v']$ where $x$ has degree 2, then by minimality of the size of $P$, we know that for the polygon $P - x$ we have $(T - x, v) \sim (T'' - x, v'')$. But then $(T, v) \sim (T'', v'')$ and the latter is in the equivalence class of $(T', v')$, so $(T, v) \sim (T', v')$, a contradiction.

So we can assume that $x$ has degree $\geq 3$ in every triangulation in $[T', v']$.

We partition this equivalence class into two sets $\mathcal{T}_1$ and $\mathcal{T}_2$. We will show that these are both empty, thus proving that no counter-example to the implication $\Leftarrow$ exists.

We define $\mathcal{T}_1$ to be the set of all signed triangulations in $[T', v']$ having a diagonal of value 0 incident with $x$. The set $\mathcal{T}_2$ are the ones where every diagonal at $x$ has value 1. These are the signed triangulations which are alternating on the subpolygon induced by $x$ and all its neighbours in $T'$. (Since the degree of $x$ is at least 3 for any signed triangulation in $[T', v']$, there is always at least one diagonal at $x$).

Claim: $\mathcal{T}_1$ is empty:

From the elements of $\mathcal{T}_1$ choose a signed triangulation $(T'', v'')$ where $x$ has minimal degree (this degree is $\geq 3$ as we have seen). The two neighbours of $x$ (along the boundary of the polygon) are adjacent in $T$ (as $x$ has degree 2 in $T$) and so have different colour. By Lemma 1.8, this means that $p''(x) \neq 0$, where $p''$ is the weighting of $(T'', v'')$. This weighting is the same as that of $(T', v')$ and as that of $(T, v)$ as their colourings are the same. If there is a diagonal of value 1 incident with $x$, say $xy_k$ (for some $k$), we flip a diagonal with value 0 next to this diagonal. Then the diagonal $xy_k$ has value 0. In this new triangulation, the degree of $x$ has gone down by one and we reach a contradiction. So all diagonals at $x$ must have value 0. We flip the first such diagonal at $x$ (e.g. going clockwise through these diagonals). The result is a triangulation where either $x$ has degree 2 (contradicting that the vertex $x$ has degree $> 2$ for all elements of $[T', v']$) or it has degree 3 and no diagonal of value 0 incident with it, implying that $p''(x) = 0$ (a contradiction to $p''(x) \neq 0$) or the resulting triangulation is an element of $\mathcal{T}_1$ where $x$ has smaller degree. Figure 19 illustrates the first two of these cases. In all three cases, this leads to a contradiction. Therefore, $\mathcal{T}_1$ is empty.

Claim: $\mathcal{T}_2$ is empty:

Recall that the signed triangulation of the subpolygon induced by $x$ and its neighbours in $T'$ is alternating (all diagonals at $x$ have value 1). For any $(Q, \varepsilon)$ an element of $\mathcal{T}_2$, we write $P(Q)$ the maximal alternating subpolygon (maximal by inclusion) which contains $x$ and its neighbours. Let $(T'', v'')$ be an element of $\mathcal{T}_2$ which minimizes the size of $P(T'')$. As $P(T'')$ is maximal as alternating signed polygon, the boundary edges of $P(T'')$ which are diagonals in the original triangulated polygon have to have value 0. Since $col(T'', v'') = col(T', v')$ and all four colours appear, there exists at least one diagonal of value 0 (so such a boundary edge of $P(T'')$ has to exist). If we flip this diagonal, we obtain a new triangulation $S$. If this diagonal is incident with two edges (two diagonals or one diagonal and a boundary edge) at $x$, $S$ belongs to $\mathcal{T}_1$ (see Figure 20 for an illustration). However, $\mathcal{T}_1$ is empty.

Otherwise, $S$ belongs to $\mathcal{T}_2$ with $P(S)$ smaller than $P(T'')$ (see Figure 21 for an illustration), also a contradiction. So $\mathcal{T}_2$ is empty. □

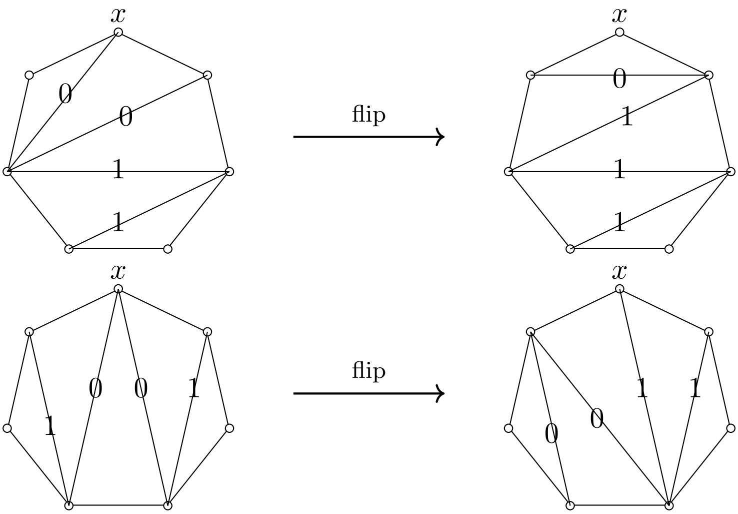



Figure 19: Examples where $x$ becomes a vertex of degree 2 respectively of degree 3.

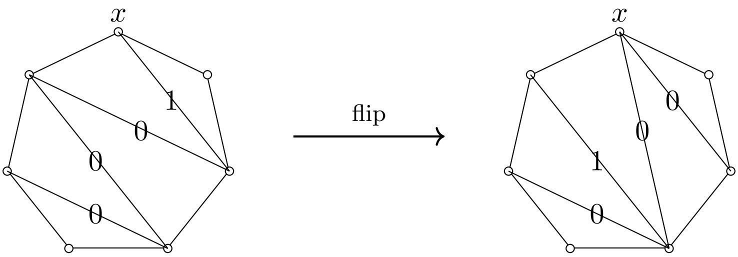



Figure 20: Example with $S \in \mathcal{T}_1$

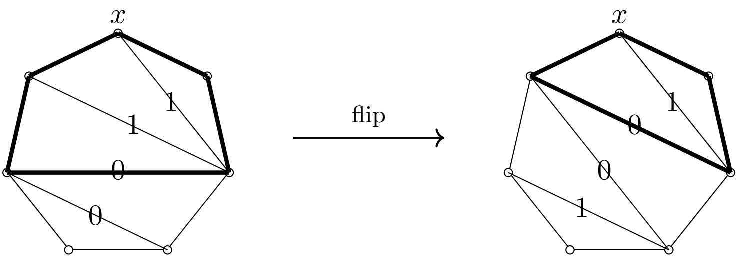



Figure 21: Example with $S \in \mathcal{T}_2$

## 2. Connected components of flip graphs

In this appendix, we describe the connected components of the 2-coloured flip graphs of $P_{n+2}$ for $n \leq 6$. We omit the isolated vertices.

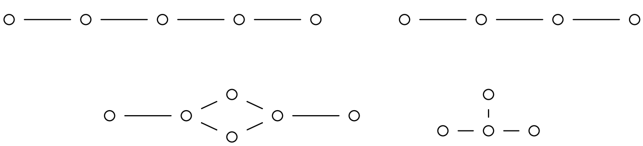
- *n = 2.* There is only one type of (non-trivial) connected components.



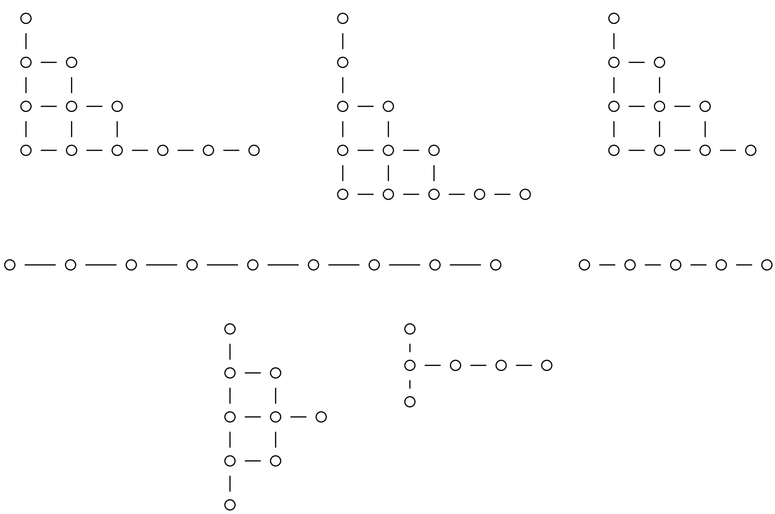
- *n = 3.* There is only one type of connected components.



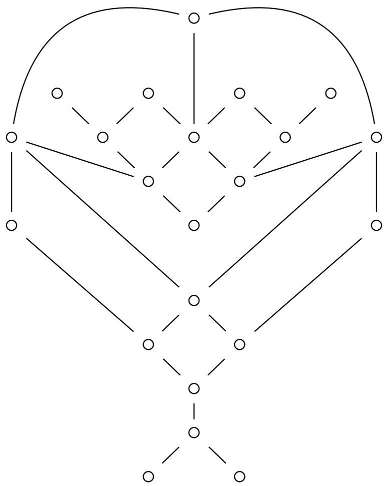
- *n = 4.* There are four different shapes of connected components.

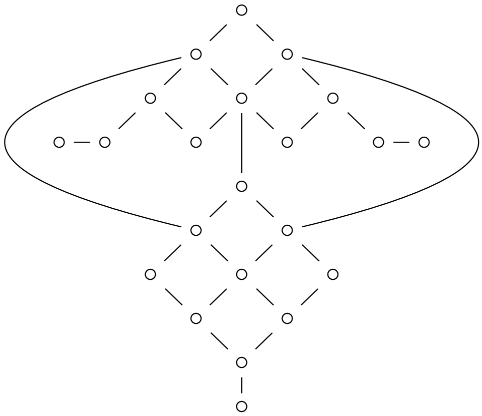


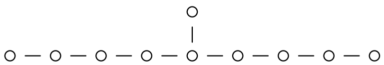
- *n = 5.* The seven shapes of the different connected components are:



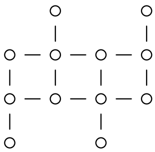
- *n = 6.* The 26 shapes of the different connected components are:

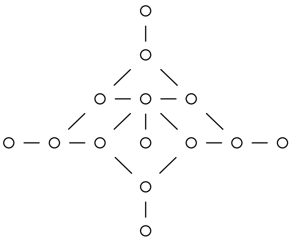


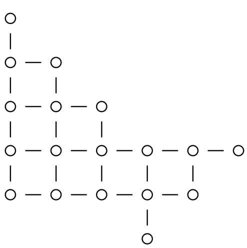


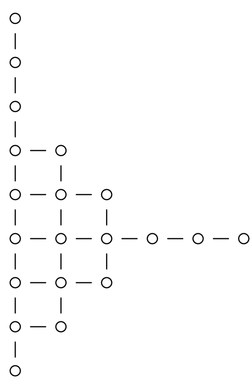


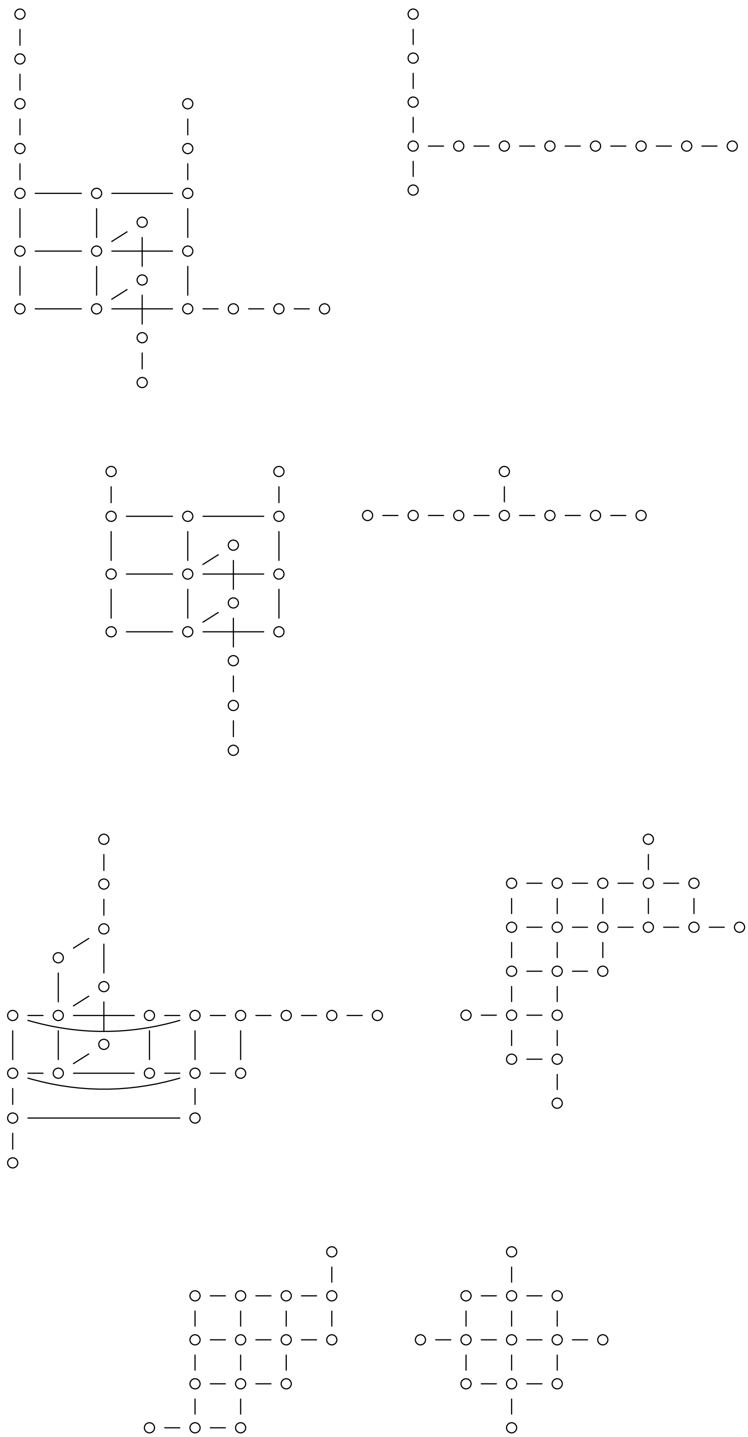


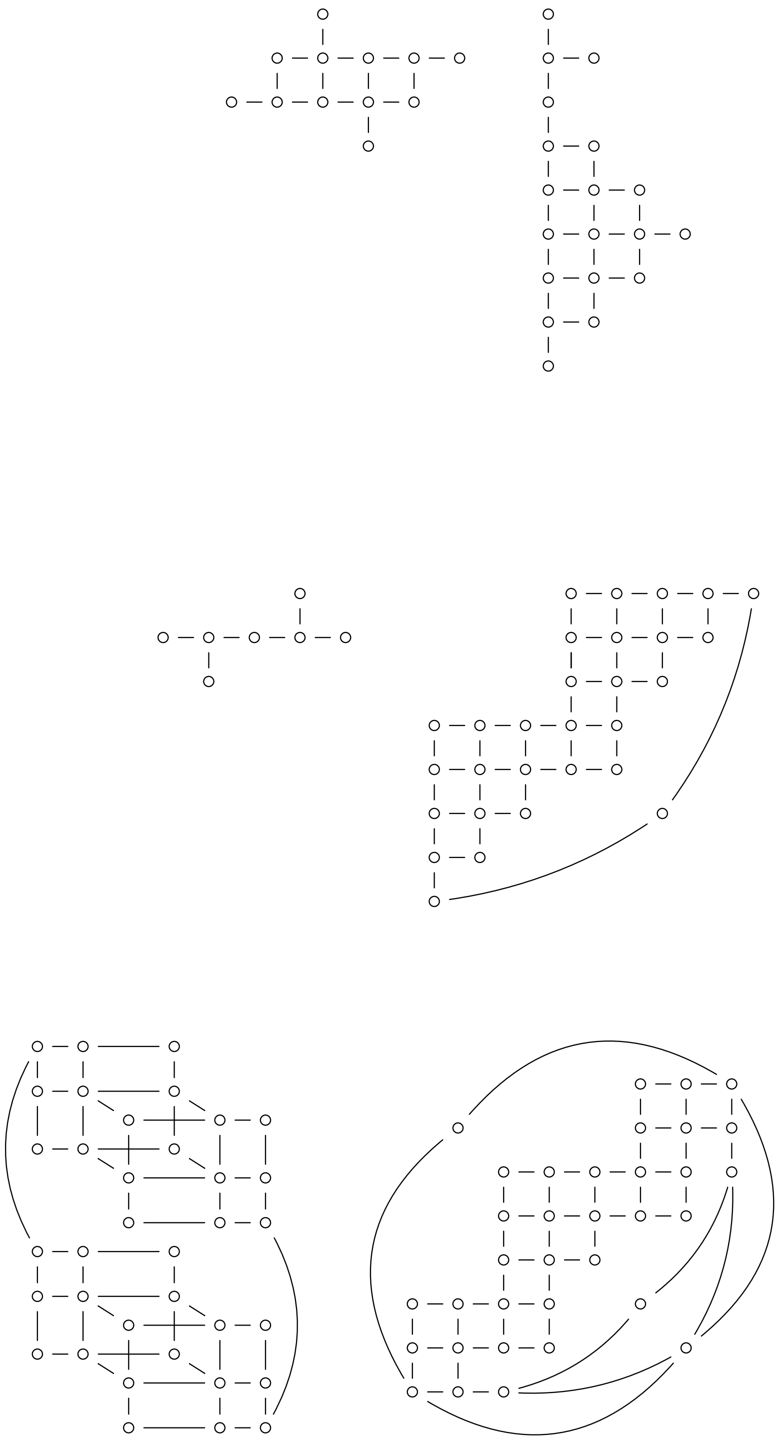


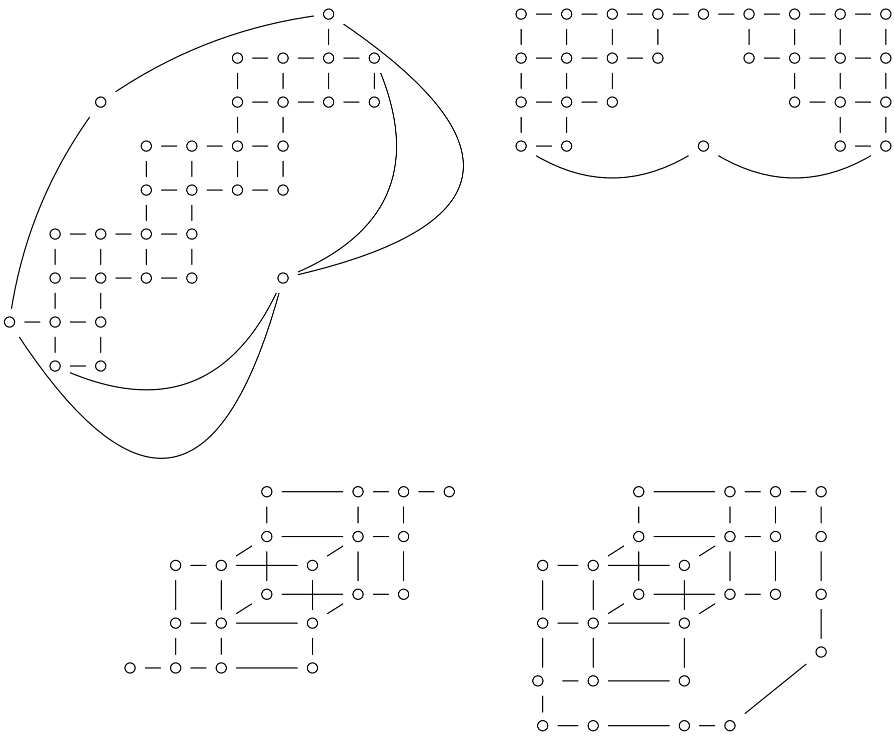












## 3. Codes for component sizes

The following tables show the number of connected components for the flip graph, for the square, pentagon, hexagon, heptagon, octagon, and nonagon. They were found using a computer search, after generating all triangulations using the same recursive method as Figure 2, and then testing which pairs differ by a flip.

Square: $n = 2$

| | | |
|---|---|---|
| size | 1 | 2 |
| number | 4 | 2 |

Pentagon: $n = 3$

| | | |
|---|---|---|
| size | 1 | 3 |
| number | 10 | 10 |

Hexagon: $n = 4$

| | | | | |
|---|---|---|---|---|
| size | 1 | 4 | 5 | 6 |
| number | 28 | 16 | 12 | 12 |

Heptagon: $n = 5$

| | | | | | | |
|---|---|---|---|---|---|---|
| size | 1 | 5 | 6 | 9 | 10 | 12 |
| number | 84 | 14 | 28 | 42 | 14 | 42 |

Octagon: $n = 6$

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| size | 1 | 6 | 7 | 8 | 10 | 12 | 13 | 14 | 15 | 16 | 18 |
| number | 264 | 16 | 16 | 16 | 16 | 64 | 8 | 8 | 16 | 32 | 32 |

Octagon, continued

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| size | 19 | 20 | 21 | 22 | 23 | 26 | 28 | 29 | 32 | 34 | 36 |
| number | 64 | 40 | 16 | 32 | 32 | 16 | 8 | 16 | 2 | 8 | 4 |

Nonagon

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| size | 1 | 7 | 9 | 13 | 15 | 17 | 18 | 21 | 23 | 27 | 28 | 29 |
| number | 858 | 18 | 36 | 36 | 54 | 36 | 36 | 18 | 72 | 126 | 72 | 6 |

Nonagon, continued

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| size | 31 | 32 | 33 | 34 | 35 | 36 | 37 | 38 | 41 | 42 | 44 |
| number | 54 | 36 | 18 | 72 | 18 | 108 | 36 | 72 | 36 | 36 | 36 |

Nonagon, continued

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| size | 45 | 46 | 53 | 55 | 57 | 59 | 61 | 66 | 70 | 71 | 79 |
| number | 108 | 36 | 54 | 36 | 18 | 54 | 36 | 36 | 36 | 18 | 6 |

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