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RESEARCH ARTICLE

Erratum for "On the LS-category of homomorphisms of groups with torsion"

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Abstract. Our previous paper [4] has an error in the proof of Theorem 1.3. In the paper, we give a counterexample for Theorem 1.3 in [4]. We present the correct formulation of Theorem 1.2, in which we prove the equality $\mathrm{cat}(\phi) = \mathrm{cd}(\phi)$ for homomorphisms $\phi : \Gamma \to \Lambda$ between finitely generated abelian groups Γ and Λ, where $\phi(T(\Gamma)) = 0$ for the torsion subgroups $T(\Gamma)$ of Γ.

Introduction

We recall that the Lusternik–Schnirelmann category $\mathrm{cat}(\phi)$ of a group homomorphism $\phi : \Gamma \to \Lambda$ is defined as the minimum number k for which there exists an open cover of $B\Gamma$ by $k+1$ subsets $U_0, \dots, U_k$ such that the restriction $\bar{\phi}|_{U_i}$ is nullhomotopic for all i where $\bar{\phi} : B\Gamma \to B\Lambda$ is a map that induces ϕ on the fundamental groups.

The cohomological dimension $\mathrm{cd}(\phi)$ of the group homomorphism ϕ is defined as the supremum of k for which there exists a Λ-module M such that the induced map on cohomology $\phi^* : H^k(\Lambda, M) \to H^k(\Gamma, M)$ is non-trivial.

Let $T(\Gamma)$ denote the torsion subgroups of an abelian group Γ. The main result of the above paper [4] is proven only for the homomorphisms of finitely generated abelian groups with $\phi(T(\Gamma)) = 0$. Thus, the proper

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formulation of the main theorem (Theorem 1.2 in the paper [4]) is the following.

Theorem 1. *For any homomorphism $\phi : \Gamma \to \Lambda$ between finitely generated abelian groups satisfying the condition $\phi(T(\Gamma)) = 0$,*

$$\mathrm{cat}(\phi) = \mathrm{cd}(\phi).$$

It turns out that Theorem 1.3 in the paper [4] claiming the equality $\mathrm{cd}(\phi) = \infty$ for nonzero homomorphisms $\phi : G \to H$ of a torsion group G is not true in view of the following.

Example 1 (A. Dranishnikov)**.** $\mathrm{cd}(\phi) = 2$ for an epimorphism

$$\phi : \mathbb{Z}_{p^2} \to \mathbb{Z}_p.$$

Proof. The proof is based on the following unpublished observation by Bestvina and Edwards: *There is a degree 0 map*

$$f : L^3_{p^2} \to L^3_p$$

of lens spaces that induces an epimorphism $f_* : \mathbb{Z}_{p^2} \to \mathbb{Z}_p$ of the fundamental groups.

Here is the construction of f. We consider two circles S^1 with $\mathbb{Z}_p$ and $\mathbb{Z}_{p^2}$ free actions. Note that a $\mathbb{Z}_{p^2}$-equivariant map $\psi : S^1 \to S^1$ between them has degree p. Then the join product $\psi * \psi : S^1 * S^1 \to S^1 * S^1$ has degree p^2. Hence it induces a map of the orbit spaces $\bar{\psi} : L^3_{p^2} \to L^3_p$ of degree p. Let $q_k : S^3 \to L^3_{p^k}$ denote the projection onto the orbit space.

We define f as the composition

$$L^3_{p^2} = L^3_{p^2} \# S^3 \to L^3_{p^2} \vee S^3 \overset{1\vee r}{\to} L^3_{p^2} \vee S^3 \overset{\bar{\psi}\vee q_1}{\to} L^3_p \vee L^3_p \overset{j}{\to} L^3_p$$

where $r : S^3 \to S^3$ has degree -1 and j identifies two copies of L^3_p. Thus, $deg(f) = p - p = 0$.

Claim 1. The induced homomorphism $f^* : H^3(L^3_p; M) \to H^3(L^3_{p^2}; M)$ is trivial for any $\mathbb{Z}_p$-module M.

Indeed, if $f^*(\alpha) \neq 0$ for $\alpha \in H^3(L^3_p; M)$ then by the Poincare duality with local coefficients $[L^3_{p^2}] \cap f^*(\alpha) \neq 0$. Since f induces an epimorphism of the fundamental groups, the induced homomorphism for 0-homology $f_* : H_0(L^3_{p^2}; M) \to H_0(L^3_p; M)$ is an isomorphism.We obtain a contradiction

$$0 \neq f_*([L^3_{p^2}] \cap f^*(\alpha)) = f_*([L^3_{p^2}]) \cap \alpha = 0.$$

The map f extends to a map $\phi : L^\infty_{p^2} \to L^\infty_p$.

Claim 2. The induced homomorphism

$$\phi^* : H^3(L_p^\infty; M) \to H^3(L_{p^2}^\infty; M)$$

is trivial for any $\mathbb{Z}_p$-module M.

This follows from the fact that the inclusion homomorphism

$$H^k(X; M) \to H^k(X^{(k)}; M)$$

is injective for any CW-complex X and a $\pi_1(X)$-module M.

The shift of dimension for the group cohomology [2] implies that $\phi^* : H^k(L_p^\infty; M) \to H^k(L_{p^2}^\infty; M)$ is trivial for all $k \geq 3$. Thus, $\mathrm{cd}(\phi) \leq 2$. Using integral cohomology one can check that $\mathrm{cd}(\phi) \geq 2$. □

Remark 1. The above homomorphism $\phi : \mathbb{Z}_{p^2} \to \mathbb{Z}_p$ has $\mathrm{cat}(\phi) = \infty$. This follows from the fact that the reduced K-theory cup-length of ϕ is unbounded. According to Atiyah's computations [1] $K(B\mathbb{Z}_{p^k}) = \mathbb{Z}[[\eta_k]]/(\eta_k^{p^k} - 1)$ where η_k is the pull-back of the canonical line bundle η over $\mathbb{C}P^\infty$ under the inclusion $\mathbb{Z}_{p^k} \to S^1$ and $\mathbb{Z}[[x]]$ is the ring of formal series. The reduced K-theory of $B\mathbb{Z}_{p^k}$ is the subring generated by $\eta_k - 1$. The induced homomorphism $\phi^* : K(B\mathbb{Z}_p) \to K(B\mathbb{Z}_{p^2})$ takes η_1 to η_2^p. Then it takes $(\eta_1 - 1)^m$ to a nonzero element $(\eta_2 - 1)^m$ for any m. This implies $\mathrm{cat}(\phi) \geq m$ for any m (see Proposition 5.1 in [3]).

Thus, Theorem 1 cannot be extended to the case when the domain contains torsions, which have nontrivial image.

The original proof of Theorem 1.3 in the paper [4] consists of two steps. Step 2 follows by Step 1 and the following observation: let $\phi : \Gamma \to \Lambda$ be an epimorphism. For a subgroup Γ' of Γ, consider the epimorphism $\phi' := \phi|_{\Gamma'} : \Gamma' \to \phi(\Gamma')$. Then $\mathrm{cd}(\phi') \leq \mathrm{cd}(\phi)$.

The proof of the observation follows from the naturality of Shapiro's Lemma, and the observation can be viewed as a generalisation of the fact that $\mathrm{cd}(\Gamma') \leq \mathrm{cd}(\Gamma)$.

However, Step 1 has a mistake when proving that $\mathrm{cd}(\phi) = \infty$ for the obvious epimorphism $\phi : \mathbb{Z}_{p^k} \to \mathbb{Z}_p$. The mistake in the paper [4] is that the fundamental class goes to the fundamental class under the induced homomorphism in the homology with integral coefficient.

Conclusion

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