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RESEARCH ARTICLE

Representations and relative Rota-Baxter operators of Hom-Leibniz-Poisson algebras

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Abstract. Representations and relative Rota-Baxter operators with respect to representations of Hom-Leibniz-Poisson algebras are introduced and studied. Some characterizations of these operators are obtained. The notion of matched pair and Nijenhuis operators of Hom-Leibniz-Poisson algebras are given and various relevant constructions of these Hom-algebras are deduced.

Introduction

Leibniz algebras are a non-commutative version of Lie algebras whose brackets satisfy the Leibniz identities rather than Jacobi identities [13]. As Poisson algebras have simultaneousely associative and Lie algebras structures, Leibniz-Poisson algebras [4] have simultaneousely associative and Leibniz algebras structures. Hence, they can be viewed as a non-commutative version of Poisson algebras. Recall that a (non-commutative) Leibniz-Poisson algebra is an associative (non-commutative) algebra $(A, \cdot)$ equipped with a binary bracket operation $[,] : A^{\times 2} \to A$ such that $(A, [,])$ is a Leibniz algebra and the following compatibility condition holds

$$[x \cdot y, z] = x \cdot [y, z] + [x, y] \cdot z \text{ for all } x, y, z \in A.$$

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The general theory of Hom-algebras takes its roots in the introduction of Hom-Lie algebras by D. Larsson, S. D. Silvestrov and J. T. Hartwig [11, 12, 9]. Starting from the well-known relation between associative algebras and Lie algebras, the notion of Hom-associative algebras was introduced [15] and it was proved that the commutator Hom-algebra of any Hom-associative algebra is a Hom-Lie algebra. Since then, other types of Hom-algebras have emerged; in particular, Hom-Leibniz algebras [15] were introduced as a non-commutative version of Hom-Lie algebras as well as Hom-Poisson algebras [18] which have simultaneousely Hom-associative algebra and Hom-Lie algebra structures and satisfying a certain compatibility condition. Hom-Leibniz algebras have been widely studied from the point of view of representation and cohomology theory [5], deformation theory [2, 16] in recent years. The same is true for Hom-associative algebras since, the definition of Hochschild-type cohomology and the study of the one parameter formal deformation theory for these type of Hom-algebras are given [1, 14].

The notion of Rota-Baxter operators on associative algebras was introduced [3] and those on Leibniz algebras are considered in [17]. It is found that, those operators have many applications in Connes-Kreimers algebraic approach to the renormalization in perturbative quantum field theory [7]. A generalization of Rota-Baxter operators called relative Rota-Baxter operators (or $\mathcal{O}$-operators) has been introduced for left Hom-Leibniz algebras [8] where the graded Lie algebra that characterizes those operators as Maurer-Cartan elements is constructed whereas in [6], the cohomology theory of $\mathcal{O}$-operators on Hom-associative algebras are found. In this paper, from the representations of Hom-associative and Hom-Leibniz algebras, we will establish those of Hom-Leibniz-Poisson algebras. We will also introduce relative Rota-Baxter operators on Hom-Leibniz-Poisson algebras after having considered them on Hom-associative and right Hom-Leibniz algebras. Most of the results for relative Rota-Baxter operators on right Hom-Leibniz algebras are established since they differ of course from those for left Hom-Leibniz algebras [8].

The paper is organized as follows. Section 2 is devoted to reminders of fundamental concepts. Some results on representations and relative Rota-Baxter operators are proved for Hom-associative algebras. Concerning section 3, similar results to those of the previous section are established for (right) Hom-Leibniz algebras. Although some of these results are proved in the case of left Hom-Leibniz algebras [8], we take them again in our case for consistency of the rest. Finally, the last section contains the main results of this work. It is a logical continuation of the results of the previous sections. Here, we introduce and study representations of Hom-Leibniz-Poisson algebras. The notions of matched pairs and relative Rota-Baxter operators of such Hom-algebras have also been discussed and interesting results have been obtained.

Throughout this paper, all vector spaces and algebras are meant over a ground field $\mathbb{K}$ of characteristic 0.

1. Preliminaries

This section is devoted to some definitions which are a very useful for next sections. Some elementary results are also proven.

Definition 1. A Hom-module is a pair (A, α_A) consisting of a $\mathbb{K}$-module A and a linear self-map $\alpha_A : A \longrightarrow A$. A morphism $f : (A, \alpha_A) \longrightarrow (B, \alpha_B)$ of Hom-modules is a linear map $f : A \longrightarrow B$ such that $f \circ \alpha_A = \alpha_B \circ f$.

Definition 2. A Hom-algebra is a triple (A, μ, α) in which (A, α) is a Hom-module, $\mu : A^{\otimes 2} \longrightarrow A$ is a linear map. The Hom-algebra (A, μ, α) is said to be multiplicative if $\alpha \circ \mu = \mu \circ \alpha^{\otimes 2}$. A morphism $f : (A, \mu_A, \alpha_A) \longrightarrow (B, \mu_B, \alpha_B)$ of Hom-algebras is a morphism of the underlying Hom-modules such that $f \circ \mu_A = \mu_B \circ f^{\otimes 2}$.

Definition 3. Let (A, μ, α) be a Hom-algebra and $\lambda \in \mathbb{K}$. Let R be a linear map satisfying

$$\mu(R(x), R(y)) = R(\mu(R(x), y) + \mu(x, R(y)) + \lambda\mu(x, y)) \quad \forall x, y \in A. \quad (1)$$

Then, R is called a Rota-Baxter operator of weight λ and (A, μ, α, R) is called a Rota-Baxter Hom-algebra of weight λ.

In the sequel, to unify our terminologies by a Rota-Baxter operator (resp. a Rota-Baxter Hom-algebra), we mean a Rota-Baxter operator (resp. a Rota-Baxter Hom-algebra) of weight $\lambda = 0$.

Definition 4 ([15])**.** A Hom-associative algebra is a multiplicative Hom-algebra (A, μ, α) satisfying the Hom-associativity condition i.e.,

$$as_\alpha(x, y, z) := \mu(\mu(x, y), \alpha(z)) - \mu(\alpha(x), \mu(y, z)) = 0 \quad \forall x, y, z \in A. \quad (2)$$

Other Hom-algebras as Hom-associative algebras, which occur regularly in this paper are Hom-Leibniz algebras.

Definition 5 ([15])**.** A (right) Hom-Leibniz algebra is a multiplicative Hom-algebra $(A, [,], \alpha)$ satisfying the Hom-Leibniz identity

$$[[x, y], \alpha(z)] = [\alpha(x), [y, z]] + [[x, z], \alpha(y)] \text{ for all } x, y, z \in A. \tag{3}$$

Recall that a morphism $f : (A, \mu_A, \alpha_A) \to (B, \mu_B, \alpha_B)$ of Hom-associative (resp. Hom-Leibniz) algebras is a morphism of underlying Hom-algebras.

Hom-Leibniz-Poisson algebras are fundamental Hom-algebras of this paper. They are a generalization of Hom-associative and Hom-Leibniz algebras. These Hom-algebras can also be viewed as a non-anticommutative version of Hom-Poisson algebras.

Definition 6. A Hom-Leibniz-Poisson algebra is a triple $(A, \cdot, [,], \alpha)$ consisting of a linear space A, two bilinear maps $\cdot, [,] : A^{\times 2} \to A$ and a linear map $\alpha : A \to A$ satisfying the following axioms:

1. $(A, \cdot, \alpha)$ is a Hom-associative algebra.
2. $(A, [,], \alpha)$ is a Hom-Leibniz algebra.
3. The following condition holds:

$$[x \cdot y, \alpha(z)] = \alpha(x) \cdot [y, z] + [x, z] \cdot \alpha(y) \text{ for all } x, y, z \in A. \tag{4}$$

A morphism $f : (A, \cdot_A, [,]_A, \alpha_A) \to (B, \cdot_B, [,]_B, \alpha_B)$ of Hom-Leibniz-Poisson algebras is a morphism of underlying Hom-associative and Hom-Leibniz algebras.

Example 1. 1. Let $\mathcal{A} := (A, \cdot, [,])$ be a Leibniz-Poisson algebra and α be a self-morphism of $\mathcal{A}$. Then, $\mathcal{A}_\alpha := (A, \cdot_\alpha := \cdot \circ \alpha^{\otimes 2}, [,]_\alpha := [,] \circ \alpha^{\otimes 2}, \alpha)$ is a Hom-Leibniz-Poisson algebra.

2. Any Hom-Poisson algebra is a Hom-Leibniz-Poisson algebra.

Definition 7. Let $(A, \cdot, [,], \alpha)$ be a Hom-Leibniz-Poisson algebra. A linear map $R : A \to A$ is called a Rota-Baxter of weight $\lambda \in \mathbb{K}$ if

$$R(x) \cdot R(y) = R(R(x) \cdot y + x \cdot R(y)) + \lambda(x \cdot y), \forall x, y \in A,$$
$$[R(x), R(y)] = R([R(x), y] + [x, R(y)]) + \lambda[x, y], \forall x, y \in A.$$

A Rota-Baxter Hom-Leibniz-Poisson algebra is a Hom-Leibniz-Poisson algebra equipped with a Rota-Baxter operator of weight $\lambda = 0$.

Definition 8. 1. A two-sided Hom-ideal of a Hom-associative (resp. Hom-Leibniz) algebra (A, μ, α) is a subspace I of A satisfying $\alpha(I) \subseteq I$, $\mu(I, A) \subseteq I$ and $\mu(A, I) \subseteq I$.
2. A two-sided Hom-ideal of a Hom-Leibniz-Poisson algebra $(A, \cdot, [,], \alpha)$ is a two-sided Hom-ideal of the Hom-associative algebra $(A, \cdot, \alpha)$ and a two-sided Hom-ideal of the Hom-Leibniz algebra $(A, [,], \alpha)$.

Let recall the definition of representations of Hom-associative algebras and some results about these notions.

Definition 9. A representation of a Hom-associative algebra $(A, \cdot, \alpha)$ is a quadruple $(V, \lambda^l, \lambda^r, \phi)$ where V is a vector space, $\phi \in gl(V)$ and $\lambda^l, \lambda^r : A \to gl(V)$ are three linear maps such that the following equalities hold for all $x, y \in A$:

$$\phi\lambda^l(x) = \lambda^l(\alpha(x))\phi; \quad \phi\lambda^r(x) = \lambda^r(\alpha(x))\phi, \tag{5}$$

$$\lambda^l(x \cdot y)\phi = \lambda^l(\alpha(x))\lambda^l(y), \tag{6}$$

$$\lambda^r(x \cdot y)\phi = \lambda^r(\alpha(y))\lambda^r(x), \tag{7}$$

$$\lambda^l(\alpha(x))\lambda^r(y) = \lambda^r(\alpha(y))\lambda^l(x). \tag{8}$$

To give examples of representations of Hom-associative algebras, let prove the following:

Proposition 1. *Let $\mathcal{A}_1 := (A_1, \mu_1, \alpha_1)$ and $\mathcal{A}_2 := (A_2, \mu_2, \alpha_2)$ be two Hom-associative algebras and $f : \mathcal{A}_1 \to \mathcal{A}_2$ a morphism of Hom-associative algebras. Then $A_2^f := (A_2, \lambda^l, \lambda^r, \alpha_2)$ is a representation of $\mathcal{A}_1$ where $\lambda^l(a)b := \mu_2(f(a), b)$ and $\lambda^r(a, b) := \mu_2(b, f(a))$ for all $(a, b) \in A_1 \times A_2$.*

Proof. Let $(x, y) \in A_1^{\times 2}$ and $z \in A_2$. First, using f is a morphism we prove (5) as follows:

$$\begin{aligned}\alpha_2(\lambda^l(x)z) &= \alpha_2\mu_2(f(x), z) = \mu_2(\alpha_2 f(x), \alpha_2(z))\\ &= \mu_2(f\alpha_1(x), \alpha_2(z)) = \lambda^l(\alpha_1(x))\alpha_2(z).\end{aligned}$$

Hence, $\alpha_2\lambda^l(x) = \lambda^l(\alpha_1(x))\alpha_2$ and similarly $\alpha_2\lambda^r(x) = \lambda^r(\alpha_1(x))\alpha_2$. Next, as f is a morphism, using (2) in $\mathcal{A}_2$, we proceed for (6) as follows:

$$\begin{aligned}\lambda^l(\mu_1(x, y))\alpha_2(z) &= \mu_2(f(\mu_1(x, y)), \alpha_2(z)) = \mu_2(\mu_2(f(x), f(y)), \alpha_2(z))\\ &= \mu_2(\alpha_2 f(x), \mu_2(f(y), z)) = \mu_2(f\alpha_1(x), \mu_2(f(y), z)) = \lambda^l(\alpha_1(x))\lambda^l(y)z.\end{aligned}$$

Therefore, $\lambda^l(\mu_1(x,y))\alpha_2 = \lambda^l(\alpha_1(x))\lambda^l(y)$. In a similar way, (7) holds. Finally, thanks to conditions $f\alpha_1 = \alpha_2 f$ and (2) in $\mathcal{A}_2$, we compute

$$\begin{aligned}\lambda^l(\alpha_1(x))\lambda^r(y)z &= \mu_2(f\alpha_1(x), \mu_2(z, f(y))) = \mu_2(\alpha_2 f(x), \mu_2(z, f(y))) \\ &= \mu_2(\mu_2(f(x), z), \alpha_2 f(y)) = \mu_2(\mu_2(f(x), z), f\alpha_1(y)) = \lambda^r(\alpha_1(y))\lambda^r(x)z\end{aligned}$$

and therefore, (8) is obtained. □

Now, using Proposition 1, we obtain the following example as applications.

Example 2. 1. Let $(A, \cdot, \alpha)$ be a Hom-associative algebra. Define a left multiplication $l : A \to gl(A)$ and a right multiplication $r : A \to gl(A)$ by $l(x)y := x \cdot y$ and $r(x)y := y \cdot x$ for all $x, y \in A$. Then (A, l, r, α) is a representation called the regular representation of $(A, \cdot, \alpha)$.

2. Let (A, μ, α) be a Hom-associative algebra and (B, α) be a two-sided Hom-ideal of (A, μ, α). Then (B, α) inherits a structure of representation of (A, μ, α) where $\rho^l(a)b := \mu(a, b)$; $\rho^r(a, b) := \mu(b, a)$ for all $(a, b) \in A \times B$.

Proposition 2. *Let $\mathcal{V} := (V, \lambda^l, \lambda^r, \phi)$ be a representation of a Hom-associative algebra $\mathcal{A} := (A, \cdot, \alpha)$ and β be a self-morphism of $\mathcal{A}$. Then $\mathcal{V}_\beta =: (V, \lambda^l_\beta := \lambda^l\beta, \lambda^r_\beta := \lambda^r\beta, \phi)$ is a representation of $\mathcal{A}$.*

Proof. Let $x, y \in A$. First, by (5) in $\mathcal{V}$ and the condition $\alpha\beta = \beta\alpha$ we have

$$\phi\lambda^l_\beta(x) = \phi\lambda^l(\beta(x)) = \lambda^l(\alpha\beta(x))\phi = \lambda^l(\beta\alpha(x))\phi = \lambda^l_\beta(\alpha(x))\phi.$$

Similarly, we prove $\phi\lambda^r_\beta(x) = \lambda^r_\beta(\alpha(x))\phi$ and hence, we obtain (5) for $\mathcal{V}_\beta$. Next, by (6) in $\mathcal{V}$ and the fact that β is a morphism, we get:

$$\begin{aligned}\lambda^l_\beta(x \cdot y)\phi &= \lambda^l(\beta(x \cdot y))\phi = \lambda^l(\beta(x) \cdot \beta(y))\phi = \lambda^l(\alpha\beta(x))\lambda^l(\beta(y)) \\ &= \lambda^l(\beta\alpha(x))\lambda^l(\beta(y)) = \lambda^l_\beta(x)\lambda^l_\beta(y),\end{aligned}$$

i.e., (6) holds and similarly, (7) holds for $\mathcal{V}_\beta$. Finally, using $\alpha\beta = \beta\alpha$ and (8) for $\mathcal{V}$, we compute

$$\begin{aligned}\lambda^l_\beta(\alpha(x))\lambda^r_\beta(y) &= \lambda^l(\beta\alpha(x))\lambda^l(\beta(y)) = \lambda^l(\alpha\beta(x))\lambda^l(\beta(y)) \\ &= \lambda^r(\alpha\beta(y))\lambda^l(\beta(x)) = \lambda^r_\beta(\alpha(y))\lambda^l_\beta(x).\end{aligned}$$

□

Let us recall the following necessary results for the last section of this paper.

Proposition 3 ([10]). *Let $(A, \cdot, \alpha)$ be a Hom-associative algebra. Then $(V, \lambda^l, \lambda^r, \phi)$ is a representation of $(A, \cdot, \alpha)$ if and only if the direct sum of vector spaces, $A \oplus V$, turns into a Hom-associative algebra with the multiplication and the linear map defined by*

$$(x + u) * (y + v) := x \cdot y + (\lambda^l(x)v + \lambda^r(y)u), \tag{9}$$

$$(\alpha \oplus \phi)(x + u) := \alpha(x) + \phi(u). \tag{10}$$

This Hom-associative algebra is called the semi-direct product of A with V.

Definition 10 ([10]). Let $\mathcal{A}_1 := (A_1, \cdot, \alpha_1)$ and $\mathcal{A}_2 := (A_2, \bullet, \alpha_2)$ two Hom-associative algebras. Let $\lambda_1^l, \lambda_1^r : A_1 \to gl(A_2)$ and $\lambda_2^l, \lambda_2^r : A_2 \to gl(A_1)$ be linear maps such that $(A_2, \lambda_1^l, \lambda_1^r, \alpha_2)$ is a representation of $\mathcal{A}_1$, $(A_1, \lambda_2^l, \lambda_2^r, \alpha_2)$ is a representation of $\mathcal{A}_2$ and the following conditions hold

$$\begin{aligned}
\lambda_1^l(\alpha_1(x))(u \bullet v) &= \lambda_1^l(\lambda_2^r(u)x)\alpha_2(v) + (\lambda_1^l(x)u) \bullet \alpha_2(v),\\
\lambda_1^r(\alpha(x))(u \bullet v) &= \lambda_1^r(\lambda_2^l(v)x)\alpha_2(u) + \alpha_2(u) \bullet (\lambda_1^r(x)v),\\
\lambda_2^l(\alpha_2(u))(x \cdot y) &= \lambda_2^l(\lambda_1^r(x)u)\alpha_1(y) + (\lambda_2^l(\alpha_2(u)x)) \cdot \alpha_1(y),\\
\lambda_2^r(\alpha_2(u))(x \cdot y) &= \lambda_2^r(\lambda_1^l(y)u)\alpha_1(x) + \alpha_1(x) \cdot (\lambda_2^r(u)y),
\end{aligned}$$

$$\lambda_1^l(\lambda_2^l(u)x)\alpha_2(v) + (\lambda_1^r(x)u) \bullet \alpha_2(v) - \lambda_1^r(\lambda_2^r(v)x)\alpha_2(u) - \alpha_2(u) \bullet (\lambda_1^l(x)v) = 0,$$

$$\lambda_2^l(\lambda_1^l(x)u)\alpha_1(y) + (\lambda_2^r(u)x) \cdot \alpha_1(y) - \lambda_2^r(\lambda_1^r(y)u)\alpha_1(x) - \alpha_1(x) \cdot (\lambda_2^l(u)y) = 0.$$

Then $((A_1, \lambda_2^l, \lambda_2^r, \alpha_1), (A_2, \lambda_1^l, \lambda_1^r, \alpha_2))$ is called a matched pair of Hom-associative algebras.

Proposition 4 ([10]). *Let $((A_1, \lambda_2^l, \lambda_2^r, \alpha_1), (A_2, \lambda_1^l, \lambda_1^r, \alpha_2))$ be a matched pair of Hom-associative algebras. Then, there is a Hom-associative algebra $(A_1 \oplus A_2, *, \alpha_1 \oplus \alpha_2)$ defined by*

$$\begin{aligned}
(x + b) * (y + b) &:= (x \cdot y + \lambda_2^l(a)y + \lambda_2^r(b)x) + \\
&\quad\ (a \bullet b + \lambda_1^l(x)b + \lambda_1^r(y)a), \qquad (11)\\
(\alpha_1 \oplus \alpha_2)(x + a) &:= \alpha_1(x) + \alpha_2(a).
\end{aligned}$$

Recall the following notion known as $\mathcal{O}$-operator for Hom-associative algebras in the literature.

Definition 11. Let $(V, \lambda^l, \lambda^r, \phi)$ be a representation of a Hom-associative algebra $(A, \cdot, \alpha)$. A linear operator $T : V \to A$ is called a relative Rota-Baxter operator on $(A, \cdot, \alpha)$ with respect to $(V, \lambda^l, \lambda^r, \phi)$ if T satisfies

$$T\phi = \alpha T, \tag{12}$$

$$(Tu) \cdot (Tv) = T(\lambda^l(Tu)v + \lambda^r(Tv)u) \text{ for all } u, v \in V. \tag{13}$$

Observe that Rota-Baxter operators on Hom-associative algebras are relative Rota-Baxter operators with respect to the regular representation.

Example 3. Consider the 2-dimensional Hom-associative algebra $(A, \cdot, \alpha)$ where the non-zero products with respect to a basis (e_1, e_2) are given by: $e_1 \cdot e_2 = e_2 \cdot e_1 := -e_1,\ e_2 \cdot e_2 := e_1 + e_2$; and $\alpha(e_1) := -e_1,\ \alpha(e_2) := e_1 + e_2$. Then a linear map $T : A \to A$ defined by $T(e_1) := a_{11}e_1 + a_{21}e_2;\ T(e_2) := a_{12}e_1 + a_{22}e_2$ is a relative Rota-Baxter on $(A, \cdot, \alpha)$ with respect to the regular representation if and only if $T\alpha = \alpha T$ and

$$(Te_i) \cdot (Te_j) = T((Te_i) \cdot e_j + e_i \cdot (Te_j)) \text{ for all } i, j \in \{1, 2\}. \tag{14}$$

The condition $\alpha T = T\alpha$ is equivalent to

$$a_{21} = 0;\ a_{11} + 2a_{12} - a_{22} = 0.$$

For $i = j = 1$, the condition (14) is satisfied trivially. Similarly, we obtain for $(i, j) \in \{(1, 2), (2, 1)\}$

$$-a_{11}a_{12} = -a_{11}a_{12} - a_{11}^2;$$

and for $i = j = 2$,

$$-2a_{12}a_{22} + a_{22}^2 = -2a_{12}a_{11} + 2a_{22}a_{11} + 2a_{22}a_{12};\ a_{22}^2 = 4a_{22}^2.$$

Summarize the above discussions, we observe that the zero-map is the only relative Rota-Baxter operator on $(A, \cdot, \alpha)$ with respect to the regular representation.

Until the end of this section, we state and prove some results affirmed in [6]. These results will be used in the last section on this paper.

Lemma 1. *Let T be a relative Rota-Baxter operator on a Hom-associative algebra $(A, \cdot, \alpha)$ with respect to a representation $(V, \lambda^l, \lambda^r, \phi)$. If define a map "$\diamond$" on V by*

$$u \diamond v := \lambda^l(Tu)v + \lambda^r(Tv)u \text{ for all } (u, v) \in V^{\times 2}, \tag{15}$$

then $(V, \diamond, \phi)$ is a Hom-associative algebra.

Proof. First, note that the multiplicativity of ϕ with respect to $\diamond$ follows from conditions (5) and (12). Next, pick $u, v, w \in V$ and observe from (15) and (13) that $T(u \diamond v) = Tu \cdot Tv$. Therefore, by a straightforward computations:

$$(u \diamond v) \diamond \phi(v) = \lambda^l(Tu \cdot Tv)\phi(w)$$
$$+\lambda^r(T\phi(w))\lambda^l(Tu)v + \lambda^r(T\phi(w))\lambda^r(Tv)u.$$

Similarly, after rearranging terms

$$\phi(u) \diamond (v \diamond w) = \lambda^l(T\phi(u))\lambda^l(Tv)w + \lambda^l(T\phi(u))\lambda^r(Tw)v$$
$$+\lambda^r(Tv \cdot Tw)\phi(u).$$

Hence, the Hom-associativity in $(V, \diamond, \phi)$ follows by (12) and (6)−(8). □

Corollary 1. *Let T be a relative Rota-Baxter operator on a Hom-associative algebra $(A, \cdot, \alpha)$ with respect to a representation $(V, \lambda^l, \lambda^r, \phi)$. Then, T is a morphism from the Hom-associative algebra $(V, \diamond, \phi)$ to the initial Hom-associative algebra $(A, \cdot, \alpha)$.*

Theorem 1. *Let T be a relative Rota-Baxter operator on a Hom-associative algebra $(A, \cdot, \alpha)$ with respect to a representation $(V, \lambda^l, \lambda^r, \phi)$. Then, $(A, \overline{\lambda^l}, \overline{\lambda^r}, \alpha)$ is a representation of the Hom-associative algebra $(V, \diamond, \phi)$ where*

$$\overline{\lambda^l}(u)x := (Tu) \cdot x - T\lambda^r(x)u \text{ for all } (x, u) \in A \times V, \tag{16}$$
$$\overline{\lambda^r}(u)x := x \cdot (Tu) - T\lambda^l(x)u \text{ for all } (x, u) \in A \times V. \tag{17}$$

Proof. First, (5) in $(A, \overline{\lambda^l}, \overline{\lambda^r}, \alpha)$ follows from the one in $(V, \lambda^l, \lambda^r, \phi)$, the multiplicativity of α with respect to $\cdot$ and (12). Next, for all $x \in A$ and $(u, v) \in V^{\times 2}$, if one observes from Corollary 1 that $T(u \diamond v) = Tu \cdot Tv$, we compute:

$$\overline{\lambda^l}(u \diamond v)\alpha(x) = (Tu \cdot Tv) \cdot \alpha(x) - T\lambda^r(\alpha(x))\lambda^l(Tu)v$$
$$-T\lambda^r(\alpha(x))\lambda^r(Tv)u = (T\phi(u)) \cdot (Tv \cdot x) - T\lambda^l(T\phi(u))\lambda^r(x)v$$
$$-T\lambda^r(\alpha(x))\lambda^r(Tv)u \text{ (by (12), (2) and (8))}.$$

Similarly, by straightforward computations:

$$\overline{\lambda^l}(\phi(u))\overline{\lambda^l}(v)x = (T\phi(u)) \cdot (Tv \cdot x) - T\phi(u) \cdot T\lambda^r(x)v$$
$$-T\lambda^r(Tv \cdot x)\phi(u) + T\lambda^r(T\lambda^r(x)v)\phi(u) = (T\phi(u)) \cdot (Tv \cdot x)$$
$$-T\lambda^l(T\phi(u))\lambda^r(x)v - T\lambda^r(T\lambda^r(x)v)\phi(u)$$
$$-T\lambda^r(Tv \cdot x)\phi(u) + T\lambda^r(T\lambda^r(x)v)\phi(u) \text{ (by (13))}$$

$$= (T\phi(u)) \cdot (Tv \cdot x) - T\lambda^l(T\phi(u))\lambda^r(x)v - T\lambda^r(Tv \cdot x)\phi(u).$$

Hence, we obtain (6) in $(A, \overline{\lambda^l}, \overline{\lambda^r}, \alpha)$ by (7) in $(V, \lambda^l, \lambda^r, \phi)$. Next to prove (7) in $(A, \overline{\lambda^l}, \overline{\lambda^r}, \alpha)$, let proceed as the previous case, i.e.,

$$\begin{aligned}\overline{\lambda^r}(u \diamond v)\alpha(x) = \alpha(x) \cdot (Tu \cdot Tv) - T\lambda^l(\alpha(x))\lambda^l(Tu)v \\ -T\lambda^l(\alpha(x))\lambda^r(Tv)u, \\ \overline{\lambda^r}(\phi(v))\overline{\lambda^r}(u)x = (x \cdot Tu) \cdot T\phi(v) - (T\lambda^l(x)u) \cdot T\phi(v) \\ -T\lambda^l(x \cdot Tu)\phi(v) + T\lambda^l(T\lambda^l(x)u)\phi(v) = (x \cdot Tu) \cdot T\phi(v) \\ -T\lambda^r(T\phi(v))\lambda^l(x)u - T\lambda^l(x \cdot Tu)\phi(v) \text{ (by (13))}.\end{aligned}$$

Hence, we obtain (7) by (12) and (2), (6), (8) in $(V, \lambda^l, \lambda^r, \phi)$. Finally, we compute:

$$\begin{aligned}\overline{\lambda^l}(\phi(u))\overline{\lambda^r}(v)x = T\phi(u) \cdot (x \cdot Tv) - T\phi(u) \cdot T\lambda^l(x)v - T\lambda^r(x \cdot Tv)\phi(u) \\ +T\lambda^r(T\lambda^l(x)v)\phi(u) = T\phi(u) \cdot (x \cdot Tv) - T\lambda^l(T\phi(u))\lambda^l(u)v \\ -T\lambda^r(T\lambda^l(x)v)\phi(u) - T\lambda^r(x \cdot Tv)\phi(u) + T\lambda^r(T\lambda^l(x)v)\phi(u) \text{ (by (13))} \\ = T\phi(u) \cdot (x \cdot Tv) - T\lambda^l(T\phi(u))\lambda^l(u)v - T\lambda^r(x \cdot Tv)\phi(u).\end{aligned}$$

Similarly, after a straightforward computation, we get also by (13):

$$\begin{aligned}\overline{\lambda^r}(\phi(v))\overline{\lambda^l}(u)x = (Tu \cdot x) \cdot T\phi(v) \\ -T\lambda^r(T\phi(v))\lambda^r(x)u - T\lambda^l(Tu \cdot x)\phi(v).\end{aligned}$$

Therefore, we obtain the desired identity by (12), (2), (6) and (7). □

2. Matched pair and relative Rota-Baxter operator on Hom-Leibniz algebras

This section is primarily dedicated to the study of relative Rota-Baxter operators for (right) Hom-Leibniz algebras. Although some of the results obtained here are treated for left Hom-Leibniz algebras [8], we will also treat them in the right Hom-Leibniz algebras. Indeed, these were needed for the understanding of the last section which is full of the fundamental results of this paper.

Definition 12 ([5]). A representation of a Hom-Leibniz algebra $(A, [,], \alpha)$ is a quadruple $(V, \phi, \rho^l, \rho^r)$ where V is a vector space, $\phi \in gl(V)$ and

$\rho^l, \rho^r : A \to gl(V)$ are three linear maps such that the following equalities hold for all $x, y \in A$:

$$\rho^l(\alpha(x))\phi = \phi\rho^l(x); \rho^l(\alpha(x))\phi = \phi\rho^l(x), \tag{18}$$

$$\rho^l([x,y])\phi = \rho^l(\alpha(x))\rho^l(y) + \rho^r(\alpha(y))\rho^l(x), \tag{19}$$

$$\rho^r(\alpha(y))\rho^l(x) = \rho^l(\alpha(x))\rho^r(y) + \rho^l([x,y])\phi, \tag{20}$$

$$\rho^r(\alpha(y))\rho^r(x) = \rho^r([x,y])\phi + \rho^r(\alpha(x))\rho^r(y). \tag{21}$$

Observe that (21) gives rise to the useful equation:

$$\rho^r([x,y])\phi + \rho^r([y,x])\phi = 0 \text{ for all } (x,y) \in A^{\times 2}. \tag{22}$$

To give examples of representations of Hom-Leibniz algebras, let prove:

Proposition 5. *Let $\mathcal{A}_1 := (A_1, [,]_1, \alpha_1)$ and $\mathcal{A}_2 := (A_2, [,]_2, \alpha_2)$ be two Hom-Leibniz algebras and $f : \mathcal{A}_1 \to \mathcal{A}_2$ be a morphism of Hom-Leibniz algebras. Then $(A_2, \rho^l, \rho^r, \alpha_2)$ is a representation of $\mathcal{A}_1$ where $\rho^l(a)b := [f(a), b]_2$ and $\rho^r(a)b := [b, f(a)]_2$ for all $(a,b) \in A \times B$.*

Proof. As (5) in Proposition 1, (18) holds. Next, let $(x,y) \in A_1^{\times 2}$ and $z \in A_2$. Then, using f is a morphism and (3), we compute

$$\begin{gathered}
\rho^l([x,y]_1)\alpha_2(z) = [f([x,y]_1), \alpha_2(z)]_2 = [[f(x), f(y)]_2, \alpha_2(z)]_2 \\
= [\alpha_2 f(x), [f(y), z]_2]_2 + [[f(x), z]_2, \alpha_2 f(y)]_2 = [f\alpha_1(x), [f(y), z]_2]_2 \\
+[[f(x), z]_2, f\alpha_1(y)]_2 = \rho^l(\alpha_1(x))\rho^l(y)z + \rho^r(\alpha_1(y))\rho^l(x)z \text{ and} \\
\rho^r(\alpha_1(y))\rho^l(x)z = [[f(x), z]_2, f\alpha_1(y)]_2 = [[f(x), z]_2, \alpha_2 f(y)]_2 \\
= [\alpha_2 f(x), [z, f(y)]_2]_2 + [[f(x), f(y)]_2, \alpha_2(z)]_2 = [f\alpha_1(x), [z, f(y)]_2]_2 \\
+[f([x,y]_1), \alpha_2(z)]_2 = \rho^l(\alpha_1(x))\rho^r(y)z + \rho^l([x,y]_1)\alpha_2(z).
\end{gathered}$$

Therefore, we get (19) and (20). In a similar way, we prove (21). □

As Hom-associative case, using Proposition 5, we get:

Example 4.

1. Let $(A, [,], \alpha)$ be a Hom-Leibniz algebra. Define a left multiplication $l : A \to gl(A)$ and a right multiplication $r : A \to gl(A)$ by $l(x)y := [x,y]$ and $r(x)y := [y,x]$ for all $x, y \in A$. Then (A, l, r, α) is a representation called the regular representation of $(A, [,], \alpha)$.

2. Let $(A, [,], \alpha)$ be a Hom-Leibniz algebra and (B, α) be a two-sided Hom-ideal of $(A, [,], \alpha)$. Then (B, α) inherits a structure of representation of $(A, [,], \alpha)$ where $\rho^l(a)b := [a,b]$; $\rho^r(b,a) := [b,a]$ for all $(a,b) \in A \times B$.

Proposition 6. *Let $\mathcal{V} := (V, \rho^l, \rho^r, \phi)$ be a representation of a Hom-Leibniz algebra $\mathcal{A} := (A, [,], \alpha)$ and β be a self-morphism of $\mathcal{A} := (A, [,], \alpha)$. Then $\mathcal{V}_\beta := (V, \rho^l_\beta := \rho^l\beta, \rho^r_\beta := \rho^r\beta, \phi)$ is a representation of $\mathcal{A}$.*

Proof. Let $x, y \in A$. First, by (18) in $\mathcal{V}$ and the condition $\alpha\beta = \beta\alpha$ we have:

$$\phi\rho^l_\beta(x) = \phi\rho^l(\beta(x)) = \rho^l(\alpha\beta(x))\phi = \rho^l(\beta\alpha(x))\phi = \rho^l_\beta(\alpha(x))\phi.$$

Similarly, we prove $\phi\rho^r_\beta(x) = \rho^r_\beta(\alpha(x))\phi$ and hence, we obtain (18) for $\mathcal{V}_\beta$. Next, by (19) in $\mathcal{V}$ and the fact that β is a morphism, we get:

$$\begin{aligned}\rho^l_\beta([x,y])\phi = \rho^l(\beta([x,y]))\phi = \rho^l([\beta(x),\beta(y)])\phi = \rho^l(\alpha\beta(x))\rho^l(\beta(y))\\ +\rho^r(\alpha\beta(y))\rho^l(\beta(x)) = \rho^l(\beta\alpha(x))\rho^l(\beta(y)) + \rho^r(\beta\alpha(y))\rho^l(\beta(x))\\ = \rho^l_\beta(\alpha(x))\rho^l_\beta(y) + \rho^r_\beta(\alpha(y))\rho^l_\beta(x),\end{aligned}$$

i.e., (19) holds for $\mathcal{V}_\beta$. Finally, using $\alpha\beta = \beta\alpha$ and (20) for $\mathcal{V}$, we compute

$$\begin{aligned}\rho^r_\beta(\alpha(y))\rho^l_\beta(x) = \rho^r(\beta\alpha(y))\rho^l(\beta(x)) = \rho^r(\alpha\beta(y))\rho^l(\beta(x))\\ = \rho^l(\alpha\beta(x))\rho^r(\beta(y))\phi + \rho^l([\beta(x),\beta(y)]) = \rho^l(\beta\alpha(x))\rho^r(\beta(y))\\ +\rho^l(\beta([x,y])) = \rho^l_\beta(\alpha(x))\rho^r_\beta(y) + \rho^l_\beta([x,y]).\end{aligned}$$

Therefore, we obtain (20) and similarly (21) for $\mathcal{V}_\beta$. □

Proposition 7. *Let $(A, [,], \alpha)$ be a Hom-Leibniz algebra and let (V, ϕ) be a Hom-module. Let $\rho^l, \rho^r : A \to gl(V)$ be two linear maps. The quadruple $(V, \rho^l, \rho^r, \phi)$ is a representation of $(A, [,], \alpha)$ if and only if the direct sum of vector spaces $A \oplus V$ turns into a Hom-Leibniz algebra by defining a multiplication and a linear map by*

$$\{(x+u),(y+v)\} := [x,y] + (\rho^l(x)v + \rho^r(y)u), \tag{23}$$

$$(\alpha \oplus \phi)(x+u) := \alpha(x) + \phi(u). \tag{24}$$

Proof. First, observe that the multiplicativity of $\alpha \oplus \phi$ with respect to $\{,\}$ is equivalent to (18). Next, pick $x, y, z \in A$ and $u, v, w \in V$. Then, by a straightforward computation, we obtain

$$\begin{aligned}\{\{x+u, y+v\}, (\alpha\oplus\phi)(z+w)\}\\ = \underbrace{[[x,y],\alpha(z)]}_{(i)} + \rho^l([x,y])\phi(w) + \rho^r(\alpha(z))\rho^l(x)v + \rho^r(\alpha(z))\rho^r(y)u,\end{aligned}$$

$$\begin{aligned}&\{(\alpha\oplus\phi)(x+u),\{y+v,z+w\}\}\\ &=[\alpha(x),[y,z]]+\rho^l(\alpha(x))\rho^l(y)w+\rho^l(\alpha(x))\rho^r(z)v+\rho^r([y,z])\phi(u),\end{aligned}$$

$$\begin{aligned}&\{\{x+u,y+v\},(\alpha\oplus\phi)(z+w)\}\\ &=[[x,z],\alpha(y)]+\rho^l([x,z])\phi(v)+\rho^r(\alpha(y))\rho^l(x)w+\rho^r(\alpha(y))\rho^r(z)u.\end{aligned}$$

Hence, by (3) for (i), we get:

$$\begin{aligned}&\{\{x+u,y+v\},(\alpha\oplus\phi)(z+w)\}-\{(\alpha\oplus\phi)(x+u),\{y+v,z+w\}\}\\ &-\{\{x+u,y+v\},(\alpha\oplus\phi)(z+w)\}=\Big(\rho^l([x,y])\phi(w)-\rho^l(\alpha(x))\rho^l(y)w\\ &-\rho^r(\alpha(y))\rho^l(x)w\Big)+\Big(\rho^r(\alpha(z))\rho^l(x)v-\rho^l(\alpha(x))\rho^r(z)v-\rho^l([x,z])\phi(v)\Big)\\ &+\Big(\rho^r(\alpha(z))\rho^r(y)u-\rho^r([y,z])\phi(u)-\rho^r(\alpha(y))\rho^r(z)u\Big).\end{aligned}$$

Therefore, (3) holds if and only if (19), (20) and (21) hold. □

Let give the following:

Definition 13. Let $\mathcal{A}_1 := (A_1,[,]_1,\alpha_1)$ and $\mathcal{A}_2 := (A_2,[,]_2,\alpha_2)$ be two Hom-Leibniz algebras. Let $\rho_1^l,\rho_1^r : A_1 \to gl(A_2)$ and $\rho_2^l,\rho_2^r : A_2 \to gl(A_1)$ be linear maps such that $(A_2,\rho_1^l,\rho_1^r,\alpha_2)$ is a representation of $\mathcal{A}_1$, $(A_1,\rho_2^l,\rho_2^r,\alpha_1)$ is a representation of $\mathcal{A}_2$ and the following conditions hold

$$\rho_1^r(\alpha_1(x))[u,v]_2-[\alpha_2(u),\rho_1^r(x)v]_2-[\rho_1^r(x)u,\alpha_2(v)]_2 -\rho_1^r(\rho_2^l(v)x)\alpha_2(u)-\rho_1^l(\rho_2^l(u)x)\alpha_2(v)=0, \tag{25}$$

$$\rho_1^l(\alpha_1(x))[u,v]_2-[\rho_1^l(x)u,\alpha_2(v)]_2+[\rho_1^l(x)v,\alpha_2(u)]_2 -\rho_1^l(\rho_2^r(u)x)\alpha_2(v)+\rho_1^l(\rho_2^r(v)x)\alpha_2(u)=0, \tag{26}$$

$$\rho_1^r(\alpha_1(x))[u,v]_2-[\rho_1^r(x)u,\alpha_2(v)]_2+[\alpha_2(u),\rho_1^l(x)v]_2 -\rho_1^l(\rho_2^l(u)x)\alpha_2(v)+\rho_1^r(\rho_2^r(v)x)\alpha_2(u)=0, \tag{27}$$

$$\rho_2^r(\alpha_2(u))[x,y]_1-[\alpha_1(x),\rho_2^r(u)y]_1-[\rho_2^r(u)x,\alpha_1(y)]_1 -\rho_2^r(\rho_1^l(y)u)\alpha_2(x)-\rho_2^l(\rho_1^l(x)u)\alpha_1(y)=0, \tag{28}$$

$$\rho_2^l(\alpha_2(u))[x,y]_1-[\rho_2^l(u)x,\alpha_1(y)]_2+[\rho_2^l(u)y,\alpha_1(x)]_1 -\rho_2^l(\rho_1^r(x)u)\alpha_1(y)+\rho_2^l(\rho_1^r(y)u)\alpha_1(x)=0, \tag{29}$$

$$\rho_2^r(\alpha_2(u))[x,y]_1-[\rho_2^r(u)x,\alpha_1(y)]_1+[\alpha_1(x),\rho_2^l(u)y]_1 -\rho_2^l(\rho_1^l(x)u)\alpha_1(y)+\rho_2^r(\rho_1^r(y)u)\alpha_1(x)=0. \tag{30}$$

Then $((A_1,\rho_2^l,\rho_2^r,\alpha_1),(A_2,\rho_1^l,\rho_1^r,\alpha_2))$ is called a matched pair of Hom-Leibniz algebras.

Remark 1. Observe that (25) and (27) give rise to

$$[\alpha_2(u),\rho_1^l(x)v]_2+\rho_1^r(\rho_2^l(v)x)\alpha_2(u)+[\alpha_2(u),\rho_1^r(x)v]_2 \\ +\rho_1^r(\rho_2^r(v)x)\alpha_2(u)=0$$

and (28) and (30) give rise to

$$[\alpha_1(x),\rho_2^l(u)y]_1+\rho_2^r(\rho_1^l(y)u)\alpha_1(x)+[\alpha_1(x),\rho_2^r(u)y]_1 \\ +\rho_2^r(\rho_1^r(y)u)\alpha_1(x)=0.$$

Proposition 8. *Let $((A_1,\rho_2^l,\rho_2^r,\alpha_1),(A_2,\rho_1^l,\rho_1^r,\alpha_2))$ be a matched pair of Hom-Leibniz algebras. Then, there is a Hom-Leibniz algebra $A_{\bowtie}:=(A_1\oplus A_2,\{,\},\alpha_1\oplus\alpha_2)$ defined by*

$$\begin{aligned}\{(x+u),(y+v)\} &:= ([x,y]_1+\rho_2^l(u)y+\rho_2^r(v)x)+([u,v] \\ &\quad +\rho_1^l(x)v+\rho_1^r(y)u), \\ (\alpha_1\oplus\alpha_2)(x+u) &:= \alpha_1(x)+\alpha_2(u).\end{aligned} \tag{31}$$

Proof. The multiplicativity of $A_{\bowtie}$ follows from the one of α_1 and α_2 with respect to $[,]_1$ and to $[,]_2$ respectively and the condition (18). Next, pick $(x,y,z)\in A_1^{\times 3}$ and $(u,v,w)\in A_2^{\times 3}$. Then by direct computations

$$\begin{aligned}&\{\{x+u,y+v\},(\alpha_1\oplus\alpha_2)(z+w)\} \\ &=\underbrace{[[x,y]_1,\alpha_1(z)]_1}_{(i_1)}+[\rho_2^l(u)y,\alpha_1(z)]_1+[\rho_2^r(v)x,\alpha_1(z)]_1 \\ &+\underbrace{\rho_2^l([u,v]_2)\alpha_1(z)}_{(i_2)}+\rho_2^l(\rho_1^l(x)v)\alpha_1(z)+\rho_2^l(\rho_1^r(y)u)\alpha_1(z) \\ &+\rho_2^r(\alpha_2(w))[x,y]_1+\underbrace{\rho_2^r(\alpha_2(w))\rho_2^l(u)y}_{(i_3)}+\underbrace{\rho_2^r(\alpha_2(w))\rho_2^l(v)x}_{(i_4)} \\ &+\underbrace{[[u,v]_2,\alpha_2(w)]_2}_{(i_1)}+[\rho_1^l(x)v,\alpha_2(w)]_2+[\rho_1^r(y)u,\alpha_2(w)]_2 \\ &+\underbrace{\rho_1^l([x,y]_1)\alpha_2(w)}_{(i_2)}+\rho_1^l(\rho_2^l(u)y)\alpha_2(w)+\rho_1^l(\rho_2^r(v)x)\alpha_2(w) \\ &+\rho_1^r(\alpha_1(z))[u,v]_2+\underbrace{\rho_1^r(\alpha_1(z))\rho_1^l(x)v}_{(i_3)}+\underbrace{\rho_1^r(\alpha_1(z))\rho_1^l(y)u}_{(i_4)},\end{aligned}$$

$$\begin{aligned}&\{(\alpha_1\oplus\alpha_2)(x+u),\{y+v,z+w\}\} \\ &=[\alpha_1(x),[y,z]_1]_1+[\alpha_1(x),\rho_2^l(v)z]_1+[\alpha_1(x),\rho_2^r(w)y]_1\end{aligned}$$

$$
\begin{aligned}
&+\rho_2^l(\alpha_2(u))[y,z]_1+\rho_2^l(\alpha_2(u))\rho_2^l(v)z+\rho_2^l(\alpha_2(u))\rho_2^r(w)y\\
&+\rho_2^r([v,w]_2)\alpha_1(x)+\rho_2^r(\rho_1^l(y)w)\alpha_1(x)+\rho_2^r(\rho_1^r(z)v)\alpha_1(x)\\
&+[\alpha_2(u),[v,w]_2]_2+[\alpha_2(u),\rho_1^l(y)w]_2+[\alpha_2(u),\rho_1^r(z)v]_2\\
&+\rho_1^l(\alpha_1(x))[v,w]_2+\rho_1^l(\alpha_1(x))\rho_1^l(y)w+\rho_1^l(\alpha_1(x))\rho_1^r(z)v\\
&+\rho_1^r([y,z]_1)\alpha_2(u)+\rho_1^r(\rho_2^l(v)z)\alpha_2(u)+\rho_1^r(\rho_2^r(w)y)\alpha_2(u),
\end{aligned}
$$

$$
\begin{aligned}
&\{\{x+u,y+v\},(\alpha_1\oplus\alpha_2)(z+w)\}\\
&=[[x,z]_1,\alpha_1(y)]_1+[\rho_2^l(u)z,\alpha_1(y)]_1+[\rho_2^r(w)x,\alpha_1(y)]_1\\
&+\rho_2^l([u,w]_2)\alpha_1(y)+\rho_2^l(\rho_1^l(x)w)\alpha_1(y)+\rho_2^l(\rho_1^r(z)u)\alpha_1(y)\\
&+\rho_2^r(\alpha_2(v))[y,z]_1+\rho_2^r(\alpha_2(v))\rho_2^l(u)z+\rho_2^r(\alpha_2(v))\rho_2^r(w)x\\
&+[[u,w]_2,\alpha_2(v)]_2+[\rho_1^l(x)w,\alpha_2(v)]_2+[\rho_1^r(z)u,\alpha_2(v)]_2\\
&+\rho_1^l([x,z]_1)\alpha_2(v)+\rho_1^l(\rho_2^l(u)z)\alpha_2(v)+\rho_1^l(\rho_2^r(w)x)\alpha_2(v)\\
&+\rho_1^r(\alpha_1(y))[v,w]_2+\rho_1^r(\alpha_1(y))\rho_1^l(x)w+\rho_1^r(\alpha_1(y))\rho_1^r(z)u.
\end{aligned}
$$

Hence, by (3), (19), (20), (21) for $(i_1),(i_2),(i_3),(i_4)$ respectively, we get:

$$
\begin{aligned}
&\{\{x+u,y+v\},(\alpha_1\oplus\alpha_2)(z+w)\}-\{(\alpha_1\oplus\alpha_2)(x+u),\{y+v,z+w\}\}\\
&\quad-\{\{x+u,y+v\},(\alpha_1\oplus\alpha_2)(z+w)\}\\
&=\Big([\rho_2^l(u)y,\alpha_1(z)]_1+\rho_2^l(\rho_1^r(y)u)\alpha_1(z)-\rho_2^l(\alpha_2(u))[y,z]_1\\
&\quad-\rho_2^l(\rho_1^r(z)u)\alpha_1(y)-[\rho_2^l(u)z,\alpha_1(y)]_1\Big)\\
&+\Big([\rho_2^r(v)x,\alpha_1(z)]_1+\rho_2^l(\rho_1^l(x)v)\alpha_1(z)-[\alpha_1(x),\rho_2^l(v)z]_1\\
&-\rho_2^r(\rho_1^r(z)v)\alpha_1(x)-\rho_2^r(\alpha_2(v))[x,z]_1\Big)+\Big(\rho_2^r(\alpha_2(w))[x,y]_1\\
&-[\alpha_1(x),\rho_2^r(w)y]_1-\rho_2^r(\rho_1^l(y)w)\alpha_1(x)-[\rho_2^r(w)x,\alpha_1(y)]_1\\
&-\rho_2^l(\rho_1^l(x)w)\alpha_1(y)\Big)+\Big([\rho_1^l(x)v,\alpha_2(w)]_2+\rho_1^l(\rho_2^r(v)x)\alpha_2(w)\\
&-\rho_1^l(\alpha_1(x))[v,w]_2-[\rho_1^l(x)w,\alpha_2(v)]_2-\rho_1^l(\rho_2^r(w)x)\alpha_2(v)\Big)\\
&+\Big([\rho_1^r(y)u,\alpha_2(w)]_2+\rho_1^l(\rho_2^l(u)y)\alpha_2(w)-[\alpha_2(u),\rho_1^l(y)w]_2\\
&-\rho_1^r(\rho_2^r(w)y)\alpha_2(u)-\rho_1^r(\alpha_1(y))[u,w]_2\Big)+\Big(\rho_1^r(\alpha_1(z))[u,v]_2\\
&\quad-[\alpha_2(u),\rho_1^r(z)v]_2-\rho_1^r(\rho_2^l(v)z)\alpha_2(u)\\
&-[\rho_1^r(z)u,\alpha_2(v)]_2-\rho_1^l(\rho_2^l(u)z)\alpha_2(v)\Big)=0\ \text{(by (25)–(30))}.
\end{aligned}
$$

□

Definition 14. Let $(V, \rho^l, \rho^r, \phi)$ be a representation of a Hom-Leibniz algebra $(A, [,], \alpha)$. A linear operator $T : V \to A$ is called a relative Rota-Baxter operator on $(A, [,], \alpha)$ with respect to $(V, \rho^l, \rho^r, \phi)$ if T satisfies

$$T\phi = \alpha T, \tag{32}$$

$$[Tu, Tv] = T(\rho^l(Tu)v + \rho^r(Tv)u) \text{ for all } u, v \in V. \tag{33}$$

Observe that Rota-Baxter operators on Hom-Leibniz algebras are relative Rota-Baxter operators with respect to the regular representation.

Example 5. Consider the 2-dimensional Hom-Leibniz algebra $(A, [,], \alpha)$ where the non-zero products with respect to a basis (e_1, e_2) are given by:

$$[e_1, e_2] = -[e_2, e_1] := e_1 \quad \text{and} \quad \alpha(e_1) := -e_1,\ \alpha(e_2) := e_1 + e_2.$$

Then a linear map $T : A \to A$ defined by $T(e_1) := a_{11}e_1 + a_{21}e_2$; $T(e_2) := a_{12}e_1 + a_{22}e_2$ is a relative Rota-Baxter on $(A, \cdot, \alpha)$ with respect to the regular representation if and only if $T\alpha = \alpha T$ and

$$[Te_i, Te_j] = T([Te_i, e_j] + [e_i, Te_j]) \text{ for all } i, j \in \{1, 2\}. \tag{34}$$

The condition $\alpha T = T\alpha$ is equivalent to

$$a_{21} = 0;\ a_{11} + 2a_{12} - a_{22} = 0.$$

For $i = j = 1$ and $i = j = 2$, the condition (34) is satisfied trivially. Similarly, we obtain for $(i, j) \in \{(1, 2), (2, 1)\}$

$$a_{11}a_{22} = a_{11}a_{22} + a_{11}^2.$$

Summarize the above discussions, we observe that $a_{11} = a_{21} = 0, a_{22} = 2a_{21}$. Hence, the linear map $T : A \to A$ defined by $T(e_2) := a_{12}e_1 + 2a_{12}e_2$ is a relative Rota-Baxter operator on $(A, [,], \alpha)$ with respect to the regular representation.

Lemma 2. *Let T be a relative Rota-Baxter operator on a Hom-Leibniz algebra $(A, [,], \alpha)$ with respect to a representation $(V, \rho^l, \rho^r, \phi)$. If define a bracket $[,]_T$ on V by*

$$[u, v]_T := \rho^l(Tu)v + \rho^r(Tv)u \text{ for all } (u, v) \in V^{\times 2}, \tag{35}$$

then, $(V, [,]_T, \phi)$ is a Hom-Leibniz algebra.

Proof. First, note that the multiplicativity of ϕ with respect to $[,]_T$ follows from conditions (18) and (32). Next, pick $u, v, w \in V$ and observe from (3) and (33) that $T[u,v]_T = [Tu, Tv]$. Therefore, we deduce:

$$\begin{aligned}[][u,v]_T, \phi(w)]_T &= \rho^l([Tu,Tv])\phi(w) + \rho^l(T\phi(w))[u,v]_T \\ &= \rho^l([Tu,Tv])\phi(w) + \rho^r(T\phi(w))\rho^l(Tu)v + \rho^r(T\phi(w))\rho^r(Tv)u.\end{aligned}$$

Similarly, we compute

$$\begin{aligned}[][\phi(u),[v,w]_T]_T &= \rho^r([Tv,Tw])\phi(u) + \rho^l(T\phi(u))\rho^l(Tv)w \\ &\quad +\rho^l(T\phi(u))\rho^l(Tw)v, \\ [[u,w]_T, \phi(v)]_T &= \rho^l([Tu,Tw])\phi(v) + \rho^r(T\phi(v))\rho^l(Tu)w \\ &\quad +\rho^r(T\phi(v))\rho^r(Tw)u.\end{aligned}$$

Hence, the Hom-Leibniz identity in $(V, [,]_T, \phi)$ follows by (32) and (19)–(21). □

Corollary 2. *Let T be a relative Rota-Baxter operator on a Hom-Leibniz algebra $(A, [,], \alpha)$ with respect to a representation $(V, \rho^l, \rho^r, \phi)$. Then, T is a morphism from the Hom-Leibniz algebra $(V, [,]_T, \phi)$ to the initial Hom-Leibniz algebra $(A, [,], \alpha)$.*

Theorem 2. *Let T be a relative Rota-Baxter operator on a Hom-Leibniz algebra $(A, [,], \alpha)$ with respect to a representation $(V, \rho^l, \rho^r, \phi)$. Then, $(A, \overline{\rho^l}, \overline{\rho^r}, \alpha)$ is a representation of the Hom-Leibniz algebra $(V, [,]_T, \phi)$ where*

$$\overline{\rho^l}(u)x := [Tu, x] - T\rho^r(x)u \text{ for all } (x,u) \in A \times V, \tag{36}$$
$$\overline{\rho^r}(u)x := [x, Tu] - T\rho^l(x)u \text{ for all } (x,u) \in A \times V. \tag{37}$$

Proof. First, observe that (18) in $(A, \overline{\rho^l}, \overline{\rho^r}, \alpha)$ follows from the one in $(V, \rho^l, \rho^r, \phi)$, the multiplicativity of α with respect to $[,]$ and (32). Next, for all $x \in A$ and $(u,v) \in V^{\times 2}$, if one observes from Corollary 2 that $T[u,v]_T = [Tu, Tv]$, we compute:

$$\begin{aligned}\overline{\rho^l}([u,v]_T)\alpha(x) &= [[Tu,Tv],\alpha(x)] - T\rho^r(\alpha(x))\rho^l(Tu)v \\ &\quad -T\rho^r(\alpha(x))\rho^r(Tv)u \text{ (by (36))} \\ &= [[Tu,Tv],\alpha(x)] - T\rho^l(T\phi(u))\rho^r(x)v - T\rho^l([Tu,x])\phi(v) \\ &\quad -T\rho^r([Tv,x])\phi(u) - T\rho^r(T\phi(v))\rho^r(x)u \\ &\quad \text{(by (20) and (21) in } (V,\rho^l,\rho^r,\phi)).\end{aligned}$$

Also, we obtain by (36) and (37):

$$\begin{aligned}\overline{\rho^l}(\phi(u))\overline{\rho^l}(v)x &= [T\phi(u),[Tv,x]] - [T\phi(u),T\rho^r(x)v]\\ &-T\rho^r([Tv,x])\phi(u) + T\rho^r(T\rho^r(x)v)\phi(u),\\ \overline{\rho^r}(\phi(v))\overline{\rho^l}(u)x &= [[Tu,x],T\phi(v)] - [T\rho^r(x)u,T\phi(v)]\\ &-T\rho^l([Tu,x])\phi(v) + T\rho^l(T\rho^r(x)u)\phi(v).\end{aligned}$$

Using $T\phi = \alpha T$ and (3), we obtain by (33):

$$\begin{aligned}&\overline{\rho^l}([u,v]_T)\alpha(x) - \overline{\rho^l}(\phi(u))\overline{\rho^l}(v)x - \overline{\rho^r}(\phi(v))\overline{\rho^l}(u)x\\ &= \Big([T\phi(u),T\rho^r(x)v] - T\rho^l(T\phi(u))\rho^r(x)v\\ &-T\rho^r(T\rho^r(x)v)\phi(u)\Big) + \Big([T\rho^r(x)u,T\phi(v)] - T\rho^l(T\rho^r(x)u)\phi(v)\\ &-T\rho^r(T\phi(v))\rho^r(x)u\Big) = 0.\end{aligned}$$

Hence, (19) holds in $(A,\overline{\rho^l},\overline{\rho^r},\alpha)$. Now, to prove (20) in $(A,\overline{\rho^l},\overline{\rho^r},\alpha)$, by straightforward computations using (36) and (37) we obtain:

$$\begin{aligned}\overline{\rho^r}(\phi(v))\overline{\rho^l}(u)x &= [[Tu,x],T\phi(v)] - [T\rho^r(x)u,T\phi(v)]\\ &-T\rho^l([Tu,x])\phi(v) + T\rho^l(T\rho^r(x)u)\phi(v)\\ &= [T\phi(u),[x,Tv]] + [[Tu,Tv],\alpha(x)] - [T\rho^r(x)u,T\phi(v)]\\ &-T\rho^l(T\phi(u))\rho^l(x)v - T\rho^r(\alpha(x))\rho^l(Tu)v\\ &+T\rho^l(T\rho^r(x)u)\phi(v) \text{ (by (32), (3) and (19))}.\end{aligned}$$

Similarly, we get:

$$\begin{aligned}\overline{\rho^l}(\phi(u))\overline{\rho^r}(v)x &= [T\phi(u),[x,Tv]] - [T\phi(u),T\rho^l(x)v]\\ &-T\rho^r([x,Tv])\phi(u) + T\rho^r(T\rho^l(x)v)\phi(u),\\ \overline{\rho^l}([u,v]_T)\alpha(x) &= [[Tu,Tv],\alpha(x)] - T\rho^r(\alpha(x))\rho^l(Tu)v\\ -T\rho^r(\alpha(x))\rho^r(Tv)u &= [[Tu,Tv],\alpha(x)] - T\rho^r(\alpha(x))\rho^l(Tu)v\\ &-T\rho^r([Tv,x])\phi(u) - T\rho^r(T\phi(v))\rho^r(x)u \text{ (by (21))}.\end{aligned}$$

Hence, after rearranging terms we come to

$$\begin{aligned}&\overline{\rho^r}(\phi(v))\overline{\rho^l}(u)x - \overline{\rho^l}(\phi(u))\overline{\rho^r}(v)x - \overline{\rho^l}([u,v]_T)\alpha(x)\\ &= \Big(-[T\rho^r(x)u,T\phi(v)] + T\rho^l(T\rho^r(x)u)\phi(v) + T\rho^r(T\phi(v))\rho^r(x)u\Big)\\ &+\Big([T\phi(u),T\rho^l(x)v] - T\rho^l(T\phi(u))\rho^l(x)v - T\rho^r(T\rho^l(x)v)\phi(u)\Big)\\ &+\Big(T\rho^r([x,Tv])\phi(v) + T\rho^r([Tv,x])\phi(v)\Big) = 0 \text{ (by (33) and (22))}.\end{aligned}$$

Finally, to prove (21), we first compute using (37):

$$\begin{aligned}\overline{\rho^r}(\phi(v))\overline{\rho^r}(u)x &= [[x,Tu],T\phi(v)]-[T\rho^l(x)u,T\phi(v)]\\ &\quad -T\rho^l([x,Tu])\phi(v)+T\rho^l(T\rho^l(x)u)\phi(v)\\ &= [\alpha(x),[Tu,Tv]]+[[x,Tv],T\phi(u)]-[T\rho^l(x)u,T\phi(v)]\\ &\quad -T\rho^l(\alpha(x))\rho^l(Tu)v-T\rho^l(T\phi(u))\rho^l(x)v\\ &\quad +T\rho^l(T\rho^l(x)u)\phi(v) \text{ (by (32), (3) and (19))},\end{aligned}$$

$$\begin{aligned}\overline{\rho^r}([u,v]_T)\alpha(x) &= [\alpha(x),T[u,v]_T]-T\rho^l(\alpha(x))\rho^l(Tu)v\\ &\quad -T\rho^l(\alpha(x))\rho^r(Tv)u\\ &= [\alpha(x),[Tu,Tv]]-T\rho^l(\alpha(x))\rho^l(Tu)v-T\rho^r(T\phi(u))\rho^l(x)u\\ &\quad +T\rho^l([x,Tv])\phi(u) \text{ (by (32), (3) and (20))},\end{aligned}$$

$$\begin{aligned}\overline{\rho^r}(\phi(u))\overline{\rho^r}(v)x &= [[x,Tv],T\phi(u)]-[T\rho^l(x)v,T\phi(u)]\\ &\quad -T\rho^l([x,Tv])\phi(u)+T\rho^l(T\rho^l(x)v)\phi(u).\end{aligned}$$

Therefore using these equations, we come to

$$\begin{aligned}&\overline{\rho^r}(\phi(v))\overline{\rho^r}(u)x-\overline{\rho^r}([u,v]_T)\alpha(x)-\overline{\rho^r}(\phi(u))\overline{\rho^r}(v)x\\ &=\Big(-[T\rho^l(x)u,T\phi(v)]+T\rho^l(T\rho^l(x)u)\phi(v)+T\rho^r(T\phi(v))\rho^l(x)u\Big)\\ &\quad+\Big([T\rho^l(x)v,T\phi(u)]-T\rho^l(T\rho^l(x)v)\phi(u)\\ &\quad-T\rho^r(T\phi(u))\rho^l(x)v\Big)=0 \text{ (by (33))}.\end{aligned}$$

□

3. Representations and relative Rota-Baxter operators of Hom-Leibniz-Poisson algebras

This section contains our interesting results. Here, we introduce and study representations of Hom-Leibniz-Poisson algebras. The notions of matched pairs and relative Rota-Baxter operators of these Hom-algebras are also treated to be in adequacy with the preceding sections. Most of the proofs are complements to the proofs performed in the previous sections.

Definition 15. A representation of a Hom-Leibniz-Poisson algebra $(A,\cdot,[,],\alpha)$ is a sextuple $(V,\lambda^l,\lambda^r,\rho^l,\rho^r,\phi)$ where V is a vector space, $\phi\in gl(V)$ and $\rho^l,\rho^r,\lambda^l,\lambda^r: A\to gl(V)$ are five linear maps such that

1. $(V, \lambda^l, \lambda^r, \phi)$ is a representation of the Hom-associative algebra, $(A, \cdot, \alpha)$.

2. $(V, \rho^l, \rho^r, \phi)$ is a representation of the Hom-Leibniz algebra, $(A, [,], \alpha)$.

3. The following equalities hold for all $x, y \in A$:

$$\rho^r(\alpha(y))\lambda^l(x) = \lambda^l(\alpha(x))\rho^r(y) + \lambda^l([x, y])\phi, \tag{38}$$

$$\rho^r(\alpha(y))\lambda^r(x) = \lambda^r([x, y])\phi + \lambda^r(\alpha(x))\rho^r(y), \tag{39}$$

$$\rho^l(x \cdot y)\phi = \lambda^l(\alpha(x))\rho^l(y) + \lambda^r(\alpha(y))\rho^l(x). \tag{40}$$

As a generalization of Proposition 1 and Proposition 5, the following result allows to give examples of representations of Hom-Leibniz-Poisson algebras.

Proposition 9. *Let* $\mathcal{A}_1 := (A_1, \mu_1, [,]_1, \alpha_1)$ *and* $\mathcal{A}_2 := (A_2, \mu_2, [,]_2, \alpha_2)$ *be two Hom-Leibniz-Poisson algebras and* $f : \mathcal{A}_1 \to \mathcal{A}_2$ *be a morphism of Hom-Leibniz-Poisson algebras. Then* $A_2^f := (A_2, \lambda^l, \lambda^r, \rho^l, \rho^r, \alpha_2)$ *is a representaion of* $\mathcal{A}_1$ *where* $\lambda^l(a)b := \mu_2(f(a), b)$, $\lambda^r(a, b) := \mu_2(b, f(a))$, $\rho^l(a)b := [f(a), b]_2$, $\rho^r(a, b) := [b, f(a)]_2$ *for all* $(a, b) \in A \times B$.

Proof. Thanks to Proposition 1 and Proposition 5, $(A_2, \lambda^l, \lambda^r, \alpha_2)$ and $(A_2, \rho^l, \rho^r, \alpha_2)$ are representations of the Hom-associative algebra (A_1, μ_1, α_1) and the Hom-Leibniz algebra $(A_1, \alpha_1, [,]_1)$ respectively. It remains to prove (38)–(40). Let $x, y \in A_1$ and $z \in A_2$. Since f is a morphism, using (4) in $\mathcal{A}_2$, we get

$$\begin{aligned}
\rho^r(\alpha_1(y))\lambda^l(x)z \quad &= [\mu_2(f(x), z), f\alpha_1(y)]_2 = [\mu_2(f(x), z), \alpha_2 f(y)]_2 \\
&= \mu_2(\alpha_2 f(x), [z, f(y)]_2) + \mu_2([f(x), f(y)]_2, \alpha_2(z)) \\
&= \mu_2(f\alpha_1(x), [z, f(y)]_2) + \mu_2(f([x, y]_1), \alpha_2(z)) \\
&= \lambda^l(\alpha_1(x))\rho^r(y)z + \lambda^l([x, y]_1)\alpha_2(z).
\end{aligned}$$

Hence, (38) holds and similarly, (39) is obtained. Finally, by same hypothesis, we have

$$\begin{aligned}
\rho^l(\mu_1(x, y))\alpha_2(z) \quad &= [f\mu_1(x, y), \alpha_2(z)]_2 = [\mu_2(f(x), f(y)), \alpha_2(z)]_2 \\
&= \mu_2(\alpha_2 f(x), [f(y), z]_2) + \mu_2([f(x), z]_2, \alpha_2 f(y)) \\
&= \mu_2(f\alpha_1(x), [f(y), z]_2) + \mu_2([f(x), z]_2), f\alpha_1(y)) \\
&= \lambda^l(\alpha_1(x))\rho^l(y)z + \lambda^r(\alpha_1(y))\rho^l(x)z
\end{aligned}$$

which means that (40) holds. □

Example 6. 1. Let $(A, \cdot, [,], \alpha)$ be a Hom-Leibniz-Poisson algebra. Define left multiplications $L, l : A \to gl(A)$ and right multiplications $R, r : A \to gl(A)$ by $L_x y := [x, y]$; $l_x y := x \cdot y$ and $R_x y := [y, x]$; $r_x y := y \cdot x$ for all $x, y \in A$. Then (A, l, r, L, R, α) is a representation called the regular representation of $(A, \cdot, [,], \alpha)$.

2. Let $(A, \cdot, [,], \alpha)$ be a Hom-Leibniz-Poisson algebra and (B, α) be a two-sided Hom-ideal of $(A, \cdot, [,], \alpha)$. Then (B, α) inherits a structure of representation of $(A, \cdot, [,], \alpha)$ where $\lambda^l(a)b := a \cdot b$; $\lambda^r(a)b := b \cdot a$ and $\rho^l(a)b := [a, b]$; $\rho^r(b, a) := [b, a]$ for all $(a, b) \in A \times B$.

Proposition 10. *Let $\mathcal{V} := (V, \lambda^l, \lambda^r, \rho^l, \rho^r, \phi)$ be a representation of a Hom-Leibniz-Poisson algebra $\mathcal{A} := (A, \cdot, [,], \alpha)$ and β be a self-morphism of $\mathcal{A} = (A, \cdot, [,], \alpha)$. Then $\mathcal{V}_\beta = (V, \lambda^l_\beta := \lambda^l \beta, \lambda^r_\beta := \lambda^r \beta, \rho^l_\beta := \rho^l \beta, \rho^r_\beta := \rho^r \beta, \phi)$ is a representation of $\mathcal{A}$.*

Proof. We know that $(V, \lambda^l_\beta := \lambda^l \beta, \lambda^r_\beta := \lambda^r \beta, \phi)$ is a representation of the Hom-associative algebra $(A, \cdot, \alpha)$ by Proposition 2 and $(V, \rho^l_\beta := \rho^l \beta, \rho^r_\beta := \rho^r \beta, \phi)$ is a representation of the Hom-Leibniz algebra $(A, [,], \alpha)$ thanks to Proposition 6. It remains to prove (38)–(40). Now, for all $x, y \in A$, since β is a morphism, we obtain by (38) and (40) in $\mathcal{V}$ respectively

$$\begin{aligned}
\rho^r_\beta(\alpha(y))\lambda^l_\beta(x) &= \rho^r(\beta\alpha(y))\lambda^l(\beta(x)) = \rho^r(\alpha\beta(y))\lambda^l(\beta(x)) \\
&= \lambda^l(\alpha\beta(x))\rho^r(\beta(y)) + \lambda^l([\beta(x), \beta(y)])\phi \\
&= \lambda^l(\beta\alpha(x))\rho^r(\beta(y)) + \lambda^l(\beta([x, y]))\phi \\
&= \lambda^l_\beta(\alpha(x))\rho^r_\beta(y) + \lambda^l_\beta([x, y])\phi,
\end{aligned}$$

$$\begin{aligned}
&\rho^l_\beta(x \cdot y)\phi = \rho^l(\beta(x) \cdot \beta(y))\phi \\
&= \lambda^l(\alpha\beta(x))\rho^l(\beta(y)) + \lambda^r(\alpha\beta(y))\rho^l(\beta(x)) \\
&= \lambda^l(\beta\alpha(x))\rho^l(\beta(y)) + \lambda^r(\beta\alpha(y))\rho^l(\beta(x)) \\
&= \lambda^l_\beta(\alpha(y))\rho^l_\beta(x) + \lambda^r_\beta(\alpha(y))\rho^l_\beta(x).
\end{aligned}$$

Hence, we get (38) and (40) for $\mathcal{V}_\beta$. As for (38), we can also check that (39) is satisfied for $\mathcal{V}_\beta$. □

Corollary 3. *Let $\mathcal{V} = (V, \lambda^l, \lambda^r, \rho^l, \rho^r, \phi)$ be a representation of a Hom-Leibniz-Poisson algebra $\mathcal{A} = (A, \cdot, [,], \alpha)$. Then*

$$\mathcal{V}^{(n)} = (V, \lambda^l \alpha^n, \lambda^r \alpha^n, \rho^l \alpha^n, \rho^r \alpha^n, \phi)$$

is a representation of $\mathcal{A}$ for each $n \in \mathbb{N}$.

Proposition 11. *Let $(A, \cdot, [,], \alpha)$ be a Hom-Leibniz-Poisson algebra, (V, ϕ) be a Hom-module and $\lambda^l, \lambda^r, \rho^l, \rho^r : A \to gl(V)$ be four linear maps. Then, the sextuple $(V, \lambda^l, \lambda^r, \rho^l, \rho^r, \phi)$ is a representation of $(A, \cdot, [,], \alpha)$ if and only if the direct sum of vector spaces, $A \oplus V$, turns into a Hom-Leibniz-Poisson algebra with the multiplications defined on $A \oplus V$ by*

$$(x+u) * (y+v) := x \cdot y + (\lambda^l(x)v + \lambda^r(y)u), \tag{41}$$

$$\{(x+u), (y+v)\} := [x,y] + (\rho^l(x)v + \rho^r(y)u), \tag{42}$$

$$(\alpha \oplus \phi)(x+u) := \alpha(x) + \phi(u). \tag{43}$$

Proof. Thanks to proofs of Proposition 3 and Proposition 7, it suffices to prove that $(V, \lambda^l, \lambda^r, \rho^l, \rho^r, \phi)$ satisfies (38), (39) and (40) if and only if $(A, \cdot, [,], \alpha)$ satisfies (4). Using the definition of $*$ and $\{,\}$, we have for any $(x,y) \in A^{\times 2},\ (u,v) \in V^{\times 2}$:

$$\begin{aligned} &\{(x+u)*(y+v), (\alpha\oplus\phi)(z+w)\} \\ &= \{x \cdot y + \lambda^l(x)v + \lambda^r(y)u, \alpha(z) + \phi(w)\} \\ &= [x\cdot y, \alpha(z)] + \rho^l(x\cdot y)\phi(w) + \rho^r(\alpha(z))\lambda^l(x)v + \rho^r(\alpha(z))\lambda^r(y)u. \end{aligned}$$

Similarly, we compute:

$$\begin{aligned} &(\alpha\oplus\phi)(x+u) * \{(y+v), (z+w)\} \\ &= \big(\alpha(x)\cdot[y,z] + \lambda^l(\alpha(x))\rho^l(y)w + \lambda^l(\alpha(x))\rho^r(z)v + \lambda^r([y,z])\phi(u)\big), \end{aligned}$$

$$\begin{aligned} &\{(x+u), (z+w)\} * (\alpha\oplus\phi)(y+v) \\ &= \big([x,z]\cdot\alpha(y) + \lambda^l([x,z])\phi(v) + \lambda^r(\alpha(y))\rho^l(x)w + \lambda^r(\alpha(y))\rho^r(z)u\big). \end{aligned}$$

Hence by (3), we get:

$$\begin{aligned} &\{(x+u)*(y+v), (\alpha\oplus\phi)(z+w)\} \\ &-(\alpha\oplus\phi)(x+u) * \{(y+v), (z+w)\} \\ &-\{(x+u), (z+w)\} * (\alpha\oplus\phi)(y+v) \\ &= \Big(\rho^l(x\cdot y)\phi(w) - \lambda^l(\alpha(x))\rho^l(y)w - \lambda^r(\alpha(y))\rho^l(x)w\Big) \\ &+\Big(\rho^r(\alpha(z))\lambda^l(x)v - \lambda^l(\alpha(x))\rho^r(z)v - \lambda^l([x,z])\phi(v)\Big) \\ &+\Big(\rho^r(\alpha(z))\lambda^r(y)u - \lambda^r([y,z])\phi(u) - \lambda^r(\alpha(y))\rho^r(z)u\Big). \end{aligned}$$

Therefore, (38), (39) and (40) hold in $(V, \lambda^l, \lambda^r, \rho^l, \rho^r, \phi)$ if and only if (4) holds in $(A, \cdot, [,], \alpha)$. □

Definition 16. Let $\mathcal{A}_1 := (A_1, \cdot, [,]_1, \alpha_1)$ and $\mathcal{A}_2 := (A_2, \bullet, [,]_2, \alpha_2)$ be two Hom-Leibniz-Poisson algebras and $\lambda_1^l, \lambda_1^r, \rho_1^l, \rho_1^r : A_1 \to gl(A_2)$, $\lambda_2^l, \lambda_2^r, \rho_2^l, \rho_2^r : A_2 \to gl(A_1)$ be linear maps such that $(A_2, \lambda_1^l, \lambda_1^r, \rho_1^l, \rho_1^r, \alpha_2)$ is a representation of $\mathcal{A}_1$, $(A_1, \lambda_2^l, \lambda_2^r, \rho_2^l, \rho_2^r, \alpha_1)$ is a representation of $\mathcal{A}_2$ and the following conditions hold:

1.$((A_1, \lambda_2^l, \lambda_2^r, \alpha_1), (A_2, \lambda_1^l, \lambda_1^r, \alpha_2))$ is a matched pair of Hom-associative algebras.

2.$((A_1, \rho_2^l, \rho_2^r, \alpha_1), (A_2, \rho_1^l, \rho_1^r, \alpha_2))$ is a matched pair of Hom-Leibniz algebras.

3. The following identities

$$\begin{aligned}
&\lambda_2^l(\alpha_2(u))[x,y]_1 + (\rho_2^l(u)y)\cdot\alpha_1(x) + \lambda_2^l(\rho_1^r(y)u)\alpha_1(x)\\
&\qquad -[\lambda_2^l(u)x, \alpha_1(y)]_1 - \rho_2^l(\lambda_1^r(x)u)\alpha_1(y) = 0, \qquad (44)\\
&\lambda_2^r(\alpha_2(u))[x,y]_1 + \alpha_1(x)\cdot(\rho_2^l(u)y) + \lambda_2^r(\rho_1^r(y)u)\alpha_1(x)\\
&\qquad -[\lambda_2^r(u)x, \alpha_1(y)]_1 - \rho_2^l(\lambda_1^l(x)u)\alpha_1(y) = 0, \qquad (45)\\
&\lambda_1^l(\alpha_1(x))[u,v]_2 + (\rho_1^l(x)v)\bullet\alpha_2(u) + \lambda_1^l(\rho_2^r(v)x)\alpha_2(u)\\
&\qquad -[\lambda_1^l(x)u, \alpha_2(v)]_2 - \rho_1^l(\lambda_2^r(u)x)\alpha_2(v) = 0, \qquad (46)\\
&\lambda_1^r(\alpha_1(x))[u,v]_2 + \alpha_2(u)\bullet(\rho_1^l(x)v) + \lambda_1^r(\rho_2^r(v)x)\alpha_2(u)\\
&\qquad -[\lambda_1^r(x)u, \alpha_2(v)]_2 - \rho_1^l(\lambda_2^l(u)x)\alpha_2(v) = 0, \qquad (47)\\
&\rho_2^r(\alpha_2(u))(x\cdot y) - \alpha_1(x)\cdot(\rho_2^r(u)y) - \lambda_2^r(\rho_1^l(y)u)\alpha_1(x)\\
&\qquad -(\rho_2^r(u)x)\cdot\alpha_1(y) - \lambda_2^l(\rho_1^l(x)u)\alpha_1(y) = 0, \qquad (48)\\
&\rho_1^r(\alpha_1(x))(u\bullet v) - \alpha_2(u)\bullet(\rho_1^r(x)v) - \lambda_1^r(\rho_2^l(v)x)\alpha_2(u)\\
&\qquad -(\rho_1^r(x)u)\bullet\alpha_2(v) - \lambda_1^l(\rho_2^l(u)x)\alpha_2(v) = 0 \qquad (49)
\end{aligned}$$

hold for all $x, y, z \in A_1$, $u, v, w \in A_2$.

Then $A^{\bowtie} := ((A_1, \lambda_2^l, \lambda_2^r, \rho_2^l, \rho_2^r, \alpha_1), (A_2, \lambda_1^l, \lambda_1^r, \rho_1^l, \rho_1^r, \alpha_2))$ is called a matched pair of Hom-Leibniz-Poisson algebras.

Proposition 12. *Let $\mathcal{A}_1 := (A_1, \cdot, [,]_1, \alpha_1)$ and $\mathcal{A}_2 := (A_2, \bullet, [,]_2, \alpha_2)$ be two Hom-Leibniz-Poisson algebras and $((A_1, \lambda_2^l, \lambda_2^r, \rho_2^l, \rho_2^r, \alpha_1), (A_2, \lambda_1^l, \lambda_1^r, \rho_1^l, \rho_1^r, \alpha_2))$ be a matched pair of Hom-Leibniz-Poisson algebras. Then, there is a Hom-Leibniz-Poisson algebra $(A_1 \oplus A_2, *, \{,\}, \alpha_1 \oplus \alpha_2)$ defined by*

$$\begin{aligned}
(x+u)*(y+v) &= (x\cdot y + \lambda_2^l(u)y + \lambda_2^r(v)x)\\
&\quad +(u\bullet v + \lambda_1^l(x)v + \lambda_1^r(y)u), \qquad (50)
\end{aligned}$$

$$\begin{aligned}\{(x+u),(y+v)\} &= ([x,y]_1+\rho_2^l(u)y+\rho_2^r(v)x)\\ &+([u,v]_2+\rho_1^l(x)v+\rho_1^r(y)u),\\ (\alpha_1\oplus\alpha_2)(x+u) &= \alpha_1(x)+\alpha_2(u).\end{aligned} \tag{51}$$

Proof. It is clear that $(A_1\oplus A_2,*,\alpha_1\oplus\alpha_2)$ is a Hom-associative algebra (see Proposition 3) and $(A_1\oplus A_2,\{,\},\alpha_1\oplus\alpha_2)$ is a Hom-Leibniz algebra (see Proposition 7). It remains to prove (4). Pick $x,y,z\in A_1$ and $u,v,w\in A_2$. Then, by a straightforward computation, we get

$$\begin{aligned}&\{(x+u)*(y+v),(\alpha_1\oplus\alpha_2)(z+w)\}\\ &=\underbrace{[x\cdot y,\alpha_1(z)]_1}_{(j_1)}+[\lambda_2^l(u)y,\alpha_1(z)]_1+[\lambda_2^r(v)x,\alpha_1(z)]_1\\ &+\underbrace{\rho_2^l(u\bullet v)\alpha_1(z)}_{(j_2)}+\rho_2^l(\lambda_1^l(x)v)\alpha_1(z)+\rho_2^l(\lambda_1^r(y)u)\alpha_1(z)\\ &+\rho_2^r(\alpha_2(w))(x\cdot y)+\underbrace{\rho_2^r(\alpha_2(w))\lambda_2^l(u)y}_{(j_3)}+\underbrace{\rho_2^r(\alpha_2(w))\lambda_2^r(v)x}_{(j_4)}\\ &+\underbrace{[u\bullet v,\alpha_2(w)]_2}_{(j_1)}+[\lambda_1^l(x)v,\alpha_2(w)]_2+[\lambda_1^r(y)u,\alpha_2(w)]_2\\ &+\underbrace{\rho_1^l(x\cdot y)\alpha_2(w)}_{(j_2)}+\rho_1^l(\lambda_2^l(u)y)\alpha_2(w)+\rho_1^l(\lambda_2^r(v)x)\alpha_2(w)\\ &+\rho_1^r(\alpha_1(z))(u\bullet v)+\underbrace{\rho_1^r(\alpha_1(z))\lambda_1^l(x)v}_{(j_3)}+\underbrace{\rho_1^r(\alpha_1(z))\lambda_1^r(y)u}_{(j_4)},\end{aligned}$$

$$\begin{aligned}&(\alpha_1\oplus\alpha_2)(x+u)*\{y+v,z+w\}\\ &=\alpha_1(x)\cdot[y,z]_1+\alpha_1(x)\cdot(\rho_2^l(v)z)+\alpha_1(x)\cdot(\rho_2^r(w)y)\\ &+\lambda_2^l(\alpha_2(u))[y,z]_1+\lambda_2^l(\alpha_2(u))\rho_2^l(v)z+\lambda_2^l(\alpha_2(u))\rho_2^r(w)y\\ &+\lambda_2^r([v,w]_2)\alpha_1(x)+\lambda_2^r(\rho_1^l(y)w)\alpha_1(x)+\lambda_2^r(\rho_1^r(z)v)\alpha_1(x)\\ &+\alpha_2(u)\bullet[v,w]_1+\alpha_2(u)\bullet(\rho_1^l(y)w)+\alpha_2(u)\bullet(\rho_1^r(z)v)\\ &+\lambda_1^l(\alpha_1(x))[v,w]_2+\lambda_1^l(\alpha_1(x))\rho_1^l(y)w+\lambda_1^l(\alpha_1(x))\rho_1^r(z)v\\ &+\lambda_1^r([y,z]_1)\alpha_2(u)+\lambda_1^r(\rho_2^l(v)z)\alpha_2(u)+\lambda_1^r(\rho_2^r(w)y)\alpha_2(u),\end{aligned}$$

$$\begin{aligned}&\{x+u,y+v\}*(\alpha_1\oplus\alpha_2)(y+v)\\ &=[x,z]_1\cdot\alpha_1(y)+(\rho_2^l(u)z)\cdot\alpha_1(y)+(\rho_2^r(w)x)\cdot\alpha_1(y)\\ &+\lambda_2^l([u,w]_2)\alpha_1(y)+\lambda_2^l(\rho_1^l(x)w)\alpha_1(y)+\lambda_2^l(\rho_1^r(z)u)\alpha_1(y)\end{aligned}$$

$$+\lambda_2^r(\alpha_2(v))[x,z]_1+\lambda_2^r(\alpha_2(v))\rho_2^l(u)z+\lambda_2^r(\alpha_2(v))\rho_2^r(w)x$$
$$+[u,w]_2\bullet\alpha_2(v)+(\rho_1^l(x)w)\bullet\alpha_2(v)+(\rho_1^r(z)u)\bullet\alpha_2(v)$$
$$+\lambda_1^l([x,z]_1)\alpha_2(v)+\lambda_1^l(\rho_2^l(u)z)\alpha_2(v)+\lambda_1^l(\rho_2^r(w)x)\alpha_2(v)$$
$$+\lambda_1^r(\alpha_1(y))[u,w]_2+\lambda_1^r(\alpha_1(y))\rho_1^l(x)w+\lambda_1^r(\alpha_1(y))\rho_1^r(z)u.$$

Therefore, applying (4), (40), (38), (39) for $(j_1),(j_2),(j_3),(j_4)$ respectively, we get:

$$\begin{aligned}
&\{(x+u)*(y+v),(\alpha_1\oplus\alpha_2)(z+w)\}\\
&-(\alpha_1\oplus\alpha_2)(x+u)*\{y+v,z+w\}\\
&-\{x+u,y+v\}*(\alpha_1\oplus\alpha_2)(y+v)\\
&=\Big([\lambda_2^l(u)y,\alpha_1(z)]_1+\rho_2^l(\lambda_1^r(y)u)\alpha_1(z)-(\rho_2^l(u)z)\cdot\alpha_1(y)\\
&-\lambda_2^l(\rho_1^r(z)u)\alpha_1(y)-\lambda_2^l(\alpha_2(u))[y,z]_1\Big)+\Big([\lambda_2^r(v)x,\alpha_1(z)]_1\\
&+\rho_2^l(\lambda_1^l(x)v)\alpha_1(z)-\lambda_2^r(\rho_1^r(z)v)\alpha_1(x)-\alpha_1(x)\cdot(\rho_2^l(v)z)\\
&-\lambda_2^r(\alpha_2(v))[x,z]_1\Big)+\Big([\lambda_1^l(x)v,\alpha_2(w)]_2+\rho_1^l(\lambda_2^r(v)x)\alpha_2(w)\\
&-(\rho_1^l(x)w)\bullet\alpha_2(v)-\lambda_1^l(\rho_2^r(w)x)\alpha_2(v)-\lambda_1^l(\alpha_1(x))[v,w]_2\Big)\\
&+\Big([\lambda_1^r(y)u,\alpha_2(w)]_2+\rho_1^l(\lambda_2^l(u)y)\alpha_2(w)-\lambda_1^r(\rho_2^r(w)y)\alpha_2(u)\\
&-\alpha_2(u)\bullet(\rho_1^l(y)w)-\lambda_1^r(\alpha_1(y))[u,w]_2\Big)+\Big(\rho_2^r(\alpha_2(w))(x\cdot y)\\
&-\alpha_1(x)\cdot(\rho_2^r(w)y)-\lambda_2^r(\rho_1^l(y)w)\alpha_1(x)-(\rho_2^r(w)x)\cdot\alpha_1(y)\\
&-\lambda_2^l(\rho_1^l(x)w)\alpha_1(y)\Big)+\Big(\rho_1^r(\alpha_1(z))(u\bullet v)-\alpha_2(u)\bullet(\rho_1^r(z)v)\\
&-\lambda_1^r(\rho_2^l(v)z)\alpha_2(u)-(\rho_1^r(z)u)\bullet\alpha_2(v)-\lambda_1^l(\rho_2^l(u)z)\alpha_2(v)\Big)=0
\end{aligned}$$

(by (44)–(49)).

□

Definition 17. Let $(V,\lambda^l,\lambda^r,\rho^l,\rho^r,\phi)$ be a representation of a Hom-Leibniz algebra $(A,\cdot,[,],\alpha)$. A linear operator $T:V\to A$ is called a relative Rota-Baxter operator on $(A,\cdot,[,],\alpha)$ with respect to $(V,\lambda^l,\lambda^r,\rho^l,\rho^r,\phi)$ if T satisfies (12), (13) and (33), i.e.,

$$T\phi=\alpha T,$$
$$Tu\cdot Tv=T(\lambda^l(Tu)v+\lambda^r(Tv)u)\text{ for all }u,v\in V,$$
$$[Tu,Tv]=T(\rho^l(Tu)v+\rho^r(Tv)u)\text{ for all }u,v\in V.$$

Observe that as Hom-associative and Hom-Leibniz algebras case, Rota-Baxter operators on Hom-Leibniz-Poisson algebras are relative Rota-Baxter operators with respect to the regular representation.

Example 7. Let $(A,\cdot,[,],\alpha)$ be a Hom-Leibniz-Poisson algebra and $(V,\lambda^l,\lambda^r,\rho^l,\rho^r,\phi)$ be a representation of $(A,\cdot,[,],\alpha)$. It is easy to verify that $A\oplus V$ is a representation of $(A,\cdot,[,],\alpha)$ under the maps $\lambda^l_{A\oplus V},\lambda^r_{A\oplus V}$, $\rho^l_{A\oplus V},\rho^r_{A\oplus V}: A\to gl(A\oplus V)$ defined by

$$\lambda^l_{A\oplus V}(a)(b+v):=a\cdot b+\lambda^l(a)v;\ \lambda^r_{A\oplus V}(a)(b+v):=\lambda^r(a)v,$$
$$\rho^l_{A\oplus V}(a)(b+v):=[a,b]+\rho^l(a)v;\ \rho^r_{A\oplus V}(a)(b+v):=\rho^r(a)v.$$

Define the linear map $T: A\oplus V\to A, a+v\mapsto a$. Then T is a relative Rota-Baxter operator on A with respect to the representation $A\oplus V$.

Also, Example 3 and Example 5 give rise to the following example of relative Rota-Baxter on a Hom-Leibniz-Poisson algebra.

Example 8. Consider the 2-dimensional Hom-Leibniz-Poisson algebra $(A,\cdot,[,],\alpha)$ where the non-zero products with respect to a basis (e_1,e_2) are given by:

$$e_1\cdot e_2=e_2\cdot e_1:=-e_1,\ e_2\cdot e_2:=e_1+e_2;\ [e_1,e_2]=-[e_2,e_1]:=e_1$$
$$\text{and}\quad \alpha(e_1):=-e_1,\ \alpha(e_2):=e_1+e_2.$$

Then, one can prove that the zero-map is the only relative Rota-Baxter operator on $(A,\cdot,[,]\alpha)$ with respect to the regular representation.

As Hom-associative algebras case [6], let give some characterizations of relative Rota-Baxter operators on Hom-Leibniz-Poisson algebras.

Proposition 13. *A linear map $T: V\to A$ is a relative Rota-Baxter operator on a Hom-Leibniz-Poisson algebra $(A,\cdot,[,],\alpha)$ with respect to the representation $(V,\lambda^l,\lambda^r,\rho^l,\rho^r,\phi)$ if and only if the graph of T,*

$$G_r(T):=\{(T(v),v),v\in V\}$$

is a subalgebra of the semi-direct product algebra $A\oplus V$.

The following result shows that a relative Rota-Baxter operator can be lifted up the Rota-Baxter operator.

Proposition 14. *Let $(A,\cdot,[,],\alpha)$ be a Hom-Leibniz-Poisson algebra, $(V,\lambda^l,\lambda^r,\rho^l,\rho^r,\phi)$ be a representation of A and $T: V\to A$ be a linear map. Define $\widehat{T}\in End(A\oplus V)$ by $\widehat{T}(a+v):=Tv$. Then T is a relative Rota-Baxter operator if and only if $\widehat{T}$ is a Rota-Baxter operator on $A\oplus V$.*

In order to give another characterization of relative Rota-Baxter operators, let introduce the following:

Definition 18. Let $(A, \cdot, [,], \alpha)$ be a Hom-Leibniz-Poisson algebra. A linear map $N : A \to A$ is said to be a Nijenhuis operator if $N\alpha = \alpha N$ and its Nijenhuis torsions vanish, i.e.,

$$N(x) \cdot N(y) = N(N(x) \cdot y + x \cdot N(y) - N(x \cdot y)) \text{ for all } x, y \in A,$$
$$[N(x), N(y)] = N([N(x), y] + [x, N(y)] - N([x, y])) \text{ for all } x, y \in A.$$

Observe that the deformed multiplications $\cdot_N, [,]_N : A \oplus A \to A$ given by

$$x \cdot_N y := N(x) \cdot y + x \cdot N(y) - N(x \cdot y),$$
$$[x, y]_N := [N(x), y] + [x, N(y)] - N([x, y]),$$

gives rise to a new Hom-Leibniz-Poisson multiplications on A, and N becomes a morphism from the Hom-Leibniz-Poisson algebra $(A, \cdot_N, [,]_N, \alpha)$ to the initial Hom-Leibniz-Poisson algebra $(A, \cdot, [,], \alpha)$.

Now, we can esealy check the following result.

Proposition 15. *Let $\mathcal{A} := (A, \cdot, [,], \alpha)$ be a Hom-Leibniz-Poisson algebra and $\mathcal{V} := (V, \lambda^l, \lambda^r, \rho^l, \rho^r, \phi)$ be a representation of $(A, \cdot, [,], \alpha)$. A linear map $T : V \to A$ is a relative Rota-Baxter operator on $\mathcal{A}$ with respect to the $\mathcal{V}$ if and only if $N_T := \begin{pmatrix} 0 & T \\ 0 & 0 \end{pmatrix} : A \oplus V \to A \oplus V$ is a Nijenhuis operator on the Hom-Leibniz-Poisson algebra $A \oplus V$.*

In the sequel, let give some results about relative Rota-Baxter operators basing on the previous sections.

Lemma 3. *Let T be a relative Rota-Baxter operator on a Hom-Leibniz-Poisson algebra $(A, \cdot, [,], \alpha)$ with respect to a representation $(V, \lambda^l, \lambda^r, \rho^l, \rho^r, \phi)$. If define a map $\diamond$ and a braket $[,]_T$ on V by (15) and (3), i.e.,*

$$u \diamond v := \lambda^l(Tu)v + \lambda^r(Tv)u \text{ for all } (u, v) \in V^{\times 2},$$
$$[u, v]_T := \rho^l(Tu)v + \rho^r(Tv)u \text{ for all } (u, v) \in V^{\times 2},$$

then, $(V, \diamond, [,]_T, \phi)$ is a Hom-Leibniz-Poisson algebra.

Proof. We know that $(V, \diamond, \phi)$ is a Hom-associative algebra and $(V, [,]_T, \phi)$ is a Hom-Leibniz algebra by Lemma 1 and Lemma 2 respectively. The result is obtained if we prove (4). For all $u, v, w \in V$, using Corollary 1 and Corollary 2 , we get:

$$\begin{aligned}
[u \diamond v, \phi(w)]_T =& \rho^l(Tu \cdot Tv)\phi(w) + \rho^r(Tw)\lambda^l(Tu)v \\
&+ \rho^r(T\phi(w))\lambda^r(Tv)u, \\
\phi(u) \diamond [v, w]_T =& \lambda^l(T\phi(u))\rho^l(Tv)w + \lambda^l(T\phi(u))\rho^l(Tw)v \\
&+ \lambda^r([Tv, Tw])\phi(u), \\
[u, w] \diamond \phi(v) =& \lambda^l([Tu, Tv])\phi(v) + \lambda^r(T\phi(v))\rho^l(Tu)w \\
&+ \lambda^r(T\phi(v))\rho^r(Tw)u.
\end{aligned}$$

Hence, we get (4) by (12), (38), (39), (40). □

Corollary 4. *Let T be a relative Rota-Baxter operator on a Hom-Leibniz-Poisson algebra $(A, \cdot, [,], \alpha)$ with respect to a representation $(V, \lambda^l, \lambda^r, \rho^l, \rho^r, \phi)$. Then, T is a morphism from the Hom-Leibniz-Poisson algebra $(V, \diamond, [,]_T, \phi)$ to the initial Hom-Leibniz-Poisson algebra $(A, [,], \alpha)$.*

Proof. It follows by Corollary 1 and Corollary 2 □

Theorem 3. *Let T be a relative Rota-Baxter operator on a Hom-Leibniz-Poisson algebra $(A, \cdot, [,], \alpha)$ with respect to a representation $(V, \lambda^l, \lambda^r, \rho^l, \rho^r, \phi)$. Then, $(A, \overline{\lambda^l}, \overline{\lambda^r}, \overline{\rho^l}, \overline{\rho^r}, \alpha)$ is a representation of the Hom-Leibniz-Poisson algebra $(V, \diamond, [,]_T, \phi)$ where $\overline{\lambda^l}, \overline{\lambda^r}, \overline{\rho^l}, \overline{\rho^r}$ are defined as (16),(17), (36),(37) respectively, i.e.,*

$$\begin{aligned}
&\overline{\lambda^l}(u)x := Tu \cdot x - T\lambda^r(x)u \text{ for all } (x, u) \in A \times V, \\
&\overline{\lambda^r}(u)x := x \cdot Tu - T\lambda^l(x)u \text{ for all } (x, u) \in A \times V, \\
&\overline{\rho^l}(u)x := [Tu, x] - T\rho^r(x)u \text{ for all } (x, u) \in A \times V, \\
&\overline{\rho^r}(u)x := [x, Tu] - T\rho^l(x)u \text{ for all } (x, u) \in A \times V.
\end{aligned}$$

Proof. We have proved that $(A, \overline{\lambda^l}, \overline{\lambda^r}, \alpha)$ is a representation of the Hom-associative algebra $(V, \diamond, \phi)$ and $(A, \overline{\rho^l}, \overline{\rho^r}, \alpha)$ is a representation of the Hom-Leibniz algebra $(V, [,]_T, \phi)$ by Theorem 1 and Theorem 2 respectively. It remains to prove (38)–(40). Let $u, v \in V$ and $x \in A$. Then

$$\begin{aligned}
\overline{\rho^r}(\phi(v))\overline{\lambda^l}(u)x = [Tu \cdot x, T\phi(v)] - [T\lambda^r(x)u, T\phi(v)] - T\rho^l(Tu \cdot x)\phi(v) \\
+T\rho^l(T\lambda^r(x)u)\phi(v) = [Tu \cdot x, T\phi(v)] - T\rho^l(T\lambda^r(x)u)\phi(v) \\
-T\rho^r(T\phi(v))\lambda^r(x)u - T\lambda^l(T\phi(u))\rho^l(x)v - T\lambda^r(\alpha(x))\rho^l(Tu)v \\
+T\rho^l(T\lambda^r(x)u)\phi(v) \text{ (by (32), (33) and (40))}
\end{aligned}$$

$$= [Tu\cdot x, T\phi(v)] - T\rho^r(T\phi(v))\lambda^r(x)u-$$
$$T\lambda^l(T\phi(u))\rho^l(x)v - T\lambda^r(\alpha(x))\rho^l(Tu)v.$$

Similarly, we compute

$$\overline{\lambda^l}(\phi(u))\overline{\rho^r}(v)x = T\phi(u)\cdot[x,Tv] - T\phi(u)\cdot T\rho^l(x)v - T\lambda^r([x,Tv])\phi(u)$$
$$+T\lambda^r(T\rho^l(x)v)\phi(u) = T\phi(u)\cdot[x,Tv] - T\lambda^l(T\phi(u))\rho^l(x)v$$
$$-T\lambda^r(T\rho^l(x)v)\phi(u) - T\rho^r(T\phi(v))\lambda^r(x)u + T\lambda^r(\alpha(x))\rho^r(Tv)u$$
$$+T\lambda^r(T\rho^l(x)v)\phi(u) \text{ (by (12), (13) and (39))}$$
$$= T\phi(u)\cdot[x,Tv] - T\lambda^l(T\phi(u))\rho^l(x)v$$
$$-T\rho^r(T\phi(v))\lambda^r(x)u + T\lambda^r(\alpha(x))\rho^r(Tv)u,$$

$$\overline{\lambda^l}([u,v]_T)\alpha(x) = [Tu,Tv]\cdot\alpha(x) - T\lambda^r(\alpha(x))\rho^l(Tu)v$$
$$-T\lambda^r(\alpha(x))\rho^r(Tv)u.$$

It follows that

$$\overline{\rho^r}(\phi(v))\overline{\lambda^l}(u)x - \overline{\lambda^l}(\phi(u))\overline{\rho^r}(v)x - \overline{\lambda^l}([u,v]_T)\alpha(x)$$
$$= \Big([Tu\cdot x, T\phi(v)] - [Tu,Tv]\cdot\alpha(x) - T\phi(u)\cdot[x,Tv]\Big) = 0$$
$$\text{(by (12) and (4))},$$

i.e., (38) holds. To get (39), we proceed as follows:

$$\overline{\rho^r}(\phi(v))\overline{\lambda^r}(u)x = [x\cdot Tu, T\phi(v)] - [T\lambda^l(x)u, T\phi(v)] - T\rho^l(x\cdot Tu)\phi(v)$$
$$+T\rho^l(T\lambda^l(x)u)\phi(v) = [x\cdot Tu, T\phi(v)] - T\rho^l(T\lambda^l(x)u)\phi(v)$$
$$-T\rho^r(T\phi(v))\lambda^l(x)u - T\lambda^l(\alpha(x))\rho^l(Tu)v - T\lambda^r(T\phi(u))\rho^l(x)v$$
$$+T\rho^l(T\lambda^l(x)u)\phi(v) \text{ (by (32), (33) and (40))}$$
$$= [x\cdot Tu, T\phi(v)] - T\rho^r(T\phi(v))\lambda^l(x)u$$
$$-T\lambda^l(\alpha(x))\rho^l(Tu)v - T\lambda^r(T\phi(u))\rho^l(x)v.$$

Similarly, we compute

$$\overline{\lambda^r}([u,v]_T)\alpha(x) = \alpha(x)\cdot[Tu,Tv] - T\lambda^l(\alpha(x))\rho^l(Tu)v - T\lambda^l(\alpha(x))\rho^r(Tv)u,$$
$$\overline{\lambda^r}(\phi(u))\overline{\rho^r}(v)x = [x,Tv]\cdot T\phi(u) - (T\rho^l(x)v)\cdot T\phi(u) - T\lambda^l([x,Tv])\phi(u)$$
$$+T\lambda^l(T\rho^l(x)v)\phi(u) = [x,Tv]\cdot T\phi(u) - T\lambda^l(T\rho^l(x)v)\phi(u)$$
$$-T\lambda^r(T\phi(u))\rho^l(x)v - T\rho^r(T\phi(v))\lambda^l(x)u + T\lambda^l(\alpha(x))\rho^r(Tv)u$$

$$+T\lambda^l(T\rho^l(x)v)\phi(u) \text{ (by (12), (13) and (38))}$$
$$= [x, Tv]\cdot T\phi(u) - T\lambda^r(T\phi(u))\rho^l(x)v - T\rho^r(T\phi(v))\lambda^l(x)u$$
$$+T\lambda^l(\alpha(x))\rho^r(Tv)u.$$

It follows that

$$\overline{\rho^r}(\phi(v))\overline{\lambda^r}(u)x - \overline{\lambda^r}([u,v]_T)\alpha(x) - \overline{\lambda^r}(\phi(u))\overline{\rho^r}(v)x$$
$$= \Big([x\cdot Tu, T\phi(v)] - \alpha(x)\cdot[Tu,Tv] - [x,Tv]\cdot T\phi(u)\Big) = 0$$
$$\text{(by (12) and (4))}$$

i.e., (39) holds. Finally, we compute as the previous case:

$$\overline{\rho^l}(u\diamond v)\alpha(x) = [Tu\cdot Tv, \alpha(x)] - T\rho^r(\alpha(x))\lambda^l(Tu)v - T\rho^r(\alpha(x))\lambda^r(Tv)u$$
$$= [Tu\cdot Tv, \alpha(x)] - T\lambda^l(T\phi(u))\rho^r(x)v - T\lambda^l([Tu,x])v - T\lambda^r([Tv,x])\phi(u)$$
$$-T\lambda^r(T\phi(v))\rho^r(x)u \text{ (by (32), (38) and (39))}.$$

Similarly,

$$\overline{\lambda^l}(\phi(u))\overline{\rho^l}(v)x = T\phi(u)\cdot[Tv,x] - T\phi(u)\cdot T\rho^r(x)v - T\lambda^r([Tv,x])\phi(u)$$
$$+T\lambda^r(T\rho^r(x)v)\phi(u) = T\phi(u)\cdot[Tv,x] - T\lambda^l(T\phi(u))\rho^r(x)v$$
$$-T\lambda^r(T\rho^r(x)v)\phi(u) - T\lambda^r([Tv,x])\phi(u) + T\lambda^r(T\rho^l(x)v)\phi(u) \text{ (by (13))}$$
$$= T\phi(u)\cdot[Tv,x] - T\lambda^l(T\phi(u))\rho^r(x)v - T\lambda^r([Tv,x])\phi(u)$$

and similarly using the same hypothesis as in the previous equation, we obtain

$$\overline{\lambda^r}(\phi(v))\overline{\rho^l}(u)x = [Tu,x]\cdot T\phi(v) - T\lambda^r(T\phi(v))\rho^r(x)u - T\lambda^l([Tu,x])\phi(v).$$

Hence, (40) follows by (32) and (4). □

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