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On left-gyrotranslation groups of gyrogroups

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Abstract. A gyrogroup is an algebraic structure whose operation is, in general, non-associative that shares some common properties with groups. In this paper, we prove that every gyrogroup induces a permutation group, called the left-gyrotranslation group, that can be used to understand the algebraic structure of the gyrogroup itself. We also show several connections between gyrogroups and their left-gyrotranslation groups and give a few related examples, especially the left-gyrotranslation group of the famous Möbius gyrogroup in the complex plane.

Introduction

A gyrogroup consists of one non-empty set G and one binary operation $\oplus$ on G satisfying the following properties: (i) there is an element e in G such that $e \oplus a = a$ for all $a \in G$; (ii) for each element a in G, there is an element b in G such that $b \oplus a = e$; (iii) for all elements a, b in G, there is an automorphism $\operatorname{gyr}[a, b]$ of $(G, \oplus)$ such that

$$a \oplus (b \oplus c) = (a \oplus b) \oplus \operatorname{gyr}[a, b](c) \tag{1}$$

for all $c \in G$; and (iv) $\operatorname{gyr}[a, b] = \operatorname{gyr}[a \oplus b, b]$ for all a, b in G. It turns out that the element e in (i) acts as a unique two-sided identity of $(G, \oplus)$ and that the element b in (ii) acts as a unique two-sided inverse of a,

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denoted by $\ominus a$. Furthermore, the automorphism $\mathrm{gyr}[a, b]$ in (iii) is completely determined by elements a and b, called the *gyroautomorphism* generated by a and b, and identity (1) is called the *left gyroassociative law*, which may be regarded as a weaker form of associativity. Property (iv) is a crucial property of a gyrogroup, called the *left loop property*. It is known that a gyrogroup forms a left Bol loop with the A_ℓ-property and vice versa (see, for instance, the remark on page 71 of [3]). In fact, a gyroautomorphism corresponds to a left inner mapping, and the left loop property is equivalent to the left Bol identity in loop theory. Therefore, the family of gyrogroups coincides with the family of left Bol loops with the A_ℓ-property. By definition, the operation of a gyrogroup is not necessarily associative. However, if the operation of a gyrogroup is associative, then that gyrogroup forms a group. A gyrogroup whose operation is not associative is called a *non-degenerate gyrogroup*.

As noted above, the family of gyrogroups properly includes the family of groups. Actually, groups and gyrogroups have strong connections in various ways, and in particular one may obtain structural information about gyrogroups by looking at their associated groups. See, for instance, [8], in which the author indicates that every gyrogroup induces a permutation group, and this associated group can be used to understand the structure of the gyrogroup itself. Moreover, in [10], the author provides one method to construct a group, called the associativization, from an arbitrary gyrogroup, and this group in some sense measures the deviation from associativity of the corresponding gyrogroup operation. For additional examples, we refer the reader to [4, 5], to name a few. This fact inspires us to studying gyrogroup structures by using tools in group theory. In this paper, we are interested in finding out relationships between gyrogroups and their corresponding left-gyrotranslation groups. We also determine the left-gyrotranslation groups of some known concrete gyrogroups (such as the Möbius gyrogroup in the complex plane).

1. Preliminaries

For basic knowledge of gyrogroup theory, we refer the reader to [7, 14]. Let G be a gyrogroup whose operation is written $\oplus$. For each element a in G, the *left gyrotranslation* by a, denoted by L_a, is the bijective self-map of G defined by the formula $L_a(x) = a \oplus x$ for all $x \in G$. The set of left gyrotranslations of G is denoted by $\widehat{G}$, that is,

$$\widehat{G} = \{L_a : a \in G\}. \tag{2}$$

It should be pointed out that the operation on G is associative (that is, G is a group) if and only if $\widehat{G}$ is closed under the function composition. Let $\mathrm{LTrs}\,(G)$ be the subgroup of the symmetric group $\mathrm{Sym}\,(G)$ generated by $\widehat{G}$, called the *left-gyrotranslation group* of G. Since $L_a^{-1} = L_{\ominus a}$ for all $a \in G$ (see, for instance, page 73 of [12]), it follows that $\mathrm{LTrs}\,(G)$ consists precisely of compositions of finitely many left gyrotranslations of G, that is,

$$\mathrm{LTrs}\,(G) = \{L_{a_1} \circ L_{a_2} \circ \cdots \circ L_{a_n} : a_1, a_2, \ldots, a_n \in G\}. \tag{3}$$

Furthermore, by the composition law of left gyrotranslations (see, for instance, part 3 of Theorem 10 of [12]), the *gyroautomorphism* of G generated by a and b is given by the formula $\mathrm{gyr}[a, b] = L_{\ominus(a \oplus b)} \circ L_a \circ L_b$ for all $a, b \in G$. Hence, the left-gyrotranslation group of G contains all the gyroautomorphisms of G. Note that $\widehat{G}$ has the following properties:

1. $\mathrm{I}_G \in \widehat{G}$, where I_G denotes the identity map on G;

2. $L_a^{-1} \in \widehat{G}$ for all $a \in G$;

3. $L_a \circ L_b \circ L_a \in \widehat{G}$ for all $a, b \in G$ (see the proof of Proposition 18 of [7]).

Therefore, $\widehat{G}$ is a *twisted subgroup* of $\mathrm{LTrs}\,(G)$. It is not difficult to prove that G is a group if and only if $\mathrm{LTrs}\,(G) = \widehat{G}$. This fact will be useful later on.

2. Main results

We begin with some known results on left-gyrotranslation groups of gyrogroups. Let G be a gyrogroup. Theorem 3.3 of [9] states that if x and y are elements in G, then there is an element T in $\mathrm{LTrs}\,(G)$ such that $T(x) = y$. Therefore, in group-theoretic terminology, the left-gyrotranslation group of G acts transitively on G by evaluation. Theorem 3.6 of [9] states that if G is non-degenerate (that is, if G is not a group), then for each element $a \in G$, there is a non-identity element R in $\mathrm{LTrs}\,(G)$ such that $R(a) = a$. This implies that if G is a non-degenerate gyrogroup, then the action of the left-gyrotranslation group of G on G by evaluation is not sharply transitive. Next, we mention a connection between subgyrogroups of G and subgroups of $\mathrm{LTrs}\,(G)$. Recall that a *subgyrogroup* H of G is a non-empty subset of G that forms a gyrogroup under the operation inherited from G with $\mathrm{gyr}[h, k](H) = H$ for all $h, k \in H$, and is *strong* if it is invariant under the gyroautomorphisms of G, that is, if $\text{gyr}[a,b](H) \subseteq H$ for all $a, b \in G$.

Proposition 1. *Let G be a gyrogroup, let λ be the map defined by the formula $\lambda(a) = L_a$ for all $a \in G$, and let H be a subgyrogroup of G.*

1. *If $\text{gyr}[a,b] = \text{I}_G$ for all $a, b \in H$, then $\lambda(H)$ is a subgroup of $\text{LTrs}\,(G)$.*

2. *If $\lambda(H)$ is a normal subgroup of $\text{LTrs}\,(G)$, then H is a strong subgyrogroup of G.*

Proof. For part 1, by assumption, λ defines a homomorphism from H to $\text{LTrs}\,(G)$. Hence, if $h, k \in H$, then $\lambda(h) \circ \lambda(k) = \lambda(h \oplus k)$ belongs to $\lambda(H)$ and $\lambda(h)^{-1} = \lambda(\ominus h)$ belongs to $\lambda(H)$. Thus, $\lambda(H)$ forms a subgroup of $\text{LTrs}\,(G)$.

To prove part 2, suppose that $\lambda(H)$ is a normal subgroup of $\text{LTrs}\,(G)$. Let $a, b \in G$, and let $h \in H$. According to the commutation relation (cf. Equation (14) of [12]), $L_{\text{gyr}[a,b](h)} = \text{gyr}[a,b] \circ L_h \circ \text{gyr}^{-1}[a,b]$. Hence, $L_{\text{gyr}[a,b](h)} \in \lambda(H)$ by normality. This implies that $\text{gyr}[a,b](h) \in H$ since λ is injective, and so $\text{gyr}[a,b](H) \subseteq H$. This proves that H is strong since a and b are arbitrary. □

Let G be a gyrogroup. Next, we will give a nice descriptive form of the left-gyrotranslation group of G. Since $\text{gyr}[a,b]$ is an automorphism of G for all $a, b \in G$, we can let $\text{GYR}\,(G)$ be the subgroup of $\text{Sym}\,(G)$ generated by all the gyroautomorphisms of G, that is,

$$\text{GYR}\,(G) = \langle \{\text{gyr}[a,b] : a, b \in G\} \rangle. \tag{4}$$

Clearly, $\text{GYR}\,(G)$ forms a subgroup of the automorphism group of $(G, \oplus)$. Moreover, G forms a group if and only if $\text{GYR}\,(G) = \{\text{I}_G\}$. Note that, since $\text{gyr}^{-1}[a,b] = \text{gyr}[b,a]$ for all $a, b \in G$ (cf. Theorem 2.34 of [14]), it follows that $\text{GYR}\,(G)$ consists precisely of compositions of finitely many gyroautomorphisms of G, that is,

$$\text{GYR}\,(G) = \{\text{gyr}[a_1, b_1] \circ \cdots \circ \text{gyr}[a_n, b_n] : a_i, b_i \in G, i = 1, 2, \ldots, n\}. \tag{5}$$

The next lemma gives one important subgroup of the symmetric group of G, which is useful in describing the left-gyrotranslation group of a gyrogroup.

Lemma 1. *Let G be a gyrogroup. Then,*

$$\widehat{G} \circ \mathrm{GYR}\,(G) = \{L_a \circ \tau : a \in G \text{ and } \tau \in \mathrm{GYR}\,(G)\}$$

forms a subgroup of $\mathrm{Sym}\,(G)$.

Proof. Set $\Gamma = \widehat{G} \circ \mathrm{GYR}\,(G)$. Clearly, $\mathrm{I}_G = L_e \circ \mathrm{gyr}[e, e] \in \Gamma$. Let $X, Y \in \Gamma$. Then, $X = L_a \circ \tau_1$ and $Y = L_b \circ \tau_2$, where $a, b \in G, \tau_1, \tau_2 \in \mathrm{GYR}\,(G)$. Using the composition law of left gyrotranslations and the commutation relation, we obtain that

$$X \circ Y^{-1} = L_{a \oplus \tau_1 \circ \tau_2^{-1}(\ominus b)} \circ \mathrm{gyr}[a, \tau_1 \circ \tau_2^{-1}(\ominus b)] \circ \tau_1 \circ \tau_2^{-1},$$

and so $X \circ Y^{-1} \in \Gamma$. From the subgroup test, it follows that Γ is a subgroup of $\mathrm{Sym}\,(G)$. □

It turns out that $\widehat{G} \circ \mathrm{GYR}\,(G)$ is indeed the left-gyrotranslation group of G, as shown in the following theorem.

Theorem 1. *Let G be a gyrogroup. Then,* $\mathrm{LTrs}\,(G) = \widehat{G} \circ \mathrm{GYR}\,(G)$.

Proof. Since $\widehat{G} \subseteq \widehat{G} \circ \mathrm{GYR}\,(G)$ and $\widehat{G} \circ \mathrm{GYR}\,(G)$ is a subgroup of $\mathrm{Sym}\,(G)$, it follows that $\mathrm{LTrs}\,(G) \subseteq \widehat{G} \circ \mathrm{GYR}\,(G)$ by minimality of $\mathrm{LTrs}\,(G)$. By the composition law, $\mathrm{gyr}[a, b] = L^{-1}_{a \oplus b} \circ L_a \circ L_b$ for all $a, b \in G$. Hence, $\mathrm{GYR}\,(G) \subseteq \mathrm{LTrs}\,(G)$, which implies that $\widehat{G} \circ \mathrm{GYR}\,(G) \subseteq \mathrm{LTrs}\,(G)$ by the closure property of $\mathrm{LTrs}\,(G)$, and so equality holds. □

Corollary 1. *Let G be a gyrogroup. Then, there exists a bijection from* $\mathrm{LTrs}\,(G)$ *to* $G \times \mathrm{GYR}\,(G)$. *In particular, if G is finite, then* $|\mathrm{LTrs}\,(G)| = |G||\mathrm{GYR}\,(G)|$.

Proof. Define a map σ by $\sigma(a, \tau) = L_a \circ \tau$ for all $a \in G$, $\tau \in \mathrm{GYR}\,(G)$. Using the Unique Factorization Theorem (cf. Theorem 11 of [12]), one can check that σ is a bijection from $G \times \mathrm{GYR}\,(G)$ to $\widehat{G} \circ \mathrm{GYR}\,(G)$, and so the corollary follows. □

Theorem 1 and Corollary 1 together illuminate the structure of the left-gyrotranslation group of a gyrogroup. In particular, they indicate that one can understand the left-gyrotranslation group of a gyrogroup G by investigating $\widehat{G}$ and $\mathrm{GYR}\,(G)$. However, the task of determining the group $\mathrm{GYR}\,(G)$ in an explicit form for a given gyrogroup G is quite complicated. A few concrete examples will be exhibited in the sequel. Now, we establish a connection between strong subgyrogroups of G and subgroups of $\mathrm{LTrs}\,(G)$ containing all the gyroautomorphisms of G.

Proposition 2. *Let G be a gyrogroup, and suppose that H is a subgyrogroup of G. If H is strong, then*

$$\widehat{H} \circ \mathrm{GYR}\,(G) = \{L_a \circ \tau : a \in H \text{ and } \tau \in \mathrm{GYR}\,(G)\}$$

is a subgroup of $\mathrm{LTrs}\,(G)$ *containing all the gyroautomorphisms of G.*

Proof. Since $\{\mathrm{gyr}[a, b] : a, b \in G\} \subseteq \widehat{H} \circ \mathrm{GYR}\,(G) \subseteq \mathrm{LTrs}\,(G)$, the proposition follows as in Lemma 1. □

The converse of Proposition 2 also holds in the sense of the following proposition.

Proposition 3. *Let G be a gyrogroup. If Σ is a subgroup of* $\mathrm{LTrs}\,(G)$ *containing all the gyroautomorphisms of G, then there is a unique strong subgyrogroup H of G such that $\Sigma = \widehat{H} \circ \mathrm{GYR}\,(G)$.*

Proof. It can be proved that the required subgyrogroup H is

$$H = \{a \in G : L_a \circ \rho \in \Sigma \text{ for some } \rho \in \mathrm{GYR}\,(G)\},$$

where the uniqueness part follows directly from the Unique Factorization Theorem. □

Next, we characterize commutativity of the left-gyrotranslation group of an arbitrary gyrogroup. This enables us to gain insight into the left-gyrotranslation group of a certain gyrogroup, as we will see shortly. We first prove the following lemma for convenience.

Lemma 2. *Let G be a gyrogroup, and let $a, b \in G$. Then, $L_a \circ L_b = L_b \circ L_a$ if and only if $a \oplus b = b \oplus a$ and $\mathrm{gyr}^2[a, b] = \mathrm{I}_G$.*

Proof. Suppose that $L_a \circ L_b = L_b \circ L_a$. By the composition law of left gyrotranslations, $L_{a \oplus b} \circ \mathrm{gyr}[a, b] = L_{b \oplus a} \circ \mathrm{gyr}[b, a]$. By the Unique Factorization Theorem, $L_{a \oplus b} = L_{b \oplus a}$ and $\mathrm{gyr}[a, b] = \mathrm{gyr}[b, a]$. This implies that $a \oplus b = b \oplus a$. Since $\mathrm{gyr}^{-1}[a, b] = \mathrm{gyr}[b, a]$, it follows that $\mathrm{gyr}^2[a, b] = \mathrm{I}_G$. Conversely, suppose that $a \oplus b = b \oplus a$ and $\mathrm{gyr}^2[a, b] = \mathrm{I}_G$. Then, $\mathrm{gyr}[a, b] = \mathrm{gyr}^{-1}[a, b] = \mathrm{gyr}[b, a]$. Hence,

$$L_a \circ L_b = L_{a \oplus b} \circ \mathrm{gyr}[a, b] = L_{b \oplus a} \circ \mathrm{gyr}[b, a] = L_b \circ L_a,$$

which completes the proof. □

In fact, Lemma 2 can be extended to a stronger result: if $a, b \in G$ and $\beta \in \operatorname{Aut}(G)$, then L_a and $L_b \circ \beta$ commute if and only if $a \oplus b = b \oplus \beta(a)$ and $\operatorname{gyr}[a, b] = \operatorname{gyr}[b, \beta(a)]$. In light of Lemma 2, we directly obtain a sufficient and necessary condition for the left-gyrotranslation group of a gyrogroup to be abelian, as shown in the following theorem.

Theorem 2. *Let G be a gyrogroup. Then, the left-gyrotranslation group* $\operatorname{LTrs}(G)$ *is abelian if and only if* $a \oplus b = b \oplus a$ *and* $\operatorname{gyr}^2[a, b] = \mathrm{I}_G$ *for all* $a, b \in G$.

Proof. Suppose that $\operatorname{LTrs}(G)$ is abelian. Let $a, b \in G$. By assumption, $L_a \circ L_b = L_b \circ L_a$. By Lemma 2, $a \oplus b = b \oplus a$ and $\operatorname{gyr}^2[a, b] = \mathrm{I}_G$. Conversely, the assumption implies that $L_a \circ L_b = L_b \circ L_a$ for all $a, b \in G$. Hence, $\operatorname{LTrs}(G)$ is abelian for its generators commute. □

A few examples of the left-gyrotranslation groups of known concrete gyrogroups are provided below.

Example 1. One of the most prominent examples of gyrogroups is the (complex) Möbius gyrogroup [15], which consists of the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ endowed with Möbius addition $\oplus_M$ defined by the formula

$$a \oplus_M b = \frac{a + b}{1 + \overline{a}b} \tag{6}$$

for all $a, b \in \mathbb{D}$. This gyrogroup is called "Möbius" because its left gyrotranslations are indeed Möbius transformations of the form

$$\tau_a(z) = \frac{a + z}{1 + \overline{a}z} \tag{7}$$

for all $a, z \in \mathbb{D}$ (cf. Lemma 6.2.2 of [1]). Moreover, the gyroautomorphisms of the Möbius gyrogroup are given by the formula

$$\operatorname{gyr}[a, b](z) = \frac{1 + a\overline{b}}{1 + \overline{a}b} z \tag{8}$$

for all $a, b, z \in \mathbb{D}$, which are indeed rotations of the unit disk because $\dfrac{1 + a\overline{b}}{1 + \overline{a}b}$ is a unimodular complex number for all $a, b \in \mathbb{D}$.

In this example, we determine the left-gyrotranslation group of the Möbius gyrogroup. It turns out that the left-gyrotranslation group of the Möbius gyrogroup is in fact the group of Möbius transformations leaving the disk fixed. As proved in [1], any Möbius transformation on $\mathbb{D}$ can be factored as $\tau_a \circ \rho_\omega$, where $a \in \mathbb{D}$, ω is a unimodular complex number, and ρ_ω is the rotation of the disk given by $\rho_\omega(z) = \omega z$ for all $z \in \mathbb{D}$. Therefore, in order to complete the example, we need only show that $\mathrm{GYR}\,(\mathbb{D})$ equals the group of rotations of the disk. To do so, it suffices to prove that if $\omega \in S^1 = \{z \in \mathbb{C} : |z| = 1\}$, then

$$\omega = \left(\frac{1 + a_1\overline{b_1}}{1 + \overline{a_1}b_1}\right)\left(\frac{1 + a_2\overline{b_2}}{1 + \overline{a_2}b_2}\right)\cdots\left(\frac{1 + a_k\overline{b_k}}{1 + \overline{a_k}b_k}\right)$$

for some $a_1, b_1, a_2, b_2, \ldots, a_k, b_k \in \mathbb{D}$. Now, choose a non-real complex number a in $\mathbb{D}$ such that $\dfrac{1 + \bar{a}}{1 + a}$ lies in the upper half of the complex plane (for example, choose $a = -0.5 - 0.5i$). Define a function f on the close interval $[0, 1]$ by the formula

$$f(c) = \frac{1 + c\overline{a}}{1 + ca}. \tag{9}$$

Since a is not real, f is well defined, and furthermore $f(c) \neq 1$ for all $0 < c \leq 1$. It follows that f is continuous and sends $[0, 1]$ to the unit circle S^1, that is, f is a path from $f(0) = 1$ to $f(1) = \dfrac{1 + \overline{a}}{1 + a}$. In fact, $f((0, 1])$ is a subset of the upper half of the complex plane. Suppose that the positive angle between $f(1)$ and the real-axis is θ. Hence, for each $r \in (0, \theta)$, there exists a number $c \in (0, 1)$ such that the positive angle between $f(c)$ and the real-axis is r by the well-known Intermediate Value Theorem. This implies that $f(s) = \cos\left(\frac{\theta}{2}\right) + i \sin\left(\frac{\theta}{2}\right)$ for some $s \in (0, 1)$. Let $\omega \in S^1$. Then, $\omega = \cos(\alpha) + i \sin(\alpha)$ for some $\alpha > 0$. One can write $\alpha = n\left(\frac{\theta}{2}\right) + \beta$ for some $n \in \mathbb{Z}$ and $0 \leq \beta < \frac{\theta}{2}$, and so $\omega = f(s)^n f(t)$, where $f(t) = \cos(\beta) + i \sin(\beta)$, as claimed. This also shows that the group of Möbius transformations preserving $\mathbb{D}$ is generated by all the Möbius translations τ_a, $a \in \mathbb{D}$.

Example 2. Here, we determine the left-gyrotranslation group of the gyrogroup $G_8 = \{0, 1, 2, 3, 4, 5, 6, 7\}$ (cf. Example 1 of [7]). Recall that the automorphism A of G_8 is decomposed as $A = (4\ 6)(5\ 7)$. Hence, $A^2 = \mathrm{I}_{G_8}$. This implies that $\{\mathrm{I}_{G_8}, A\}$ forms a subgroup of $\mathrm{Sym}\,(G_8)$, and so $\mathrm{GYR}\,(G_8) = \{\mathrm{I}_{G_8}, A\}$ by minimality of $\mathrm{GYR}\,(G_8)$. It follows by Corollary 1 that $\mathrm{LTrs}\,(G_8)$ has order 16. By Theorem 2, $\mathrm{LTrs}\,(G_8)$ is not abelian since $1 \oplus 4 \neq 4 \oplus 1$, for example. Hence, $\mathrm{LTrs}\,(G_8)$ forms a non-abelian group of order 16. We have by inspection that the center of $\mathrm{LTrs}\,(G_8)$ is

$$Z(\mathrm{LTrs}\,(G_8)) = \{L_0, L_3, L_1 \circ A, L_2 \circ A\}.$$

Recall that there exists a non-abelian group of order 16 with presentation

$$\langle x, y, z : x^4 = y^2 = z^2 = 1, xy = yx, xz = zx, yzy^{-1} = zx^2 \rangle; \qquad (10)$$

see, for instance, Table B.2 of [2]. Set $x = L_1 \circ A$, $y = A$, and $z = L_5$. Then, x, y, and z are generators of $\mathrm{LTrs}\,(G_8)$. Furthermore, $x^4 = \mathrm{I}_{G_8}$, $y^2 = \mathrm{I}_{G_8}$, $z^2 = \mathrm{I}_{G_8}$, and $y \circ z \circ y^{-1} = z \circ x^2$. Since $x \in Z(\mathrm{LTrs}\,(G_8))$, we obtain that $x \circ y = y \circ x$ and $x \circ z = z \circ x$. Therefore, by von Dyck's Theorem, $\mathrm{LTrs}\,(G_8)$ has presentation (10).

Example 3. Here, we determine the left-gyrotranslation group of the gyrogroup G_{15} (cf. Example 8 of [7]). As in [7], the non-trivial gyroautomorphisms of G_{15} are A, B, C, and D, given by formula (72) of [7]. A direct computation shows that $\{\mathrm{I}_{G_{15}}, A, B, C, D\}$ is a subgroup of $\mathrm{Sym}\,(G_{15})$ so that $\mathrm{GYR}\,(G_{15}) = \{\mathrm{I}_{G_{15}}, A, B, C, D\}$ by minimality of $\mathrm{GYR}\,(G_{15})$. This implies by Corollary 1 that $\mathrm{LTrs}\,(G_{15})$ has order 75. Since $A^2 \neq \mathrm{I}_{G_{15}}$, it follows from Theorem 2 that $\mathrm{LTrs}\,(G_{15})$ is not abelian. It is worth noting that there is a unique non-abelian group of order 75. Usually, the proof of this fact makes use of the Sylow Theorems by showing that any group of order 75 is necessarily a semidirect product of its 5-Sylow subgroup and its 3-Sylow subgroup and then finding a non-trivial homomorphism from $\mathbb{Z}_3$ to the automorphism group of $\mathbb{Z}_5 \times \mathbb{Z}_5$. Here, we directly give a concrete example of a non-abelian group of order 75, which is $\mathrm{LTrs}\,(G_{15})$. In fact, $\mathrm{LTrs}\,(G_{15})$ can be realized as a subgroup of the symmetric group S_{15} under the identification $0 \leftrightarrow 1$ and $i \leftrightarrow i+1$ for $i = 1, 2, \dots, 14$. Furthermore, if x, y, and z are assigned as

$$\begin{aligned} x &= (1\ 5\ 14\ 13\ 10)(2\ 8\ 6\ 11\ 7) \\ y &= (2\ 8\ 6\ 11\ 7)(3\ 4\ 9\ 12\ 15) \\ z &= (1\ 3\ 2)(4\ 7\ 5)(6\ 13\ 12)(8\ 10\ 15)(9\ 11\ 14), \end{aligned} \qquad (11)$$

then it can be verified that $\mathrm{LTrs}\,(G_{15})$ has a presentation

$$\langle x, y, z : x^5 = y^5 = z^3 = 1, xy = yx, zxz^{-1} = x^{-1}y, zyz^{-1} = x^{-1} \rangle \qquad (12)$$

by invoking von Dyck's Theorem.

Examples 1 and 3 are good examples to see how the study of gyrogroups leads to a better understanding of group structures. In a similar fashion to Examples 2 and 3, one can verify that the left-gyrotranslation groups of the following gyrogroups:

- K_{16} (cf. Example 2.13 of [13]);
- D_{16}^{gyr} (cf. Example 5.1 of [11]);
- Q_{16}^{gyr} (cf. Example 5.2 of [11]);
- SD_{16}^{gyr} (cf. Example 5.3 of [11]);
- $\text{Dih}(G_8)$ (cf. Example 5 of [6])

are non-abelian groups of order 32 since these gyrogroups are of order 16 and each gyrogroup has only one non-trivial gyroautomorphism of order 2.

Next, we will prove that left-gyrotranslation groups are invariant objects of gyrogroups: isomorphic gyrogroups possess isomorphic left-gyrotranslation groups. Therefore, the notion of left-gyrotranslation groups may be used to distinguish gyrogroups, up to isomorphism, in certain circumstances.

Theorem 3. *Suppose that G and H are gyrogroups. If $G \cong H$ as gyrogroups, then* $\text{LTrs}\,(G) \cong \text{LTrs}\,(H)$ *as groups.*

Proof. Suppose that $\phi : G \to H$ is a gyrogroup isomorphism. First, let us show that $\phi \circ \text{gyr}[a, b] \circ \phi^{-1} = \text{gyr}[\phi(a), \phi(b)]$ for all $a, b \in G$. Let $a, b \in G$, and let $h \in H$. By the gyrator identity (cf. part 10 of Theorem 2.10 of [14]),

$$\begin{aligned}\phi \circ \text{gyr}[a, b] \circ \phi^{-1}(h) &= \phi(\text{gyr}[a, b](\phi^{-1}(h)))\\ &= \phi(\ominus(a \oplus b) \oplus (a \oplus (b \oplus \phi^{-1}(h))))\\ &= \ominus(\phi(a) \oplus \phi(b)) \oplus (\phi(a) \oplus (\phi(b) \oplus h))\\ &= \text{gyr}[\phi(a), \phi(b)](h),\end{aligned}$$

as required. In view of Equation (5), this implies that if τ lies in $\text{GYR}\,(G)$, then $\phi \circ \tau \circ \phi^{-1}$ lies in $\text{GYR}\,(H)$ and that if η lies in $\text{GYR}\,(H)$, then $\phi^{-1} \circ \eta \circ \phi$ lies in $\text{GYR}\,(G)$.

Define $\tilde{\phi}$ as $\tilde{\phi}(L_a \circ \tau) = L_{\phi(a)} \circ \phi \circ \tau \circ \phi^{-1}$ for all $a \in G, \tau \in \text{GYR}\,(G)$. By the uniqueness part of the Unique Factorization Theorem, $\tilde{\phi}$ is well defined and injective. As proved above, ϕ maps $\text{LTrs}\,(G)$ to $\text{LTrs}\,(H)$. To see that $\tilde{\phi}$ is surjective, let $b \in H$, and let $\eta \in \text{GYR}\,(H)$. By the surjectivity of ϕ, $\phi(a) = b$ for some $a \in G$. Note that $L_a \circ \phi^{-1} \circ \eta \circ \phi$ is in $\text{LTrs}\,(G)$, and furthermore $\tilde{\phi}(L_a \circ \phi^{-1} \circ \eta \circ \phi) = L_b \circ \eta$. This shows that $\tilde{\phi}$ acts as a group isomorphism from $\text{LTrs}\,(G)$ to $\text{LTrs}\,(H)$, which completes the proof. □

We close this section with the remark that the converse of Theorem 3 is not, in general, true. In fact, let G be the gyrogroup G_{15} in Example 3, and let H be the group $\mathrm{LTrs}\,(G_{15})$. As noted earlier, $\mathrm{LTrs}\,(H) = \widehat{H}$. Furthermore, $\widehat{H}$ is isomorphic to H by Cayley's Theorem in abstract algebra. It follows that $\mathrm{LTrs}\,(G)$ and $\mathrm{LTrs}\,(H)$ are isomorphic. However, G and H are not isomorphic.

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