

Dissemination of zeros of generalized derivative of a polynomial via matrix approach

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ABSTRACT. A polynomial of degree n is denoted by $p(z) := \sum_{j=0}^n a_j z^j$. Then, by classical Cauchy's result,

$$|z| \leq 1 + \max \left(\left| \frac{a_{n-1}}{a_n} \right|, \left| \frac{a_{n-2}}{a_n} \right|, \left| \frac{a_{n-3}}{a_n} \right|, \dots, \left| \frac{a_0}{a_n} \right| \right)$$

contains all of $p(z)$'s zeros.

In order to improve on classical Cauchy finding, we will extend such results in this study to the polar derivative of an algebraic polynomial using matrix technique.

1. History and basic results

While using results on the characteristic values of M to acquire information on the distribution of zeros in p , it may appear strange to describe polynomial p as a characteristic polynomial of a matrix M . In actuality, practically all of the conclusions obtained using this kind of matrix approach can be verified using a straightforward polynomial approach. However, using matrix techniques leads to several helpful adjustments that happen automatically. Numerous researchers like the approach of Matrix Analysis [3, 8], as it is a well-established subject.

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Similar matrices. Let M and N be two square matrices of the same order, if there exists an invertible (where P is change of basis matrix) P of same order, such that $N = P^{-1}MP$, then M is similar to N . Similar matrices have same eigen values, trace, determinant, algebraic and geometric multiplicities. Similarly M^n and etc. M and $P^{-1}MP$ are similar matrices.

Let C_k be k -th degree polynomial for $k = 0, 1, 2, \dots$. Clearly,

$$C_0(z), C_1(z), \dots, C_{k-1}(z), zC_{k-1}(z)$$

form a basis of the vector space of polynomials of degree at most k . Therefore there exist constants P_{mk} , so that

$$C_k = (z - P_{k-1,k})C_{k-1}(z) - P_{k-2,k}C_{k-2}(z) - \dots - P_{0,k}C_0(z).$$

We have the following matrix representation of C_k (see [6, p. 264]) as a consequence of above representation:

Theorem 1.1. *Let $C_0, C_1, \dots$, be a sequence of monic polynomials with degree of $C_k(z)$ is k for $k = 0, 1, 2, \dots$. Then $(-1)^n C_n(z)$ is the characteristic polynomial of the matrix*

$$C = \begin{bmatrix} P_{01} & P_{02} & \cdots & P_{0,n-1} & P_{0,n} \\ 1 & P_{12} & \cdots & P_{1,n-1} & P_{1,n} \\ 0 & 1 & \cdots & P_{2,n-1} & P_{2,n} \\ . & . & \cdots & . & . \\ . & . & \cdots & . & . \\ . & . & \cdots & . & . \\ 0 & 0 & \cdots & 1 & P_{n-1,n} \end{bmatrix}.$$

Equivalently

$$C_n(z) = (-1)^n \det(P_n - zI_n),$$

where I_n is identity matrix of order n .

We have the following more general result (see [6, p. 264]):

Theorem 1.2. *Every polynomial $p(z) := \sum_{j=0}^n a_j C_j(z)$, ($a_n \neq 0$) may be represented as*

$$p(z) = (-1)^n \det(G_n - zI_n),$$

where $G_n := P_n + A_n$ and

$$A_n = \begin{bmatrix} 0 & \cdots & 0 & \frac{-a_0}{a_n} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ 0 & \cdots & 0 & \frac{-a_{n-1}}{a_n} \end{bmatrix}.$$

A matrix C whose all characteristic values [3] coincide with roots of polynomial equation $p(z) = 0$ is called a companion matrix of p . Specifically we say that G_n in Theorem 1.2, is a companion matrix of p with respect to basis $C_0, C_1, \dots, C_n$. If $C_z = z^k$ for $k = 0, 1, \dots, n$ and the constants P_{mk} (defined above) are all zero, so we get

$$G_{0n} = \begin{bmatrix} 0 & \cdots & 0 & q_0 \\ 1 & \cdots & 0 & q_1 \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdots & 1 & q_{n-1} \end{bmatrix}.$$

Here $(q_k := -\frac{a_k}{a_n}, k = 0, 1, \dots, n-1)$. Frobenius matrix of p is defined as the companion matrix [6] of $p(z) := \sum_{j=0}^n a_j z^j$. By the application of Gershgorin Disc Theorem [6] to G_{0n} , we will observe that distribution of all zeros of $p(z) := \sum_{j=0}^n a_j z^j$ lie in

$$\left\{ z \in \mathbb{C} : \left| z + \frac{a_{n-1}}{a_n} \right| \leq 1 \right\} \\ \cup \left\{ z \in \mathbb{C} : |z| \leq \max \left\{ 1 + \left| \frac{a_{n-2}}{a_n} \right|, \dots, 1 + \left| \frac{a_1}{a_n} \right|, \left| \frac{a_0}{a_n} \right| \right\} \right\}.$$

If all coefficients of p are distinct from zero, we may use the following matrix

$$D := \left(\frac{a_1}{a_n}, \frac{a_2}{a_n}, \dots, \frac{a_{n-1}}{a_n}, 1 \right)$$

in order to construct the similar matrix

$$D^{-1}G_{0n}D = \begin{bmatrix} 0 & \cdots & 0 & \frac{a_0}{a_n} \\ \frac{a_1}{a_2} & \cdots & 0 & -\frac{a_1}{a_2} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdots & \frac{a_{n-1}}{a_n} & -\frac{a_{n-1}}{a_n} \end{bmatrix},$$

whose characteristic values too coincide with the zeros of p . Using Gershgorin Disc Theorem, we will obtain refinement of the Cauchy bound. Let $p(z) := \sum_{j=0}^n a_j z^j$, $a_n \neq 0$ be a polynomial of degree n over $\mathbb{C}$. Then the companion matrix of $P(z)$ is defined as

$$C = \begin{bmatrix} 0 & 0 & \cdots & \frac{-a_0}{a_n} \\ 1 & 0 & \cdots & \frac{-a_1}{a_n} \\ 0 & 1 & \cdots & \frac{-a_2}{a_n} \\ \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & \frac{-a_{n-1}}{a_n} \end{bmatrix}.$$

The characteristic polynomial of C is the polynomial $P(z)$, and roots of $p(z) = 0$ are called characteristic roots (eigenvalues). Then Polar derivative of $p(z)$ [1] with respect α , which is denoted by $D_\alpha p(z)$ and is defined by

$$D_\alpha p(z) = np(z) + (\alpha - z)p'(z).$$

The polynomial $D_\alpha p(z)$ is of degree at most $n - 1$ and it generalizes the ordinary derivative, where α is real or complex number, in the sense that

$$\lim_{\alpha \rightarrow \infty} \frac{D_\alpha p(z)}{\alpha - z} = p'(z).$$

Here we first mention result of Cauchy [2], concerning the bounds for the moduli of zeros of algebraic polynomial.

Theorem 1.3. *Let $p(z) := \sum_{j=0}^n a_j z^j$ be a polynomial of degree n . Then all zeros of $p(z)$ lie in*

$$|z| \leq 1 + \max \left(\left| \frac{a_{n-1}}{a_n} \right|, \left| \frac{a_{n-2}}{a_n} \right|, \left| \frac{a_{n-3}}{a_n} \right|, \dots, \left| \frac{a_0}{a_n} \right| \right).$$

This result is well known in the theory of zero distribution of polynomials.

2. Lemmas

The following lemma, due to Hadamard [5], is necessary.

Lemma 2.1. *Let $A = [a_{ij}]$ be an $n \times n$ matrix, then $|A| \neq 0$, if*

$$|a_{ij}| > P_i = \sum_{i \neq j; j=1}^n |a_{ij}|, i = 1, 2, 3, \dots, n.$$

We deduce the following lemma from Lemma 2.1 (see [4, 5, 6, 7, 9]).

Lemma 2.2. *Let $A = [a_{ij}]$ be an $n \times n$ matrix, then characteristic roots of A lie in the union of disks*

$$|z - a_{ii}| \leq P_i, i = 1, 2, 3, \dots, n,$$

where

$$P_i = \sum_{i \neq j; j=1}^n |a_{ij}|, i = 1, 2, 3, \dots, n.$$

3. Main findings

Using the polar derivative of an algebraic polynomial, we prove a result.

Theorem 3.1. *Let $p(z) := \sum_{j=0}^n a_j z^j, a_n \neq 0$ be a polynomial of degree n . Then all zeros of $D_\alpha p(z)$ lie in*

$$|z| \leq 1 + \max(|\zeta_{n-1}|, |\zeta_{n-2}|, |\zeta_{n-3}|, \dots, |\zeta_1|),$$

where $\zeta_{n-j} = \frac{(n-j)\alpha a_{n-j} + (j+1)a_{n-j-1}}{n\alpha a_n + a_{n-1}}, j = 1, 2, 3, \dots, n-1$.

Proof. Suppose $P(z) := \sum_{j=0}^n a_j z^j$. Then by definition of polar derivative

$$\begin{aligned} D_\alpha P(z) &= (n\alpha a_n + a_{n-1})z^{n-1} + ((n-1)\alpha a_{n-1} + 2a_{n-2})z^{n-2} \\ &\quad + ((n-2)\alpha a_{n-2} + 3a_{n-3})z^{n-3} + \dots + (3\alpha a_3 + (n-2)a_2)z^2 \\ &\quad + (2\alpha a_2 + (n-1)a_1)z + (\alpha a_1 + na_0). \end{aligned} \tag{3.1}$$

Therefore the companion matrix of the (3.1) is

$$C = \begin{bmatrix} 0 & 0 & \cdots & \frac{-k_1}{k_n} \\ 1 & 0 & \cdots & \frac{-k_2}{k_n} \\ 0 & 1 & \cdots & \frac{-k_3}{k_n} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & \frac{-k_{n-1}}{k_n} \end{bmatrix},$$

where $k_j = (n-j)\alpha a_{n-j} + (j+1)a_{n-j-1}$, $j = 1, 2, 3, \dots, n-1$ and $k_n = n\alpha a_n + a_{n-1}$.

Applying Lemma 2.2 to the rows of C , it follows that all the characteristic roots of C lie in the union of disks

$$\begin{aligned} |z| &\leq \left| \frac{k_1}{k_n} \right|, \\ |z| &\leq 1 + \left| \frac{k_j}{k_n} \right|, \quad j = 2, 3, \dots, n-2, \\ \left| z + \frac{k_{n-1}}{k_n} \right| &\leq 1. \end{aligned}$$

Thus all characteristic roots of lie in the union of disks

$$|z| \leq 1 + \left| \frac{k_j}{k_n} \right|, \quad j = 1, 2, 3, \dots, n-1.$$

Since all these disks are contained in

$$|z| \leq 1 + \max(|\zeta_{n-1}|, |\zeta_{n-2}|, |\zeta_{n-3}|, \dots, |\zeta_1|),$$

where $\zeta_{n-j} = \frac{(n-j)\alpha a_{n-j} + (j+1)a_{n-j-1}}{n\alpha a_n + a_{n-1}}$, $j = 1, 2, 3, \dots, n-1$. This completes the proof of the Theorem 3.1. $\square$

Dividing numerator and denominator by α and letting $\alpha \rightarrow \infty$ on right side of inequality in Theorem 3.1, we get a sharper result than Cauchy [2]:

Corollary 3.2. *Let $p(z) := \sum_{j=0}^n a_j z^j$, $a_n \neq 0$ be a polynomial of degree n . Then all zeros of $D_\alpha p(z)$ lie in*

$$|z| \leq 1 + \max(|\zeta_{n-1}|, |\zeta_{n-2}|, |\zeta_{n-3}|, \dots, |\zeta_1|),$$

where $\zeta_{n-j} = \frac{(n-j)a_{n-j}}{na_n}$, $j = 1, 2, 3, \dots, n-1$.

Set $\alpha = 0$, Theorem 3.1 reduces to:

Corollary 3.3. Let $p(z) := \sum_{j=0}^n a_j z^j$ be a polynomial of degree n . Then all zeros of $D_0 p(z)$ lie in

$$|z| \leq 1 + \max(|\zeta_{n-1}|, |\zeta_{n-2}|, |\zeta_{n-3}|, \dots, |\zeta_1|),$$

where $\zeta_{n-j} = \frac{(j+1)a_{n-j-1}}{a_{n-1}}, j = 1, 2, 3, \dots, n-1$.

If $a_j \in \mathbb{R}, \forall j = 0, 1, 2, \dots, n$, Corollary 3.3 yields:

Corollary 3.4. Let $p(z) := \sum_{j=0}^n a_j z^j$ be a polynomial of degree n . Then all zeros of $D_0 p(z)$ lie in

$$|z| \leq 1 + \max(\zeta_{n-1}, \zeta_{n-2}, \zeta_{n-3}, \dots, \zeta_1),$$

where $\zeta_{n-j} = \frac{(j+1)a_{n-j-1}}{a_{n-1}}, j = 1, 2, 3, \dots, n-1$.

Theorem 3.5. Let $p(z) := a_n z^n + a_p z^p + \dots + a_1 z^1 + a_0$, $a_n \neq 0$, $0 \leq p \leq n-1$ be a polynomial of degree n over $\mathbb{C}$. Then for every $r \in \mathbb{R}^+$, all zeros of $D_\alpha p(z)$, $\alpha \neq 0$ lie in

$$|z| \leq 1 + \max \left(r, \sum_{j=1}^p \left| \frac{na_{j-1} + j\alpha a_j - (j-1)a_{j-1}}{n\alpha a_n} \right| \frac{1}{r^{n-j-1}} \right).$$

Proof. Suppose $p(z) := a_n z^n + a_p z^p + \dots + a_1 z^1 + a_0$, $a_n \neq 0$, $0 \leq p \leq n-1$ be a polynomial of degree n over $\mathbb{C}$. Then

$$\begin{aligned} D_\alpha p(z) &= n\alpha a_n z^{n-1} + p\alpha a_p z^p + (na_p + p\alpha a_p - (p-1)a_{p-1})z^{p-1} \\ &+ \dots + (na_1 + 2\alpha a_2 - 2a_1)z + (na_0 + \alpha a_1). \end{aligned}$$

Then the companion matrix of $D_\alpha p(z)$ is

$$S = \begin{bmatrix} 0 & 0 & \dots & 0 & \frac{-(na_0 + \alpha a_1)}{n\alpha a_n} \\ 1 & 0 & \dots & 0 & \frac{-(na_1 + 2\alpha a_2 - 2a_1)}{n\alpha a_n} \\ 0 & 1 & \dots & 0 & \frac{-k_3}{k_n} \\ \cdot & \cdot & \dots & \cdot & \cdot \\ \cdot & \cdot & \dots & \cdot & \cdot \\ \cdot & \cdot & \dots & \cdot & \cdot \\ 0 & 0 & \dots & 0 & \frac{-(na_p + p\alpha a_p - (p-1)a_{p-1})}{n\alpha a_n} \\ \cdot & \cdot & \dots & \cdot & \cdot \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix}.$$

Consider a matrix

$$P = \text{diag}(r^{n-2}, r^{n-3}, \dots, r, 1),$$

where $r \in \mathbb{R}^+$. Therefore

$$P^{-1}SP = \begin{bmatrix} 0 & 0 & \dots & 0 & \frac{-(na_0 + \alpha a_1)}{n\alpha a_n r^{n-2}} \\ r & 0 & \dots & 0 & \frac{-(na_1 + 2\alpha a_2 - a_1)}{n\alpha a_n r^{n-3}} \\ \cdot & \cdot & \dots & \cdot & \cdot \\ \cdot & \cdot & \dots & \cdot & \cdot \\ \cdot & \cdot & \dots & \cdot & \cdot \\ 0 & 0 & \dots & 0 & \frac{-(na_p + p\alpha a_p - (p-1)a_{p-1})}{n\alpha a_n r^{n-p-1}} \\ \cdot & \cdot & \dots & \cdot & \cdot \\ 0 & 0 & \dots & r & 0 \end{bmatrix}.$$

Apply Gershgorin Theorem (Lemma 2.2) to the columns of $P^{-1}SP$, it follows that all the eigenvalues of $P^{-1}SP$ lie in

$$|z| \leq 1 + \max \left(r, \sum_{j=1}^p \left| \frac{na_{j-1} + j\alpha a_j - (j-1)a_{j-1}}{n\alpha a_n} \right| \frac{1}{r^{n-j-1}} \right). \quad (3.2)$$

Since the matrix $P^{-1}SP$ is similar to the matrix S and the eigenvalues of S are the zeros of $D_\alpha P(z)$ lie in the circle defined in (3.2). This completes the proof. $\square$

Set $r = 1$, Theorem 3.5 yields:

Corollary 3.6. Let $p(z) := a_n z^n + a_p z^p + \dots + a_1 z^1 + a_0$, $a_n \neq 0$, $0 \leq p \leq n-1$ be a polynomial of degree n over $\mathbb{C}$. Then all zeros of $D_\alpha p(z)$, $\alpha \neq 0$ lie in

$$|z| \leq 1 + \max \left(1, \sum_{j=1}^p \left| \frac{na_{j-1} + j\alpha a_j - (j-1)a_{j-1}}{n\alpha a_n} \right| \right).$$

Letting $\alpha \rightarrow 0$ on the right side of the inequality in Theorem 3.5 and dividing the numerator and denominator by α yields:

Corollary 3.7. Let $p(z) := a_n z^n + a_p z^p + \dots + a_1 z^1 + a_0$, $a_n \neq 0$, $0 \leq p \leq n-1$ be a polynomial of degree n over $\mathbb{C}$. Then for every $r \in \mathbb{R}^+$, all zeros of $D_\alpha p(z)$, $\alpha \neq 0$ lie in

$$|z| \leq 1 + \max \left(r, \sum_{j=1}^p \left| \frac{ja_j}{na_n} \right| \frac{1}{r^{n-j-1}} \right).$$

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