© Algebra and Discrete Mathematics
Volume **37** (2024). Number 2, pp. 287–303
DOI:10.12958/adm2274 RESEARCH ARTICLE

Certain verbal congruences on the free trioid

Anatolii V. Zhuchok

Abstract. Loday and Ronco introduced the notion of a trioid as an algebra defined on a set with three binary associative operations. Every trialgebra is a linear analog of a trioid. Our paper is devoted to the study of verbal congruences on trioids. We characterize the least abelian dimonoid congruences, the least n-nilpotent dimonoid congruences, the least left (right) n-trinilpotent dimonoid congruence, the least n-nilpotent semigroup congruence and the least left (right) n-nilpotent semigroup congruence on the free trioid. The obtained results can be useful in trialgebra theory.

1. Introduction

Trioids and trialgebras as well free monogenic trioids and free trialgebras first appeared in the paper of J.-L. Loday and M.O. Ronco [11]. We recall that a *trioid* (respectively, a *trialgebra*) is a nonempty set (respectively, a vector space) equipped with three binary associative operations $\dashv$, $\vdash$, and $\perp$ satisfying the following axioms:

$$(x \dashv y) \dashv z = x \dashv (y \vdash z), \tag{T1}$$
$$(x \vdash y) \dashv z = x \vdash (y \dashv z), \tag{T2}$$
$$(x \dashv y) \vdash z = x \vdash (y \vdash z), \tag{T3}$$
$$(x \dashv y) \dashv z = x \dashv (y \perp z), \tag{T4}$$
$$(x \perp y) \dashv z = x \perp (y \dashv z), \tag{T5}$$
$$(x \dashv y) \perp z = x \perp (y \vdash z), \tag{T6}$$

2020 Mathematics Subject Classification: *08B20, 20M10, 20M50, 17A30, 17D99.*

Key words and phrases: *trioid, free trioid, dimonoid, semigroup, congruence.*

$$(x \vdash y) \perp z = x \vdash (y \perp z), \tag{T7}$$

$$(x \perp y) \vdash z = x \vdash (y \vdash z). \tag{T8}$$

It is obvious that every trialgebra is just a linear analog of a trioid and the results obtained for trioids can be applied to trialgebras. The main motivation for the introduction of trioids was the study of ternary planary trees [11]. Trioids appeared to be naturally related to several areas such as algebraic topology, trialgebra theory, dialgebra theory and semigroup theory. Trioids are also investigated because of they generalize the concept of a dimonoid (see, e.g., [10, 14, 20, 25]). Digroups are a type of dimonoids that generalize groups and have close relationships with trioids (see, e.g., [16, 17, 35, 39]). There are some connections of trioids with doppelsemigroups (see, e.g., [19, 27, 45]). More precisely, any commutative trioid $(T, \dashv, \vdash, \perp)$ gives rise to a commutative doppelsemigroup $(T, \dashv, \vdash)$. Free monogenic trioids turn out to be very useful in constructing free trialgebras [11]. Trialgebras have been studied by various authors (see, e.g., [1, 5, 6, 8]). It is well-known that any trialgebra gives rise to a noncommutative version of Poisson algebra by $[x, y] := x \dashv y - y \vdash x$ and $xy := x \perp y$. There are also several studies on Poisson algebraic structures, see, for example, [3]. Dendriform trialgebras [15] are the operadic duals of trialgebras, and the free dendriform trialgebra has an interesting Hopf algebra structure. Cubical trialgebras are operadically self-dual and they are a generalisation of cubical dialgebras [11]. Nowadays, trioid theory is one of actively developing areas of modern algebra. Some results of this theory were presented in the survey papers [29, 32]. One of the first steps in trioid theory is the description of congruences.

Let A be an algebra. A congruence θ on A is *verbal* if there exists a variety V such that θ is the least congruence on A such that $A/\theta \in V$. Every verbal congruence relation is known to be fully invariant and every fully invariant congruence on a free algebra is verbal [2]. Any solution of the word problem for a free object in a variety provides a copy of the free object as a quotient structure. The problem of the characterization of verbal congruences on free algebras is equivalent to the word problem.

The following types of verbal congruences on trioids were studied in [22, 24, 26]: the least idempotent congruence on a trioid with a commutative operation, the least semilattice congruence on a trioid with an idempotent operation, the least separative congruence on a trioid with a commutative operation, the least group congruence on a trioid with an inverse semigroup, the least group congruence on a trioid with an orthodox semigroup, the least group congruence on a trioid with a regular semigroup and the least semilattice congruence on an arbitrary trioid. Certain least congruences on relatively free trioids have been investigated, too. In particular, the problem of the characterization of the least dimonoid congruences and the least semigroup congruence on free trioids, free commutative trioids, free rectangular trioids, free n-nilpotent trioids, free left (right) n-trinilpotent trioids as well as free abelian trioids was solved in [30, 31, 38]. The least left (right) zero congruence and the least rectangular band congruence on the free rectangular trioid were presented in [42]. For free trioids, characterizations of the least left (right) zero congruence, the least rectangular band congruence and the least n-nilpotent congruence were given in [40] and [41], respectively. The description of the least rectangular triband congruence on the free trioid follows from Theorem 3.1 (i) in [40]. The least commutative congruence, the least commutative dimonoid congruences and the least commutative semigroup congruence on the free trioid were presented in [18]. The problem of the description of the least left (right) n-trinilpotent congruence on the free trioid was solved in [34]. The least abelian congruence and the least idempotent congruence on the free trioid were given in [44] and [7], respectively.

This paper continues to investigate verbal congruences on free trioids. Section 2 gives two constructions of the free trioid needed in the paper. In Section 3, the least abelian dimonoid congruences on the free trioid are characterized (Theorem 3.1). In Section 4, we present the least n-nilpotent dimonoid congruences and the least n-nilpotent semigroup congruence on the free trioid (Theorem 4.1). The final section concentrates to descriptions of the least left (right) n-trinilpotent dimonoid congruence and the least left (right) n-nilpotent semigroup congruence on the free trioid (Theorem 5.1).

2. Free trioids

Loday and Ronco described the free trioid of rank 1 [11]. In this section, we give two constructions of the free trioid of an arbitrary rank (see [32, 40]).

Let X be an arbitrary nonempty set, $\overline{X} = \{\overline{x} \mid x \in X\}$, and let $F[X]$ be the free semigroup on X. Let further $P \subset F[X \cup \overline{X}]$ be a subsemigroup which contains words w with the element $\overline{x}$ $(x \in X)$ occuring in w at least one time. For every $w \in P$ denote by $\widetilde{w}$ the word obtained from w by change of all letters $\overline{x}$ $(x \in X)$ by x. For instance, if $w = x\overline{x}\,\overline{y}x\overline{z}$ then

$\widetilde{w} = xxyxz$. Obviously, $\widetilde{w} \in F[X \cup \overline{X}] \backslash P$.

Define operations $\dashv$, $\vdash$, and $\perp$ on P by

$$w \dashv u = w\widetilde{u}, \quad w \vdash u = \widetilde{w}u, \quad w \perp u = wu$$

for all $w, u \in P$. The algebra $(P, \dashv, \vdash, \perp)$ is denoted by $Frt(X)$.

Proposition 2.1 ([40, Proposition 1])**.** *$Frt(X)$ is the free trioid.*

Now we give another representation of the free trioid of an arbitrary rank.

As usual, $\mathbb{N}$ denotes the set of all positive integers. For every word ω over X the length of ω is denoted by ℓ_ω. For any $n, k \in \mathbb{N}$ and $L \subseteq \{1, 2, \ldots, n\}, L \neq \varnothing$, we let $L + k = \{m + k \,|\, m \in L\}$. Define operations $\dashv$, $\vdash$, and $\perp$ on the set

$$F = \{(w, L) \,|\, w \in F[X],\ L \subseteq \{1, 2, \ldots, \ell_w\},\ L \neq \varnothing\}$$

by

$$(w, L) \dashv (u, R) = (wu, L), \quad (w, L) \vdash (u, R) = (wu, R + \ell_w),$$

$$(w, L) \perp (u, R) = (wu, L \cup (R + \ell_w))$$

for all $(w, L), (u, R) \in F$. The algebra $(F, \dashv, \vdash, \perp)$ is denoted by $FT(X)$.

Theorem 2.1 ([32, Theorem 7.1])**.** *The free trioid $Frt(X)$ is isomorphic to the trioid $FT(X)$.*

Further, we will use both constructions of the free trioid.

3. The least abelian dimonoid congruences on the free trioid

In this section, we characterize the least abelian dimonoid congruences on the free trioid. We will use notations of Section 2.

We begin with recalling required definitions.

A trioid $(T, \dashv, \vdash, \perp)$ is called *abelian* [44] if $x \dashv y = y \vdash x$ for all $x, y \in T$. A congruence ρ on a trioid $(T, \dashv, \vdash, \perp)$ is called *abelian* [44] if $(T, \dashv, \vdash, \perp)/\rho$ is an abelian trioid. A semigroup S is a *rectangular band* if $xyx = x$ for all $x, y \in S$. Equivalently, a semigroup S is a *rectangular band* if $x^2 = x$, $xyz = xz$ for all $x, y, z \in S$. A trioid $(T, \dashv, \vdash, \perp)$ is called a *rectangular trioid* [42] if semigroups $(T, \dashv)$, $(T, \vdash)$ and $(T, \perp)$ are rectangular bands. A congruence ρ on a trioid $(T, \dashv, \vdash, \perp)$ is called a *rectangular triband congruence* if $(T, \dashv, \vdash, \perp)/\rho$ is a rectangular trioid. If ρ is a congruence on a trioid $(T, \dashv, \vdash, \perp)$ such that the operations of $(T, \dashv, \vdash, \perp)/\rho$ coincide and $(T, \dashv, \vdash, \perp)/\rho$ is a rectangular band, we say that ρ is a *rectangular band congruence* [40].

Following Loday [10], a *dimonoid* is a nonempty set equipped with two binary associative operations $\dashv$ and $\vdash$ satisfying the axioms $(T1)$–$(T3)$ of a trioid. We remark that a dialgebra (see, e.g., [4, 9, 10, 12]) is just a linear analog of a dimonoid. If ρ is a congruence on a trioid $(T, \dashv, \vdash, \perp)$ such that two operations of $(T, \dashv, \vdash, \perp)/\rho$ coincide and $(T, \dashv, \vdash, \perp)/\rho$ is a dimonoid, we say that ρ is a *dimonoid congruence* [18]. A dimonoid congruence ρ on a trioid $(T, \dashv, \vdash, \perp)$ is called a $d^{\perp}_{\dashv}$*-congruence* (respectively, $d^{\perp}_{\vdash}$*-congruence*) [18] if the operations $\dashv$ and $\perp$ (respectively, $\vdash$ and $\perp$) of $(T, \dashv, \vdash, \perp)/\rho$ coincide.

We will consistently use the following symbolism.

Let ω be an arbitrary nonempty word over the alphabet X and $w \in Frt(X)$.

$\omega^{(0)}$: the first letter of ω.

$\omega^{(1)}$: the last letter of ω.

θ: the empty word.

$d_x(\omega)$: the number of occurrences of the element x $(x \in X)$ in ω.

Suppose that u is the initial subword of w with the minimal length such that $u^{(1)} \in \overline{X}$. In this case,

$w^{[0]}$: the letter $\widetilde{u^{(1)}}$.

$w[0]$: the word obtained from w by deleting $u^{(1)}$. We will regard $w[0] = \theta$ if $w \in \overline{X}$.

Let u be the terminal subword of w with the minimal length such that $u^{(0)} \in \overline{X}$. Then

$w^{[1]}$: the letter $\widetilde{u^{(0)}}$.

$w[1]$: the word obtained from w by deleting $u^{(0)}$. By definition, $w[1] = \theta$ if $w \in \overline{X}$.

We will regard $\widetilde{\omega} = \omega$ and $\widetilde{\theta} = \theta$.

Let $T = (T, \dashv, \vdash)$ be a dimonoid. Then

$(T)^{\dashv}$: the trioid $(T, \dashv, \vdash, \dashv)$.

$(T)^{\vdash}$: the trioid $(T, \dashv, \vdash, \vdash)$.

Clearly, $(T)^{\dashv}$ and $(T)^{\vdash}$ are distinct as trioids while they coincide as dimonoids.

Let $f : T_1 \to T_2$ be a homomorphism of trioids. In this case,

Δ_f: the kernel of f, that is, $\Delta_f = \{(x, y) \in T_1 \times T_1 \mid xf = yf\}$.

The following theorem characterizes the least abelian dimonoid congruences on $Frt(X)$.

Theorem 3.1. *Let $Frt(X)$ be the free trioid and $w, u \in Frt(X)$.*

(i) Define a relation $\widetilde{\alpha_{\dashv}^{\perp}}$ on $Frt(X)$ by $w\widetilde{\alpha_{\dashv}^{\perp}}u$ if and only if

$$(w^{[0]}, d_x(\widetilde{w[0]})) = (u^{[0]}, d_x(\widetilde{u[0]}))$$

for all $x \in X$. Then $\widetilde{\alpha_{\dashv}^{\perp}}$ is the least abelian $d_{\dashv}^{\perp}$-congruence on $Frt(X)$.

(ii) Define a relation $\widetilde{\alpha_{\vdash}^{\perp}}$ on $Frt(X)$ by $w\widetilde{\alpha_{\vdash}^{\perp}}u$ if and only if

$$(w^{[1]}, d_x(\widetilde{w[1]})) = (u^{[1]}, d_x(\widetilde{u[1]}))$$

for all $x \in X$. Then $\widetilde{\alpha_{\vdash}^{\perp}}$ is the least abelian $d_{\vdash}^{\perp}$-congruence on $Frt(X)$.

Proof. We denote the free abelian dimonoid $(FAd(X), \dashv, \vdash)$ given in [43] by $FAd[X]$.

(i) Define a map $\alpha_{\dashv}^{\perp} : Frt(X) \to (FAd[X])^{\dashv}$ by $w\alpha_{\dashv}^{\perp} = (w^{[0]}, \widetilde{w[0]})$. Our aim is to show that $\alpha_{\dashv}^{\perp}$ is an epimorphism. We have

$$\begin{aligned}
&(w \dashv u)\alpha_{\dashv}^{\perp} = (w\widetilde{u})\alpha_{\dashv}^{\perp} = ((w\widetilde{u})^{[0]}, \widetilde{(w\widetilde{u})[0]}) \\
&= (w^{[0]}, \widetilde{w[0]}\widetilde{u}) = (w^{[0]}, \widetilde{w[0]}u^{[0]}\widetilde{u[0]}) \\
&= (w^{[0]}, \widetilde{w[0]}) \dashv (u^{[0]}, \widetilde{u[0]}) = w\alpha_{\dashv}^{\perp} \dashv u\alpha_{\dashv}^{\perp}, \\
&(w \vdash u)\alpha_{\dashv}^{\perp} = (\widetilde{w}u)\alpha_{\dashv}^{\perp} = ((\widetilde{w}u)^{[0]}, \widetilde{(\widetilde{w}u)[0]}) \\
&= (u^{[0]}, \widetilde{w}\widetilde{u[0]}) = (u^{[0]}, w^{[0]}\widetilde{w[0]}\widetilde{u[0]}) \\
&= (w^{[0]}, \widetilde{w[0]}) \vdash (u^{[0]}, \widetilde{u[0]}) = w\alpha_{\dashv}^{\perp} \vdash u\alpha_{\dashv}^{\perp}, \\
&(w \perp u)\alpha_{\dashv}^{\perp} = (wu)\alpha_{\dashv}^{\perp} = ((wu)^{[0]}, \widetilde{(wu)[0]}) \\
&= (w^{[0]}, \widetilde{w[0]})\widetilde{u}) = (w^{[0]}, \widetilde{w[0]}u^{[0]}\widetilde{u[0]}) \\
&= (w^{[0]}, \widetilde{w[0]}) \dashv (u^{[0]}, \widetilde{u[0]}) = w\alpha_{\dashv}^{\perp} \dashv u\alpha_{\dashv}^{\perp}.
\end{aligned}$$

Thus, $\alpha_{\dashv}^{\perp}$ is a homomorphism. Since

$$(\overline{x}\nu)\alpha_{\dashv}^{\perp} = ((\overline{x}\nu)^{[0]}, \widetilde{(\overline{x}\nu)[0]}) = (x, \widetilde{\nu}) = (x, \nu)$$

for any $(x, \nu) \in FAd[X]$, the map $\alpha_{\dashv}^{\perp}$ is surjective. By Theorem 1 from [43], $FAd[X]$ is the free abelian dimonoid, hence $(FAd[X])^{\dashv}$ is the trioid which is free in the variety of abelian trioids with $\dashv = \perp$. It means that $\Delta_{\alpha_{\dashv}^{\perp}}$ is the least abelian $d_{\dashv}^{\perp}$-congruence on $Frt(X)$. From the definition of $\alpha_{\dashv}^{\perp}$ it follows that $w\alpha_{\dashv}^{\perp} = u\alpha_{\dashv}^{\perp}$ if and only if $w^{[0]} = u^{[0]}$ and $\widetilde{w[0]} = \widetilde{u[0]}$ in the free commutative monoid on X. The latter equality is equivalent to $d_x(\widetilde{w[0]}) = d_x(\widetilde{u[0]})$ for all $x \in X$. So, we conclude that $\Delta_{\alpha_{\dashv}^{\perp}} = \widetilde{\alpha_{\dashv}^{\perp}}$.

(ii) Define a map $\alpha_{\vdash}^{\perp} : Frt(X) \to (FAd[X])^{\vdash}$ by $w\alpha_{\vdash}^{\perp} = (w^{[1]}, \widetilde{w[1]})$. We can show that $\alpha_{\vdash}^{\perp}$ is a surjective homomorphism. We get

$$\begin{aligned}
&(w \dashv u)\alpha_{\vdash}^{\perp} = (w\widetilde{u})\alpha_{\vdash}^{\perp} = ((w\widetilde{u})^{[1]}, \widetilde{(w\widetilde{u})[1]}) \\
&= (w^{[1]}, \widetilde{w[1]})\widetilde{u}) = (w^{[1]}, \widetilde{w[1]}u^{[1]}\widetilde{u[1]}) \\
&= (w^{[1]}, \widetilde{w[1]}) \dashv (u^{[1]}, \widetilde{u[1]}) = w\alpha_{\vdash}^{\perp} \dashv u\alpha_{\vdash}^{\perp}, \\
&(w \vdash u)\alpha_{\vdash}^{\perp} = (\widetilde{w}u)\alpha_{\vdash}^{\perp} = ((\widetilde{w}u)^{[1]}, \widetilde{(\widetilde{w}u)[1]}) \\
&= (u^{[1]}, \widetilde{\widetilde{w}u[1]}) = (u^{[1]}, w^{[1]}\widetilde{w[1]}\widetilde{u[1]}) \\
&= (w^{[1]}, \widetilde{w[1]}) \vdash (u^{[1]}, \widetilde{u[1]}) = w\alpha_{\vdash}^{\perp} \vdash u\alpha_{\vdash}^{\perp}, \\
&(w \perp u)\alpha_{\vdash}^{\perp} = (wu)\alpha_{\vdash}^{\perp} = ((wu)^{[1]}, \widetilde{(wu)[1]}) \\
&= (u^{[1]}, \widetilde{\widetilde{w}u[1]}) = (u^{[1]}, w^{[1]}\widetilde{w[1]}\widetilde{u[1]}) \\
&= (w^{[1]}, \widetilde{w[1]}) \vdash (u^{[1]}, \widetilde{u[1]}) = w\alpha_{\vdash}^{\perp} \vdash u\alpha_{\vdash}^{\perp}.
\end{aligned}$$

So, $\alpha_{\vdash}^{\perp}$ is a homomorphism. As above, the map $\alpha_{\vdash}^{\perp}$ is surjective. According to Theorem 1 from [43], $FAd[X]$ is the free abelian dimonoid. It means that $(FAd[X])^{\vdash}$ is the trioid which is free in the variety of abelian trioids with $\vdash=\perp$, and that is why $\Delta_{\alpha_{\vdash}^{\perp}}$ is the least abelian $d_{\vdash}^{\perp}$-congruence on $Frt(X)$. Similarly to (i), by the construction of $\alpha_{\vdash}^{\perp}$, we have $\Delta_{\alpha_{\vdash}^{\perp}} = \widetilde{\alpha_{\vdash}^{\perp}}$. □

At the end of this section we present the least rectangular triband dimonoid congruences and the least rectangular band congruence on the free trioid as kernels of the corresponding known homomorphisms.

Let $FRct(X)$ be the free rectangular dimonoid suggested in [21] (see also [28]).

Remark 3.1. Define a relation $\zeta_{\dashv}^{\perp}$ on the free trioid $Frt(X)$ by

$$w\zeta_{\dashv}^{\perp}u \quad \text{if and only if} \quad (\widetilde{w}^{(0)}, w^{[0]}, \widetilde{w}^{(1)}) = (\widetilde{u}^{(0)}, u^{[0]}, \widetilde{u}^{(1)}).$$

By Theorem 3.2 (i) from [40], a map

$$\varphi_{FRct}^{\dashv} : Frt(X) \to (FRct(X))^{\dashv} \quad \text{defined by}$$

$$w \mapsto (\widetilde{w}^{(0)}, w^{[0]}, \widetilde{w}^{(1)}),\ w \in Frt(X)$$

is a surjective homomorphism. Since by Theorem 1 from [21] $FRct(X)$ is the free rectangular dimonoid, $(FRct(X))^{\dashv}$ is the trioid which is free in the variety of rectangular trioids with $\dashv = \perp$. Then $\Delta_{\varphi_{FRct}^{\dashv}}$ is the least rectangular triband $d_{\dashv}^{\perp}$-congruence on $Frt(X)$. From the construction of $\varphi_{FRct}^{\dashv}$ it follows that $\Delta_{\varphi_{FRct}^{\dashv}} = \zeta_{\dashv}^{\perp}$. Then $\zeta_{\dashv}^{\perp}$ is the least rectangular triband $d_{\dashv}^{\perp}$-congruence on $Frt(X)$.

Remark 3.2. Define a relation $\zeta_{\vdash}^{\perp}$ on the free trioid $Frt(X)$ by

$$w\zeta_{\vdash}^{\perp}u \quad \text{if and only if} \quad (\widetilde{w}^{(0)}, w^{[1]}, \widetilde{w}^{(1)}) = (\widetilde{u}^{(0)}, u^{[1]}, \widetilde{u}^{(1)}).$$

According to Theorem 3.2 (ii) from [40], a map

$$\varphi_{FRct}^{\vdash} : Frt(X) \to (FRct(X))^{\vdash} \text{ defined by}$$

$$w \mapsto (\widetilde{w}^{(0)}, w^{[1]}, \widetilde{w}^{(1)}),\ w \in Frt(X)$$

is an epimorphism. Due to Theorem 1 from [21], we have that $(FRct(X))^{\vdash}$ is the trioid which is free in the variety of rectangular trioids with $\vdash = \perp$. Hence $\Delta_{\varphi_{FRct}^{\vdash}}$ is the least rectangular triband $d_{\vdash}^{\perp}$-congruence on $Frt(X)$. By definition of $\varphi_{FRct}^{\vdash}$, we have $\Delta_{\varphi_{FRct}^{\vdash}} = \zeta_{\vdash}^{\perp}$. Then $\zeta_{\vdash}^{\perp}$ is the least rectangular triband $d_{\vdash}^{\perp}$-congruence on $Frt(X)$.

Remark 3.3. Define a relation $\widetilde{\varphi_{rb}}$ on the free trioid $Frt(X)$ by

$$w\widetilde{\varphi_{rb}}u \quad \text{if and only if} \quad (\widetilde{w}^{(0)}, \widetilde{w}^{(1)}) = (\widetilde{u}^{(0)}, \widetilde{u}^{(1)}).$$

From Corollary 1 of [40] it follows that $\widetilde{\varphi_{rb}}$ is the least rectangular band congruence on $Frt(X)$.

The least commutative semigroup congruence on the free trioid $FT(X)$ was characterized in [18].

4. The least n-nilpotent dimonoid congruences on the free trioid

In this section, we characterize the least n-nilpotent dimonoid congruences and the least n-nilpotent semigroup congruence on the free trioid. We will use notations of Sections 2 and 3.

We only recall the basic definitions.

A semigroup S is called *nilpotent* if $S^{n+1} = 0$ for some $n \in \mathbb{N}$. The least such n is called the *nilpotency index* of S. For $k \in \mathbb{N}$ a nilpotent semigroup of nilpotency index $\leq k$ is called *k-nilpotent*. If ρ is a congruence on a trioid $(T, \dashv, \vdash, \perp)$ such that the operations of $(T, \dashv, \vdash, \perp)/\rho$ coincide and $(T, \dashv, \vdash, \perp)/\rho$ is an n-nilpotent semigroup, we say that ρ is an *n-nilpotent semigroup congruence*.

An element 0 of a trioid $(T, \dashv, \vdash, \perp)$ is called *zero* [32] if $x * 0 = 0 = 0 * x$ for all $x \in T$ and $* \in \{\dashv, \vdash, \perp\}$. A trioid $(T, \dashv, \vdash, \perp)$ with zero is called *nilpotent* if for some $n \in \mathbb{N}$ and any $x_i \in T$, $1 \leq i \leq n+1$, and $*_j \in \{\dashv, \vdash, \perp\}$, $1 \leq j \leq n$, any parenthesizing of

$$x_1 *_1 x_2 *_2 \ldots *_n x_{n+1}$$

gives $0 \in T$. The least such n is called the *nilpotency index* of $(T, \dashv, \vdash, \perp)$. For $k \in \mathbb{N}$ a nilpotent trioid of nilpotency index $\leq k$ is said to be *k-nilpotent* [41]. If ρ is a congruence on a trioid $(T, \dashv, \vdash, \perp)$ such that $(T, \dashv, \vdash, \perp)/\rho$ is an n-nilpotent trioid, we say that ρ is an *n-nilpotent congruence* [41].

If $y_1 y_2 \ldots y_m \in Frt(X)$, where $y_i \in X \cup \overline{X}, 1 \leq i \leq m$, then the least (greatest) number $i \in \{1, 2, \ldots, m\}$ such that $y_i \in \overline{X}$ will be denoted by $min(y_1 y_2 \ldots y_m)$ ($max(y_1 y_2 \ldots y_m)$). For $L \subseteq \{1, 2, \ldots, n\}$, $L \neq \varnothing$, denote the least (greatest) number of L by L_{min} (L_{max}).

Now we can formulate the main result of this section.

Theorem 4.1. *Let $Frt(X)$ be the free trioid, $n \in \mathbb{N}$ and $w, u \in Frt(X)$.*

(i) Define a relation $\beta_{\dashv}^{\perp}$ on $Frt(X)$ by $w \beta_{\dashv}^{\perp} u$ if and only if

$$(\widetilde{w}, min(w)) = (\widetilde{u}, min(u)), \quad or \quad \ell_w > n \quad and \quad \ell_u > n.$$

Then $\beta_{\dashv}^{\perp}$ is the least n-nilpotent $d_{\dashv}^{\perp}$-congruence on $Frt(X)$.

(ii) Define a relation $\beta_{\vdash}^{\perp}$ on $Frt(X)$ by $w \beta_{\vdash}^{\perp} u$ if and only if

$$(\widetilde{w}, max(w)) = (\widetilde{u}, max(u)), \quad or \quad \ell_w > n \quad and \quad \ell_u > n.$$

Then $\beta_{\vdash}^{\perp}$ is the least n-nilpotent $d_{\vdash}^{\perp}$-congruence on $Frt(X)$.

(iii) Define a relation β on $Frt(X)$ by $w \beta u$ if and only if

$$\widetilde{w} = \widetilde{u}, \quad or \quad \ell_w > n \quad and \quad \ell_u > n.$$

Then β is the least n-nilpotent semigroup congruence on $Frt(X)$.

Proof. Let $P_n^0(X)$ and $FNT_n(X)$ be free n-nilpotent trioids from [41] and [30], respectively, let $FN_n(X)$ be the free n-nilpotent dimonoid [20] and $FNS_n(X)$ the free n-nilpotent semigroup [30]. Define a map

$$\xi : Frt(X) \to P_n^0(X) \quad \text{by}$$

$$w\xi = \begin{cases} w, & \ell_w \leq n, \\ 0, & \ell_w > n \end{cases} \qquad (w \in Frt(X)).$$

By Theorem 5 from [41], ξ is a surjective homomorphism. According to Theorem 3 from [30], $P_n^0(X) \cong FNT_n(X)$ and the corresponding isomorphism is defined by the rule

$$\varphi : P_n^0(X) \to FNT_n(X), \quad h \mapsto h\varphi$$

$$= \begin{cases} (\widetilde{h}, V_h), \quad h = a_1a_2...a_k \in P_n, \ V_h \subseteq \{1, 2, \ldots, k\} \quad \text{and} \\ \qquad\qquad\qquad\qquad\qquad\qquad j \in V_h \Leftrightarrow a_j \in \overline{X}, \\ 0, \quad h = 0. \end{cases}$$

(i) Consider a map $\kappa_{\dashv}^{\perp} : FNT_n(X) \to (FN_n(X))^{\dashv}$ defined by

$$v\kappa_{\dashv}^{\perp} = \begin{cases} (w', L_{min}), & v = (w', L), \\ 0, & v = 0. \end{cases}$$

This map is an epimorphism by Theorem 4 (i) of [30]. Since ξ and $\kappa_{\dashv}^{\perp}$ are surjective homomorphisms and φ is an isomorphism, the composition $\xi \circ \varphi \circ \kappa_{\dashv}^{\perp}$ is a surjective homomorphism of $Frt(X)$ onto $(FN_n(X))^{\dashv}$. One can check that $\xi \circ \varphi \circ \kappa_{\dashv}^{\perp}$ is defined by

$$w \mapsto \begin{cases} (\widetilde{w}, L_{min}), & \ell_w \leq n, \\ 0, & \ell_w > n \end{cases} \qquad (w \in Frt(X)).$$

We have that $P_n^0(X)$ and $FNT_n(X)$ are isomorphic free n-nilpotent trioids, and Theorem 1 of [20] implies that $(FN_n(X))^{\dashv}$ is the trioid which is free in the variety of n-nilpotent trioids with $\dashv = \perp$. Using these facts, we get that $\Delta_{\xi \circ \varphi \circ \kappa_{\dashv}^{\perp}}$ is the least n-nilpotent $d_{\dashv}^{\perp}$-congruence on $Frt(X)$. From the construction of $\xi \circ \varphi \circ \kappa_{\dashv}^{\perp}$ it follows that $\Delta_{\xi \circ \varphi \circ \kappa_{\dashv}^{\perp}} = \beta_{\dashv}^{\perp}$.

(ii) By Theorem 4 (ii) from [30], a map $\kappa_{\vdash}^{\perp} : FNT_n(X) \to (FN_n(X))^{\vdash}$ defined by

$$v\kappa_{\vdash}^{\perp} = \begin{cases} (w', L_{max}), & v = (w', L), \\ 0, & v = 0 \end{cases}$$

is a surjective homomorphism. The composition $\xi \circ \varphi \circ \kappa_{\vdash}^{\perp}$ is a surjective homomorphism of $Frt(X)$ onto $(FN_n(X))^{\vdash}$ since ξ, $\kappa_{\vdash}^{\perp}$ are surjective homomorphisms and φ is an isomorphism. We can show that $\xi \circ \varphi \circ \kappa_{\vdash}^{\perp}$ is defined by

$$w \mapsto \begin{cases} (\widetilde{w}, L_{max}), & \ell_w \leq n, \\ 0, & \ell_w > n \end{cases} \qquad (w \in Frt(X)).$$

As above, using Theorem 1 of [20] and Theorem 3 of [30], we deduce that $\Delta_{\xi\circ\varphi\circ\kappa_{\vdash}^{\perp}}$ is the least n-nilpotent $d_{\vdash}^{\perp}$-congruence on $Frt(X)$. By definition of $\xi \circ \varphi \circ \kappa_{\vdash}^{\perp}$ we get $\Delta_{\xi\circ\varphi\circ\kappa_{\vdash}^{\perp}} = \beta_{\vdash}^{\perp}$.

(iii) Define a map $\kappa : FNT_n(X) \to FNS_n(X)$ by

$$v\kappa = \begin{cases} w', & v = (w', L), \\ 0, & v = 0. \end{cases}$$

According to Theorem 4 (iii) of [30], κ is a surjective homomorphism. It can be shown that the composition $\xi \circ \varphi \circ \kappa$ of the surjective homomorphism ξ, the isomorphism φ and the surjective homomorphism κ is a surjective homomorphism of $Frt(X)$ onto $FNS_n(X)$ which is defined by

$$w \mapsto \begin{cases} \widetilde{w}, & \ell_w \leq n, \\ 0, & \ell_w > n \end{cases} \qquad (w \in Frt(X)).$$

As above, $P_n^0(X)$ and $FNT_n(X)$ are isomorphic free n-nilpotent trioids. Using it and the fact that $FNS_n(X)$ is the free n-nilpotent semigroup, we deduce that $\Delta_{\xi\circ\varphi\circ\kappa}$ is the least n-nilpotent semigroup congruence on $Frt(X)$. By the construction of $\xi \circ \varphi \circ \kappa$, we have $\Delta_{\xi\circ\varphi\circ\kappa} = \beta$. ☐

5. The least left n-trinilpotent dimonoid congruence on the free trioid

In this section, we characterize the least left n-trinilpotent dimonoid congruence and the least left n-nilpotent semigroup congruence on the free trioid. We will use notations of Sections 2–4.

Following Schein [13], a semigroup S is called a *left (right) nilpotent semigroup of rank m* if the product of any m elements from this semigroup gives a left (right) zero. The class of all left nilpotent semigroups of rank m is characterized by the identity $g_1g_2 \dots g_mg_{m+1} = g_1g_2 \dots g_m$. The least such m is called the *left nilpotency index* of a semigroup S [37]. Following [37], for $k \in \mathbb{N}$ a left nilpotent semigroup of left nilpotency index $\leq k$ is called a *left k-nilpotent semigroup*. Right k-nilpotent semigroups are defined dually [37]. If ρ is a congruence on a trioid $(T, \dashv, \vdash, \perp)$ such that the operations of $(T, \dashv, \vdash, \perp)/\rho$ coincide and $(T, \dashv, \vdash, \perp)/\rho$ is a left (right) n-nilpotent semigroup, we say that ρ is a *left (right) n-nilpotent semigroup congruence*.

Recall the definition of a left k-trinilpotent trioid [33].

By $\Omega^{'}$ denote the signature of a trioid. Let $a_1, \ldots, a_n$ be individual variables. By $P(a_1, \ldots, a_n)$ we will denote the set of all terms of the signature $\Omega^{'}$ having the form $a_1 \circ_1 \ldots \circ_{n-1} a_n$ with parenthesizing, where $\circ_1, \ldots, \circ_{n-1} \in \Omega^{'}$. A trioid $(T, \dashv, \vdash, \perp)$ is called *left trinilpotent* if for some $n \in \mathbb{N}$, any $a \in T$ and any $p(a_1, \ldots, a_n) \in P(a_1, \ldots, a_n)$ the following identities hold:

$$p(a_1, \ldots, a_n) * a = p(a_1, \ldots, a_n),$$

$$p(a_1, \ldots, a_n) \vdash a = a_1 \vdash \ldots \vdash a_n,$$

where $* \in \{\dashv, \perp\}$. The least such n is called the *left trinilpotency index* of $(T, \dashv, \vdash, \perp)$. For $k \in \mathbb{N}$ a left trinilpotent trioid of left trinilpotency index $\leq k$ is said to be *left k-trinilpotent*. Right k-trinilpotent trioids are defined dually [33]. If ρ is a congruence on a trioid $(T, \dashv, \vdash, \perp)$ such that $(T, \dashv, \vdash, \perp)/\rho$ is a left (right) n-trinilpotent trioid, we say that ρ is a *left (right) n-trinilpotent congruence* [34].

Let $w' \in F[X]$ and $n \in \mathbb{N}$. Following [33], if $\ell_{w'} \geq n$, by $\overset{n}{\overrightarrow{w'}}$ we denote the initial subword with the length n of w', and if $\ell_{w'} < n$, let $\overset{n}{\overrightarrow{w'}} = w'$. Following [34], let

$$L^{(n)} = \begin{cases} \{n\}, & n \leq L_{min}, \\ \{m \in L \mid m \leq n\}, & n > L_{min} \end{cases}$$

for any nonempty $L \subseteq \{1, 2, \ldots, k\}$ and $k \in \mathbb{N}$.

The main result of this section is the following theorem.

Theorem 5.1. *Let $FT(X)$ be the free trioid, $n \in \mathbb{N}$ and (w_1, L), $(w_2, R) \in FT(X)$.*

(i) Define a relation $\gamma_{\dashv}^{\perp}$ on $FT(X)$ by

$$(w_1, L)\gamma_{\dashv}^{\perp}(w_2, R) \quad \textit{if and only if}$$

$$\overset{n}{\overrightarrow{w}}_1 = \overset{n}{\overrightarrow{w}}_2 \quad \textit{and} \quad (L^{(n)})_{min} = (R^{(n)})_{min}.$$

Then $\gamma_{\dashv}^{\perp}$ *is the least left* n*-trinilpotent* $d_{\dashv}^{\perp}$*-congruence on* $FT(X)$*.*

(ii) Define a relation γ *on* $FT(X)$ *by*

$$(w_1, L)\gamma(w_2, R) \quad \textit{if and only if} \quad \overset{n}{\overrightarrow{w_1}} = \overset{n}{\overrightarrow{w_2}} .$$

Then γ *is the least left* n*-nilpotent semigroup congruence on* $FT(X)$*.*

Proof. Consider the free left n-trinilpotent trioid $FT_n^l(X)$ [33], the free left n-dinilpotent dimonoid $FD_n^l(X)$ [23, 36] and the free left n-nilpotent semigroup $FLNS_n(X)$ [37]. By the Main Theorem of [34], a map

$$d_n : FT(X) \to FT_n^l(X) \quad \text{defined by}$$

$$(w_1, L) \mapsto (w_1, L)d_n = (\overset{n}{\overrightarrow{w_1}}, L^{(n)})$$

is a surjective homomorphism.

(i) Define a map

$$\mu_{\dashv}^{\perp} : FT_n^l(X) \to (FD_n^l(X))^{\dashv} \quad \text{by} \quad (w_1, L)\mu_{\dashv}^{\perp} = (w_1, L_{min}).$$

By Theorem 3 (i) of [31], $\mu_{\dashv}^{\perp}$ is an epimorphism. Since d_n and $\mu_{\dashv}^{\perp}$ are surjective homomorphisms, the composition $d_n \circ \mu_{\dashv}^{\perp}$ is a surjective homomorphism of $FT(X)$ onto $(FD_n^l(X))^{\dashv}$. One can guess that $d_n \circ \mu_{\dashv}^{\perp}$ is defined by

$$(w_1, L) \mapsto (\overset{n}{\overrightarrow{w_1}}, (L^{(n)})_{min}), (w_1, L) \in FT(X).$$

Since by Theorem 3.4 from [36] $FD_n^l(X)$ is the free left n-dinilpotent dimonoid, $(FD_n^l(X))^{\dashv}$ is the trioid which is free in the variety of left n-trinilpotent trioids with $\dashv = \perp$. Then $\Delta_{d_n \circ \mu_{\dashv}^{\perp}}$ is the least left n-trinilpotent $d_{\dashv}^{\perp}$-congruence on $FT(X)$. From the construction of $d_n \circ \mu_{\dashv}^{\perp}$ it follows that $\Delta_{d_n \circ \mu_{\dashv}^{\perp}} = \gamma_{\dashv}^{\perp}$.

(ii) By Theorem 3 (ii) from [31], a map $\mu : FT_n^l(X) \to FLNS_n(X)$ defined by $(w_1, L)\mu = w_1$ is a surjective homomorphism. The composition $d_n \circ \mu$ is a surjective homomorphism of $FT(X)$ onto $FLNS_n(X)$ since d_n and μ are surjective homomorphisms. One can show that $d_n \circ \mu$ is defined by

$$(w_1, L) \mapsto \overset{n}{\overrightarrow{w_1}}, (w_1, L) \in FT(X).$$

According to Lemma 3.2 from [37], $FLNS_n(X)$ is the free left n-nilpotent semigroup. Then $\Delta_{d_n \circ \mu}$ is the least left n-nilpotent semigroup congruence on $FT(X)$. By definition of $d_n \circ \mu$, we have $\Delta_{d_n \circ \mu} = \gamma$. □

Remark 5.1. By Remark 2 of [31], $(FD_n^l(X))^{\vdash}$ is not a left n-trinilpotent trioid. It means that we cannot characterize the least left n-trinilpotent $d_{\vdash}^{\perp}$-congruence on the free trioid with the help of a homomorphism from $FT_n^l(X)$ to $(FD_n^l(X))^{\vdash}$.

Remark 5.2. In order to characterize the least right n-trinilpotent dimonoid congruence and the least right n-nilpotent semigroup congruence on the free trioid we use the duality principle.

Since trialgebras are linear analogs of trioids, the main results of this paper can be useful for the description of congruences on trialgebras.

Acknowledgments

The present paper was written during the research stay of the author at the Institute of Mathematics of the University of Potsdam (Germany) within the framework of the Philipp Schwartz Initiative program of the Alexander von Humboldt Foundation. The author would like to thank Dr. Jörg Koppitz for inviting to the Institute of Mathematics to work on this project and the University of Potsdam for hospitality.

References

[1] Bagherzadeha, F., Bremnera, M., Madariagab, S.: Jordan trialgebras and post-Jordan algebras. J. Algebra. **486**, 360–395 (2017). https://doi.org/10.1016/j.jalgebra.2017.04.022

[2] Bergman, C., Berman, J.: Fully invariant and verbal congruence relations. Algebra Univers. **70**, 71–94 (2013). https://doi.org/10.1007/s00012-013-0238-z

[3] Bock, W., Futorny, V., Neklyudov, M.: A. Poisson algebra on the Hida Test functions and a quantization using the Cuntz algebra. Lett. Math. Phys. **112**, 24 (2022). https://doi.org/10.1007/s11005-022-01507-4

[4] Bokut, L.A., Chen, Y., Liu, C.: Gröbner–Shirshov bases for dialgebras. Int. J. Algebra Comput. **20**(3), 391–415 (2010). https://doi.org/10.1142/S0218196710005753

[5] Casas, J.M.: Trialgebras and Leibniz 3-algebras. Bol. Soc. Mat. Mex. **12**(2), 165–178 (2006).

[6] Huang, J., Chen, Y., Zhang, Z.: Free products of trialgebras. In https://doi.org/10.48550/arXiv.2107.00459

[7] Huang, J., Bai, Y., Chen, Y. et al.: Constructions of free dibands and tribands. Semigroup Forum. **104**, 647–666 (2022). https://doi.org/10.1007/s00233-022-10270-w

[8] Huang, J., Chen, Y.: Gröbner–Shirshov bases theory for trialgebras. Mathematics. **9**, 1207 (2021). https://doi.org/10.3390/math9111207

[9] Kolesnikov, P.S., Voronin, V.Yu.: On special identities for dialgebras. Linear Multilinear Algebra. **61**(3), 377–391 (2013). https://doi.org/10.1080/03081087.2012.686108

[10] Loday, J.-L.: Dialgebras. In: Dialgebras and related operads. Lect. Notes Math. Springer-Verlag, Berlin. **1763**, 7–66 (2001).

[11] Loday, J.-L., Ronco, M.O.: Trialgebras and families of polytopes. Contemp. Math. **346**, 369–398 (2004). https://doi.org/10.1090/conm/346/06296

[12] Sanchez-Ortega, J.: On the definitions of nucleus for dialgebras. J. Algebra. **392**(15), 244–264 (2013). https://doi.org/10.1016/j.jalgebra.2013.06.022

[13] Schein, B.M.: One-sided nilpotent semigroups. Uspekhi Mat. Nauk. **19**:1(115), 187–189 (1964) (in Russian).

[14] Smith, J.D.H.: Cayley theorems for Loday algebras. Results Math. **77**, 218 (2022). https://doi.org/10.1007/s00025-022-01748-8

[15] Yau, D.: Gerstenhaber structure and Deligne's conjecture for Loday algebras. J. Pure Applied Algebra. **209**(3), 739–752 (2007). https://doi.org/10.1016/j.jpaa.2006.07.003

[16] Zhang, G., Chen, Y.: A construction of the free digroup. Semigroup Forum. **102**, 553–567 (2021). https://doi.org/10.1007/s00233-021-10161-6

[17] Zhang, G., Chen, Y., Zhang, Z.: Free products of digroups. Algebra Univers. **83**(3) (2022). https://doi.org/10.1007/s00012-021-00761-2

[18] Zhuchok, A.V.: Free commutative trioids. Semigroup Forum. **98**(2), 355–368 (2019). https://doi.org/10.1007/s00233-019-09995-y

[19] Zhuchok, A.V.: Free left n-dinilpotent doppelsemigroups. Commun. Algebra. **45**(11), 4960–4970 (2017). https://doi.org/10.1080/00927872.2017.1287274

[20] Zhuchok, A.V.: Free n-nilpotent dimonoids. Algebra Discrete Math. **16**(2), 299–310 (2013).

[21] Zhuchok, A.V.: Free rectangular dibands and free dimonoids. Algebra Discrete Math. **11**(2), 92–111 (2011).

[22] Zhuchok, A.V.: Group congruences on trioids. Proc. Inst. Applied Math. and Mech. **27**, 148–158 (2013) (in Ukrainian).

[23] Zhuchok, A.V.: Relatively free dimonoids and bar-units. Int. J. Algebra Comput. **31**(08), 1587–1599 (2021). https://doi.org/10.1142/S0218196721500570

[24] Zhuchok, A.V.: Semilattice decompositions of trioids. Bul. Acad. Stiinte Repub. Mold. Mat. **71**(1), 130–134 (2013).

[25] Zhuchok, A.V.: Semilatties of subdimonoids. Asian-Eur. J. Math. **4**(2), 359–371 (2011). https://doi.org/10.1142/S1793557111000290

[26] Zhuchok, A.V.: Some congruences on trioids. J. Math. Sci. **187**(2), 138–145 (2012). https://doi.org/10.1007/s10958-012-1056-0

[27] Zhuchok, A.V.: Structure of free strong doppelsemigroups. Commun. Algebra. **46**(8), 3262–3279 (2018). https://doi.org/10.1080/00927872.2017.1407422

[28] Zhuchok, A.V.: Structure of relatively free dimonoids. Commun. Algebra. **45**(4), 1639–1656 (2017). https://doi.org/10.1080/00927872.2016.1222404

[29] Zhuchok, A.V.: Structure of relatively free trioids. Algebra Discrete Math. **31**(1), 152–166 (2021). https://doi.org/10.12958/adm1732

[30] Zhuchok, A.V.: The least dimonoid congruences on free n-nilpotent trioids. Lobachevskii J. Math. **41**(9), 1747–1753 (2020). https://doi.org/10.1134/S199508022009036X

[31] Zhuchok, A.V.: The least dimonoid congruences on relatively free trioids. Mat. Stud. **57**(1), 23–31 (2022). https://doi.org/10.30970/ms.57.1.23-31

[32] Zhuchok, A.V.: Trioids. Asian-European J. Math. **8**(4), 1550089 (23 p.), (2015). https://doi.org/10.1142/S1793557115500898

[33] Zhuchok, A.V., Kryklia, Y.A.: Free left n-trinilpotent trioids. Commun. Algebra. **49**(2), 467–481 (2021). https://doi.org/10.1080/00927872.2020.1802472

[34] Zhuchok, A.V., Kryklia, Y.A.: The least left n-trinilpotent congruence on the free trioid. Algebra Univers. **83**(4) (2022). https://doi.org/10.1007/s00012-021-00758-x

[35] Zhuchok, A.V., Pilz, G.F.: New models for some free algebras of small ranks. Carpathian Math. Publ. **15**(1), 295–305 (2023). https://doi.org/10.15330/cmp.15.1.295-305

[36] Zhuchok, A.V., Zhuchok, Yul.V.: Free left n-dinilpotent dimonoids. Semigroup Forum. **93**(1), 161–179 (2016). https://doi.org/10.1007/s00233-015-9743-z

[37] Zhuchok, A.V., Zhuchok, Yul.V., Odintsova, O.O.: Free left k-nilpotent n-tuple semigroups. Bul. Acad. Stiinte Repub. Mold. Mat. **94**(3), 29–38 (2020).

[38] Zhuchok, A.V., Zhuchok, Yul.V., Zhuchok, Yu.V.: Certain congruences on free trioids. Commun. Algebra. **47**(12), 5471–5481 (2019). https://doi.org/10.1080/00927872.2019.1631322

[39] Zhuchok, A.V., Zhuchok, Yu.V.: On two classes of digroups. Sao Paulo J. Math. Sci. **11**(1), 240–252 (2017). https://doi.org/10.1007/s40863-016-0038-4

[40] Zhuchok, Yul.V.: Decompositions of free trioids. Bulletin of Taras Shevchenko National University of Kyiv. Series: Physics and Mathematics. **4**, 28–34 (2014).

[41] Zhuchok, Yul.V.: Free n-nilpotent trioids. Mat. Stud. **43**(1), 3–11 (2015). https://doi.org/10.15330/ms.43.1.3-11

[42] Zhuchok, Yul.V.: Free rectangular tribands. Bul. Acad. Stiinte Repub. Mold. Mat. **78**(2), 61–73 (2015).

[43] Zhuchok, Yu.V.: Free abelian dimonoids. Algebra Discrete Math. **20**(2), 330–342 (2015).

[44] Zhuchok, Yu.V.: Free abelian trioids. Algebra Discrete Math. **32**(1), 147–160 (2021). https://doi.org/10.12958/adm1860

[45] Zhuchok, Yu.V., Koppitz, J.: Representations of ordered doppelsemigroups by binary relations. Algebra Discrete Math. **27**(1), 144–154 (2019).

Contact information

A. V. Zhuchok

Department of Mathematics and Informatics, Luhansk Taras Shevchenko National University, Ivan Bank Street, 3, Poltava, 36014, Ukraine;
Institute of Mathematics, University of Potsdam, Karl-Liebknecht-Strasse 24-25, Potsdam, D-14476, Germany
E-Mail: zhuchok.av@gmail.com

Received by the editors: 15.05.2024.