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On a variation of $\oplus$-supplemented modules

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Abstract. Let R be a ring and M be an R-module. M is called $\oplus_{ss}$*-supplemented* if every submodule of M has a ss-supplement that is a direct summand of M. In this paper, the basic properties and characterizations of $\oplus_{ss}$-supplemented modules are provided. In particular, it is shown that (1) if a module M is $\oplus_{ss}$-supplemented, then $Rad(M)$ is semisimple and $Soc(M) \trianglelefteq M$; (2) every direct sum of ss-lifting modules is $\oplus_{ss}$-supplemented; (3) a commutative ring R is an artinian serial ring with semisimple radical if and only if every left R-module is $\oplus_{ss}$-supplemented.

Introduction

In homological algebra, semisimple modules and the varieties of supplemented modules, which are generalizations of semisimple modules, have a very important place, and some important characterizations of ring classes are given in terms of homological algebra via these modules. For example, a ring R is semisimple if and only if every left (right) R-module is semisimple if and only if every left (right) R-module is injective, that is, every module is a direct summand of its extensions. R is left (semi) perfect if and only if every (finitely generated) left R-module is supplemented if and only if every left R-module is srs(strongly radical supplemented). $\frac{R}{P(R)}$ is left perfect, where $P(R)$ is the sum of all radical left ideals of R if and only if every left R-module is Rad-supplemented. R is semilocal

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if and only if every left R-module is weakly Rad-supplemented, that is, semilocal. R is a left and right artinian serial ring with $Rad(R)^2 = 0$ if and only if every left R-module is lifting if and only if every left R-module is extending. A commutative ring R is artinian serial if and only if every left R-module is $\oplus$-supplemented if and only if every left R-module is Rad-$\oplus$-supplemented if and only if every left R-module is srs$^{\oplus}$. The main purpose of this paper is to develop the concept of $\oplus_{ss}$-supplemented modules as a new type of the class of supplemented modules. We introduce $\oplus_{ss}$-supplemented modules and focus on basic properties of these modules. We show that if a module M is $\oplus_{ss}$-supplemented, then $Rad(M)$ is semisimple and $Soc(M) \trianglelefteq M$. We prove that every direct sum of ss-lifting modules is $\oplus_{ss}$-supplemented. Over a left WV-ring every $\oplus$-supplemented module is $\oplus_{ss}$-supplemented. We also show that a ring R is semiperfect ring with semisimple radical, that is, Soc_s-semiperfect, if and only if every left free R-module is $\oplus_{ss}$-supplemented. In particular, we give a characterization of artinian serial rings using $\oplus_{ss}$-supplemented modules.

1. First section

In this section, we briefly recall the main concepts and results related to types of supplements and variations of supplemented modules. For a better understanding of the topic, we start with some fundamental definitions of module and ring theory presented in books [6], [14], [19] and [28].

Throughout this paper, we consider associative rings with identity, denoted as R, and modules unital left R-modules. Let M be an R-module. We use the notation $U \leq M$ to mean U is a submodule of M. We write $Rad(M)$ and $Soc(M)$ for the radical and the socle of M, respectively (see [28]). A submodule E of M is said to be *essential* in M, denoted as $E \trianglelefteq M$, if $E \cap N \neq 0$ for every nonzero submodule N of M. Dually, a submodule U of M is *small* in M, denoted by the notation $U \ll M$, if $M \neq U + K$ for every proper submodule K of M. A module M is called *hollow* if every proper submodule of M is small in M, and it is called *local* if it is a finitely generated nonzero hollow module.

As a generalization of direct summands, one defines supplement submodules as follows. Let U and V be submodules of a module M. V is called *supplement* of U in M if it is minimal with respect to the property $U + V = M$. In this case, U is said to have a supplement V in M. Equivalently, V is a supplement of U in M if and only if $M = U + V$

and $U \cap V \ll V$. Following [28, 19.3 (4)], a submodule V is called *weak supplement* of U in M if $M = U + V$ and $U \cap V \ll M$. A module M is called *(weakly) supplemented* if every submodule of M has a (weak) supplement in M. It is shown in [28, 42.6 and 43.9] that a ring R is (semi) perfect if and only if every (finitely generated) left R-module is supplemented. As a proper generalization of supplemented modules, srs-modules are introduced in the paper [4]. In the same paper, the characterization of left (semi) perfect rings is given in terms of srs-modules (see [4, Corollary 2.5 and Corollary 2.6]).

Let M be a module. M is called $\oplus$*-supplemented* if every submodule of M has a supplement that is a direct summand of M ([19]). Every hollow module is $\oplus$-supplemented and $\oplus$-supplemented modules are supplemented. It is shown in [17, Corollary 3.13] that a commutative ring R is artinian serial if and only if every left R-module is $\oplus$-supplemented. Over a Dedekind domain, it is proven in [19, Proposition A.7 and Proposition A.8] that every supplemented module is $\oplus$-supplemented. For the basic properties, characterizations and some generalizations of $\oplus$-supplemented modules, we recommend the book [19] and the papers [12,13,17, 22,24,31].

Since $Rad(M)$ is the sum of all small submodules of a module M, Rad-supplement submodules are defined as a generalization of supplement submodules. Let U and V be submodules of a module M with $M = U + V$. V is called *Rad-supplement* of U in M in case $U \cap V \subseteq Rad(V)$ (see [6, 10.14]). M is called *Rad-supplemented* if its submodules have a Rad-supplement in M. It follows from [3, Theorem 6.1] that, for a ring R, $\frac{R}{P(R)}$ is left perfect, where $P(R)$ is the sum of all left ideals I of R such that $I = Rad(I)$ if and only if every left R-module is Rad-supplemented. In [27], a module M is called *Rad-*$\oplus$*-supplemented* if every submodule of M has a Rad-supplement that is a direct summand of M. It is clear that every $\oplus$-supplemented module is Rad-$\oplus$-supplemented. For the concept of Rad-$\oplus$-supplemented, we refer to [10] and [27].

It is well known that a simple submodule of a module M is a direct summand of M or small in M. Following this fact, Zhou and Zhang define the submodule $Soc_s(M)$ as the sum of all simple submodules that are small in M (see [29]).

The following lemma follows from [15, Lemma 2] and we will use it throughout the paper.

Lemma 1. *Let M be a module. Then $Soc_s(M) = Soc(M) \cap Rad(M)$.*

Let X be a module. Since $Soc_s(X) \subseteq Rad(X)$, it is of interest to investigate the analogue of this notion by replacing "$Rad(X)$" with "$Soc_s(X)$". ss-supplement submodules, which are between supplements and direct summands, are defined as a special type of supplements as follows.

Lemma 2 (see [15, Lemma 3]). *Let M be a module and U, V be submodules of M. Then the following statements are equivalent:*

(1) $M = U + V$ and $U \cap V \subseteq Soc_s(V)$.

(2) $M = U + V$, $U \cap V \subseteq Rad(V)$ and $U \cap V$ is semisimple.

(3) $M = U + V$, $U \cap V \ll V$ and $U \cap V$ is semisimple.

As in [15], we say that V is an *ss-supplement* of U in M if the equal conditions in the above lemma are satisfied. A module M is called *ss-supplemented* if every submodule of M has an ss-supplement in M. Every semisimple module is ss-supplemented. The authors give in the same paper the various properties and characterizations of these modules. It follows from [15, Theorem 41] that a ring R is semiperfect with semisimple radical if and only if every left R-module is ss-supplemented.

δ-supplement submodules, δ_{ss}-supplement submodules, sa-supplement submodules, extended S-supplement submodules and wsa-supplement submodules are extensively studied by many authors as varieties of supplement submodules. In a series of articles [7–9, 21, 30], the authors have obtained detailed information about variations of supplement submodules and related rings.

2. $\oplus_{ss}$-supplemented modules

In this section, we define the concept of $\oplus_{ss}$-supplemented modules. Our aim is introduce $\oplus_{ss}$-supplemented modules as a special case of ss-supplemented modules. We provide the various properties of such modules. In particular, we prove that a commutative ring R is an artinian serial ring with semisimple radical if and only if every left R-module is $\oplus_{ss}$-supplemented, and a ring R is Soc_s-semiperfect if and only if every free R-module is $\oplus_{ss}$-supplemented.

Definition 1. *Let R be a ring and M be an R-module. M is called* $\oplus_{ss}$-supplemented *if every submodule of M has an ss-supplement that is a direct summand of M by [16].*

It is clear that every $\oplus_{ss}$-supplemented module is $\oplus$-supplemented. However, usually a $\oplus$-supplemented module does not have to be $\oplus_{ss}$-supplemented. We will now give an example for this below. First we need the following fact. Recall from [15] that a module M is *strongly local* if it is local and its radical is semisimple.

Proposition 1. *Let M be a local module. Then the following statements are equivalent:*

(1) M is strongly local.

(2) M is $\oplus_{ss}$-supplemented.

Proof. (1) $\Rightarrow$ (2) Let U be any proper submodule of M. Since M is a strongly local module, we can write $U \subseteq Rad(M) \subseteq Soc(M)$. Therefore U is semisimple and thus M is an ss-supplement of U in M. Hence M is $\oplus_{ss}$-supplemented.

(2) $\Rightarrow$ (1) Since $\oplus_{ss}$-supplemented modules are ss-supplemented, the proof follows from [15, Proposition 15]. □

Example 1. Let M be the local $\mathbb{Z}$-module $\mathbb{Z}_{p^k}$, for p is any prime integer and $k \geq 3$. Since local modules are $\oplus$-supplemented, M is $\oplus$-supplemented. Note that $Soc_s(\mathbb{Z}_{p^k}) = Soc(\mathbb{Z}_{p^k}) \cong \mathbb{Z}_p$ and $Rad(M) = p\mathbb{Z}_{p^k}$. Hence M is not strongly local and so it is not $\oplus_{ss}$-supplemented by Proposition 1.

In [26], a ring R is called a *left WV-ring* if every simple left R-module is $\frac{R}{I}$-injective, where $\frac{R}{I} \ncong R$ and I is any ideal of R. Clearly left WV-rings are generalizations of V-rings. It is shown in [26, Lemma 6.12] that if a ring R is a left WV-ring, then it is a left V-ring or $Rad(R)$ is a simple left R-module. We will use this fact freely in this article without reference.

Proposition 2. *Let R be a left WV-ring. Then every Rad-$\oplus$-supplemented R-module is $\oplus_{ss}$-supplemented.*

Proof. Let M be a Rad-$\oplus$-supplemented R-module and U be any submodule of M. By the assumption, there exists a direct summand V of M such that $M = U + V$ and $U \cap V \subseteq Rad(V)$. If R is a left V-ring, then $U \cap V \subseteq Rad(V) = 0$ and so U is a direct summand of M. Therefore M is semisimple and then it is trivially $\oplus_{ss}$-supplemented.

Suppose that R is not a left V-ring. Consider the epimorphism $\psi : F \to V$ for some free R-module F. Since R is a left WV-ring, $Rad(R)$ is semisimple and so, by [28, 21.17 (2)], we obtain $Rad(F) = Rad(R)F \subseteq Soc({}_RR)F = Soc(F)$. Thus $Rad(F)$ is trivially a semisimple module. It follows from [26, Corollary 6.8] that $\frac{R}{Rad(R)}$ is a V-ring. So, by [28, 23.7], we can write $Rad(V) = \psi(Rad(F))$. It means that $Rad(V)$ is semisimple as a homomorphic image of the semisimple module $Rad(F)$. Hence V is an ss-supplement of U in M. □

Now, we have the following result:

Corollary 1. *Let R be a left WV-ring. Then*

(1) Every ⊕-supplemented R-module is $\oplus_{ss}$-supplemented.

(2) Every local R-module is $\oplus_{ss}$-supplemented.

(3) Every local R-module is strongly local.

Proof. (1) By Proposition 2.

(2) Let M be any local R-module. Since local modules are ⊕-supplemented, it follows from (1) that M is $\oplus_{ss}$-supplemented.

(3) It follows from (2) and Proposition 1. □

The following theorem shows the different between the class of ⊕-supplemented modules and the class of $\oplus_{ss}$-supplemented modules, and that a nonzero radical module cannot be $\oplus_{ss}$-supplemented.

Theorem 1. *Let M be a $\oplus_{ss}$-supplemented module. Then $Rad(M)$ is semisimple. In particular, $Soc_s(M) = Rad(M)$.*

Proof. Since M is a $\oplus_{ss}$-supplemented module, there exists a decomposition $M = M_1 \oplus M_2$ such that $M = Rad(M) + M_1$, $Rad(M) \cap M_1 \ll M_1$ and $Rad(M) \cap M_1$ is semisimple. According to [28, 41.1 (5)], we can write $Rad(M_1) = Rad(M) \cap M_1$ and so $Rad(M_1)$ is semisimple. Note that, by [28, 21.6 (5)], $Rad(M) = Rad(M_1) \oplus Rad(M_2)$. Therefore

$$\begin{aligned} M &= Rad(M) + M_1 \\ &= Rad(M_1) \oplus Rad(M_2) + M_1 \\ &= M_1 \oplus Rad(M_2) \end{aligned}$$

and thus $M_2 = M_2 \cap M = M_2 \cap (M_1 \oplus Rad(M_2)) = Rad(M_2)$ by modularity law. It follows from [15, Proposition 26] that M_2 is a ss-supplemented as a factor module of M. Since $M_2 = Rad(M_2)$, by [15, Proposition 16], we obtain that $M_2 = 0$. Hence $Rad(M) = Rad(M_1)$ is semisimple. □

A module M is called *lifting* if there is a decomposition $M = M_1 \oplus M_2$ such that $M_1 \leq U$ and $U \cap M_2 \ll M_2$ for every submodule U of M. The equivalence of M being lifting is given by [28, 41.11 and 41.15] in the form of M is amply supplemented and every supplement submodule of M is a direct summand of M. Following [11], a module M is called *ss-lifting* if for every submodule U of M, there is a decomposition $M = M_1 \oplus M_2$ such that $M_1 \leq U$ and $U \cap M_2 \subseteq Soc_s(M)$. Every ss-lifting module is $\oplus_{ss}$-supplemented and lifting. It is shown in [11, Theorem 2] that every π-projective and ss-supplemented module is ss-lifting .

As a result of Theorem 1 we obtain the following result.

Corollary 2. *If a module M is ss-lifting, then $Rad(M)$ is semisimple.*

Proof. Since ss-lifting modules are $\oplus_{ss}$-supplemented, the proof follows from Theorem 1. □

We remove the small radical condition in [11, Theorem 4] by using Corollary 2 in the following theorem.

Theorem 2. *Let M be a module. Then M is ss-lifting if and only if it is a lifting module with semisimple radical.*

Proof. (⇒) By Corollary 2, $Rad(M)$ is semisimple. This completes the proof.

(⇐) Let U be any submodule of M. Since M is lifting, there is a decomposition $M = U' \oplus V$ such that $U' \leq U$ and $U \cap V$ is a small submodule of V. It follows that $U \cap V \subseteq Rad(V) \subseteq Rad(M) \subseteq Soc(M)$. This implies $U \cap V \subseteq Soc_s(M)$. It means that M is ss-lifting. □

It is well known that $Soc(M)$ is the intersection of all essential submodules of a module M.

Theorem 3. *Let M be a $\oplus_{ss}$-supplemented module. Then $Soc(M) \trianglelefteq M$.*

Proof. Since M is a $\oplus_{ss}$-supplemented module, by [6, 17.2], there is a decomposition $M = M_1 \oplus M_2$ such that M_1 is semisimple and M_2 is ss-supplemented with $Rad(M_2) \trianglelefteq M_2$. It follows that $Soc(M) = Soc(M_1) \oplus Soc(M_2) = M_1 \oplus Soc(M_2) \trianglelefteq M_1 \oplus M_2 = M$. □

In general, the socle of a $\oplus$-supplemented module need not be essential. We can see this reality in the example below.

Example 2. Given the ring $\mathbb{Z}_{(2)}$ containing all rational numbers of the form $\frac{a}{b}$ with $2 \nmid b$. Therefore $R = \mathbb{Z}_{(2)}$ is a local Dedekind domain and its fractions field K is hollow as a left R-module. It follows that $_RK$ is $\oplus$-supplemented. On the other hand, the socle $Soc(_RK)$ is zero since R is a commutative domain. Hence $Soc(_RK)$ is not essential in $_RK$.

Now we will give that the class of projective $\oplus_{ss}$-supplemented modules are the same as ss-lifting modules.

Theorem 4. *Let M be a projective module. The following statement are equivalent.*

(1) M is ss-supplemented.

(2) M is $\oplus_{ss}$-supplemented.

(3) M is ss-lifting.

Proof. It follows from [23, Theorem 2.18]. □

We will give an analogue of the finite direct sum of the types of supplemented modules in the following theorem for $\oplus_{ss}$-supplemented modules.

Theorem 5. *Let R be an arbitrary ring. Then every finite direct sum of $\oplus_{ss}$-supplemented R-modules is $\oplus_{ss}$-supplemented.*

Proof. The proof is straightforward. □

The following result is crucial.

Theorem 6. *For any ring R, every direct sum of strongly local R-modules is $\oplus_{ss}$-supplemented.*

Proof. Let $\{M_i\}_{i\in I}$ be a collection of strongly local R-modules and $M = \bigoplus_{i\in I} M_i$. Put $\overline{M} = \frac{M}{Rad(M)}$. Note that by [28, 41.1 (5)], $Rad(M_i) = M_i \cap Rad(M)$ for each $i \in I$. Defining $\overline{M_i} = \frac{M_i+Rad(M)}{Rad(M)}$, we obtain for each $i \in I$

$$\overline{M_i} \cong \frac{M_i}{M_i\cap Rad(M)} = \frac{M_i}{Rad(M_i)}.$$

Since M_i is strongly local for every $i \in I$, it follows that $\frac{M_i}{Rad(M_i)}$ is simple. This implies that

$$\overline{M} = \frac{M}{Rad(M)} = \bigoplus_{i\in I} \frac{M_i}{Rad(M_i)} \cong \bigoplus_{i\in I} \overline{M_i}$$

and thus $\overline{M}$ is semisimple since the class of semisimple modules is closed under direct sums. Let U be any submodule of M. There exists a subset $J \subseteq I$ such that $\overline{M} = \overline{U} \oplus (\bigoplus_{i\in J} \frac{M_i}{Rad(M_i)})$. Let $V = \bigoplus_{i\in J} M_i$. Clearly, V is a direct summand of M. Then $M = U + V$ and $U \cap V \subseteq Rad(M)$. By [28, 21.6 (5)], $Rad(M) = \bigoplus_{i\in I} Rad(M_i)$ and so $Rad(M)$ is semisimple. Therefore V is an ss-supplement of U in M. Hence M is $\oplus_{ss}$-supplemented. □

Example 3. Given the left $\mathbb{Z}$-module $M = \mathbb{Z}_9$. Then the only submodules of M are $\{\overline{0}\}$, $\{\overline{0}, \overline{3}, \overline{6}\}$ and $M = \mathbb{Z}_9$, and so $Rad(M) = Soc(M) = \{\overline{0}, \overline{3}, \overline{6}\}$ is semisimple. Since M is local, it is a strongly local module. Now we consider the left $\mathbb{Z}$-module $N = \bigoplus_{i\in I} \mathbb{Z}_9$ for any index set I. By Theorem 6, N is $\oplus_{ss}$-supplemented.

The following theorem shows that the direct sum of the lifting modules under one condition is $\oplus$-supplemented.

Theorem 7 (see [17, Theorem 2.12])**.** *Let R be any ring and let M be an R-module such that $M = \bigoplus_{i\in I} M_i$, where M_i is a lifting module for each $i \in I$. Suppose further that $Rad(M) \ll M$. Then M is $\oplus$-supplemented.*

Now we give an analogous characterization of this fact for $\oplus_{ss}$-supplemented modules without condition.

Theorem 8. *Let R be a ring. Then every direct sum of ss-lifting R-modules is $\oplus_{ss}$-supplemented.*

Proof. Let $\{M_i\}_{i\in I}$ be a family of ss-lifting R-modules and $M = \bigoplus_{i\in I} M_i$. Since each M_i $(i \in I)$ is ss-lifting, it follows from Corollary 2 that $Rad(M_i)$ is semisimple and so

$$Soc_s(M_i) = Rad(M_i) \cap Soc(M_i) = Rad(M_i).$$

According to [28, 21.6 (5)], we have $Rad(M)$ is semisimple. By [23, Theorem 3.1], we obtain that

$$\frac{M_i}{Rad(M_i)} = \frac{M_i + Rad(M)}{Rad(M)}$$

is semisimple for all $i \in I$. Therefore $\frac{M}{Rad(M)} = \sum_{i\in I} \frac{M_i+Rad(M)}{Rad(M)}$ is semisimple as a sum of these semisimple modules $\frac{M_i+Rad(M)}{Rad(M)}$.

Let U be any submodule of M. Then there is an index set $\lambda \subseteq I$ and a submodule $(i \in \lambda)$ $N_i \subseteq M_i$ such that

$$\frac{M}{Rad(M)} = (\frac{U+Rad(M)}{Rad(M)}) \bigoplus (\underset{i\in I}{\oplus} \frac{N_i+Rad(M)}{Rad(M)}).$$

By the hypothesis, there is a decomposition $(i \in \lambda)$ $M_i = L_i \oplus V_i$ such that $L_i \subseteq N_i \subseteq L_i + Rad(M_i)$ and $N_i \cap V_i \subseteq Soc_s(M_i) = Rad(M_i)$. Put $V = \bigoplus_{i\in\lambda} V_i$ and therefore V is a direct summand of M. Since $Rad(M)$ is semisimple, it is a small submodule of M and so $M = U+V+Rad(M) = U+V$. On the other hand, $U \cap V \subseteq (U+Rad(M)) \cap (\sum_{i\in\lambda} N_i+Rad(M)) \subseteq Rad(M)$ and that $U \cap V$ is semisimple and a small submodule of M. Following [28, 19.3 (5)], we obtain that $U \cap V \subseteq Soc_s(V)$. Hence M is $\oplus_{ss}$-supplemented. □

Theorem 9. *Let M be a module. Then the following statements are equivalent:*

(1) M is $\oplus_{ss}$-supplemented.

(2) M is a Rad-$\oplus$-supplemented module with semisimple radical.

Proof. (1) ⇒ (2) It is clear that M is Rad-$\oplus$-supplemented. Then there exists a decomposition $M_1 \oplus M_2 = M$ such that $M = Rad(M) + M_1$, $Rad(M) \cap M_1 \ll M_1$ and $Rad(M) \cap M_1$ is semisimple. By the proof of Theorem 1, $M_2 = 0$ and then $Rad(M_1) = Rad(M)$ is semisimple.

(2) ⇒ (1) Since the class of semisimple modules is closed under submodules, it is clear. □

Corollary 3. *For a module M, the following are equivalent:*

(1) M is $\oplus_{ss}$-supplemented.

(2) M is a Rad-$\oplus$-supplemented module with semisimple radical.

(3) M is a $\oplus$-supplemented module with semisimple radical.

Proof. (1) ⇒ (3) and (3) ⇒ (2) are clear.

(2) ⇒ (1) By Theorem 9. □

Let R be an arbitrary ring. A functor τ from the category of left R-modules to itself is called a *preradical* if it satisfies the following properties.

(1) $\tau(M)$ is a submodule of any R-module M.

(2) If $f : M^{'} \to M$ is an R-module homomorphism, then $f(\tau(M^{'})) \subseteq \tau(M)$ and $\tau(f)$ is the restriction of f to $\tau(M^{'})$.

Proposition 3. *Let R be a ring and τ be a preradical of the category of the left R-modules. If M is a $\oplus_{ss}$-supplemented R-module, then*

(1) $\frac{M}{\tau(M)}$ is $\oplus_{ss}$-supplemented.

(2) If $\tau(M)$ is a direct summand of M, then $\tau(M)$ is also $\oplus_{ss}$-supplemented.

Proof. (1) Let $\frac{U}{\tau(M)}$ be any submodule of $\frac{M}{\tau(M)}$. By the hypothesis, there is a decomposition $M = V \oplus V^{'}$ such that V is an ss-supplement of U in M. It follows from the proof of [15, Proposition 26] that $\frac{V+\tau(M)}{\tau(M)}$ is an ss-supplement of $\frac{U}{\tau(M)}$ in $\frac{M}{\tau(M)}$. Since τ is a preradical in the category of left R-modules, it follows from [13, Lemma 2.4] that we can write the decomposition $\tau(M) = V \cap \tau(M) \oplus V^{'} \cap \tau(M)$. Therefore, by the modularity law,

$$\begin{aligned}
\frac{V+\tau(M)}{\tau(M)} \cap \frac{V^{'}+\tau(M)}{\tau(M)} &= \frac{(V+\tau(M))\cap(V^{'}+\tau(M))}{\tau(M)} \\
&= \frac{(V+(V\cap\tau(M)\oplus V^{'}\cap\tau(M)))\cap(V^{'}+(V\cap\tau(M)\oplus V^{'}\cap\tau(M)))}{\tau(M)} \\
&= \frac{(V+V^{'}\cap\tau(M))\cap(V^{'}+V\cap\tau(M))}{\tau(M)} \\
&= 0.
\end{aligned}$$

It means that $\frac{V+\tau(M)}{\tau(M)}$ is a direct summand of $\frac{M}{\tau(M)}$. Hence $\frac{M}{\tau(M)}$ is $\oplus_{ss}$-supplemented.

(2) Assume that there is a decomposition $M = \tau(M) \oplus L$ for some submodule L of M. Let T be any submodule of $\tau(M)$. Since M is a $\oplus_{ss}$-supplemented module, there exist submodules Y, Z of M such that $M = Y \oplus Z$ and Y is an ss-supplement of T in M. Then, by the modularity law, we get that $\tau(M) = \tau(M) \cap M = \tau(M) \cap (T+Y) = T + Y \cap \tau(M)$. Again applying [13, Lemma 2.4], we obtain that $\tau(M) = Y \cap \tau(M) \oplus Z \cap \tau(M)$. Let $m \in T \cap (Y \cap \tau(M)) = T \cap Y$. Since $Y \cap Z \subseteq Soc_s(Y)$, Rm is semisimple and a small submodule of Y. So, by [28, 19.3 (5)], $m \in Rm \subseteq Soc_s(Y \cap \tau(M))$. Therefore $T \cap Y \subseteq Soc_s(Y \cap \tau(M))$. It means that $\tau(M)$ is $\oplus_{ss}$-supplemented. □

Let R be a ring and τ be a preradical of the category of left R-modules. In [2], M is called *τ-lifting* if every submodule N of M has a decomposition $N = A \oplus (B \cap N)$ such that $M = A \oplus B$ and $B \cap N \subseteq \tau(B)$ and also they called that M is *τ-semiperfect* if every factor module of M has a projective τ-cover, that is, for any submodule N of M, there exist a projective module P and an epimorphism $\psi : P \longrightarrow \frac{M}{N}$ such that $ker(\psi) \subseteq \tau(P)$.

In [23], a module M is called *ss-semilocal* if $\frac{M}{Soc_s(M)}$ is semisimple. The rings with the property that every left module is ss-semilocal are called ss-perfect. Note that $Soc_s(M)$ is the largest semisimple and small submodule of any module M and so Soc_s is preradical in the category of R-modules. Using Theorem 4, we get the following theorem.

Theorem 10. *Let M be a projective module. The following statements are equivalent.*

(1) M is ss-supplemented.

(2) M is $\oplus_{ss}$-supplemented.

(3) M is Soc_s-lifting, that is, ss-lifting.

(4) M is Soc_s-semiperfect.

Proof. By Theorem 4. □

For a ring R, we obtain the next result:

Corollary 4. *Let R be a ring. The following statements are equivalent.*

(1) R is Soc_s-semiperfect.

(2) ${}_RR$ is $\oplus_{ss}$-supplemented.

(3) ${}_RR$ is Soc_s-lifting, that is, ss-lifting.

(4) R is left ss-perfect ring.

Let U be a submodule of an R-module M. Following [25], U is called *strongly lifting* in M if whenever $\frac{M}{U} = \frac{A+U}{U} \oplus \frac{B+U}{U}$, then M has a decomposition M. In [1], Alkan M. expanded this definition and presented a new definition as follows. The submodule U is called *quasi strongly lifting* (QSL) in M, if whenever $\frac{A+U}{U}$ is a direct summand of $\frac{M}{U}$, M has a direct summand P such that $P \subseteq A$ and $P + U = A + U$. Using [1, Proposition 3.6.], we get the following fact.

Proposition 4. *A module M is ss-lifting if and only if it is $\oplus_{ss}$-supplemented and $Rad(M)$ is QSL.*

Proof. It is obtained from [1, Proposition 3.6] and Theorem 1. □

We now characterize the rings over which all (projective) modules are $\oplus_{ss}$-supplemented. Let R be a ring and M be an R-module. Following [19], we consider the following condition:

(D_3) For any direct summands M_1, M_2 of M with $M = M_1 + M_2$, $M_1 \cap M_2$ is also a direct summand of M.

Note that every (self) projective module satisfies the condition (D_3).

Lemma 3. *Let M be a $\oplus_{ss}$-supplemented module with (D_3). Then every direct summand of M is $\oplus_{ss}$-supplemented.*

Proof. Let N be a direct summand of M and U be a submodule of N. Since M is $\oplus_{ss}$-supplemented, there exists a direct summand V of M such that $M = U + V$ and $U \cap V \subseteq Soc_s(V)$. It follows from the modularity law that $N = U + (N \cap V)$. Since $M = U + V$ has (D_3), $N \cap V$ is also a direct summand of M and so we can write $M = (N \cap V) \oplus L$ for some submodule L of M. Again using the modularity law,

$$\begin{aligned} N &= N \cap M = N \cap ((N \cap V) \oplus L) \\ &= (N \cap V) \oplus (N \cap L). \end{aligned}$$

It means that $N \cap V$ is also a direct summand of N. Note that $U \cap (N \cap V) = U \cap V \subseteq Rad(V)$. Let $m \in U \cap V$. Therefore the cyclic submodule Rm is a small submodule of M. By [28, 19.3 (5)], Rm is small in $N \cap V$ and so $m \in Rad(N \cap V)$. Since $U \cap V$ is semisimple, we obtain that $m \in Soc_s(N \cap V)$. Therefore $U \cap (N \cap V) = U \cap V \subseteq Soc(N \cap V)$. Hence N is $\oplus_{ss}$-supplemented. □

Corollary 5. *The following statements are equivalent for a ring R.*

(1) R is Soc_s-semiperfect.

(2) Every free R-module is $\oplus_{ss}$-supplemented.

(3) Every projective R-module is $\oplus_{ss}$-supplemented.

Proof. (1) ⇒ (2) Let F be any free R-module. It follows from Corollary 4 that ${}_RR$ is ss-lifting. Therefore F is $\oplus_{ss}$-supplemented as a direct sum of copies of the ss-lifting module ${}_RR$ by Theorem 8.

(2) ⇒ (3) Let M be a projective R-module. Then M is isomorphic to a direct summand of some free R-module F. Using Lemma 3, M is $\oplus_{ss}$-supplemented.

(3) ⇒ (1) By Corollary 4. □

It is shown in [17, Theorem 1.1] that a commutative ring R is an artinian serial ring if and only if every left R-module is ⊕-supplemented. Now we generalize this fact in the next corollary, characterizing the commutative rings in which modules are $\oplus_{ss}$-supplemented.

Corollary 6. *A commutative ring R is an artinian serial ring with semisimple radical if and only if every left R-module is $\oplus_{ss}$-supplemented.*

Proof. (⇒) Let M be an R-module. It follows from [27, Corollary 2.15] that M is Rad-⊕-supplemented. Since $Rad(R)$ is semisimple, we can write $Rad(M)$ is a semisimple R-module with a method similar to the proof of Proposition 2. Hence, by Theorem 9, M is $\oplus_{ss}$-supplemented.

(⇐) By [17, Theorem 1.1], we get R is an artinian serial ring. Since ${}_RR$ is $\oplus_{ss}$-supplemented, $Rad(R)$ is semisimple according to Theorem 1. □

Remark 1. Let R be a Dedekind domain and M be an R-module. M is *reduced* if M has no nonzero injective submodules. If M is $\oplus_{ss}$-supplemented, it follows from Theorem 1 that M is reduced.

(1) Let R be a local ring which is not a field. Combining Theorem 9, [15, Proposition 11] and [27, Corollary 3.3], we have M is $\oplus_{ss}$-supplemented if and only if M is isomorphic to a bounded R-module with semisimple radical.

(2) Let R be a non-local ring. By Theorem 9, [27, Theorem 3.2], [3, Proposition 7.3] and [31, Theorem 3.1], M is $\oplus_{ss}$-supplemented if and only if M is a torsion module with semisimple radical and every $\mathfrak{p}$-component of M is supplemented.

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