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Groups of nilpotency class 2 of order p^4 as additive groups of local nearrings

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Abstract. The paper is devoted to the study of local nearrings (those with identity, for which all non-invertible elements form subgroups of their additive group). A study of such nearrings was first initiated by C. J. Maxson in 1968, and the problem on the determination of the finite p-groups, which are the additive groups of local nearrings have become one of the most important. Particular cases of this (still unsolved) problem have been studied in many works. In previous papers the authors have shown that, up to isomorphism, there exist at least p local nearrings on elementary abelian additive groups of order p^3, which are not nearfields, and at least $p+1$ on each non-metacyclic non-abelian or metacyclic abelian groups of order p^3. In this paper we study the groups of nilpotency class 2 of order p^4, which are the additive groups of local nearrings. It is proved that, for odd p, 4 out of total number 6 of such groups are the additive groups of local nearrings. Explicit examples of the corresponding local nearrings are provided.

Introduction

The problem of finding the groups that can be additive groups for the nearrings with identity is studied from the late 1960s. One of the first results in this direction was obtained in [8], where it was shown that there exists a unique nearring with identity whose additive group is cyclic and that, in fact, this nearring is a commutative ring. It was also proved

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that the symmetric group S_n with $n \geq 3$ cannot be an additive group of a nearring with identity. It was shown that the alternating group A_4 also cannot be an additive group of a nearring with identity [6]. There is no nearring with identity whose additive group is isomorphic to the quaternion group Q_8 [7].

A study of local nearrings was first initiated in [11] and it was found that the additive group of a finite zero-symmetric local nearring is a p-group. In [12] it is shown that, up to isomorphism, there exist $p-1$ local zero-symmetric nearrings with elementary abelian additive groups of order p^2, in which the subgroups of non-invertible elements have order p, that is, those nearrings which are not nearfields. Together with the fundamental paper [25] and [8], a complete description of all zero-symmetric local nearrings of order p^2 is obtained. The dihedral group D_4 of order 8 cannot be the additive group of local nearrings [13]. The existence of local nearrings on finite abelian p-groups is proved in [14], i.e. every non-cyclic abelian p-group of order $p^n > 4$ is the additive group of a zero-symmetric local nearring which is not a ring. Also, it is established in [18] that an arbitrary non-metacyclic Miller–Moreno p-group of order $p^n > 8$ is the additive group of some local nearring. Nearrings with identity and local nearrings on Miller–Moreno groups were studied in [18], [16] and [17].

Boykett and Nöbauer [4] classified all non-abelian groups of order less than 32 that can be the additive groups of a nearring with identity and found the number of non-isomorphic nearrings with identity on such groups. The package SONATA [1] of the computer algebra system GAP [10] contains a library of all non-isomorphic nearrings of order at most 15 and nearrings with identity of order up to 31, among which 698 are local.

The classification of nearrings of higher orders requires much more complex calculations. For local nearrings they were realized in the form of a new GAP package called LocalNR [22]. Its current version (not yet distributed with GAP) contains 37599 non-isomorphic local nearrings of order at most 361, except orders 128, 256 and some of orders 32, 64 and 243. We have already calculated some classes of local nearrings of orders 32, 64, 128, 243 and 625.

However, it is not true that any finite group is the additive group of a nearring with identity. Therefore it is important to determine such groups and to classify some classes of nearrings with identity on these groups, for example, local nearrings.

In [19] it was shown that on each group of order p^3 with $p > 2$ there exists a local nearring. Moreover, lower bounds for the number of local nearrings on groups of order p^3 are obtained. It is established that on each non-metacyclic non-abelian or metacyclic abelian groups of order p^3 there exist at least $p + 1$ non-isomorphic local nearrings. In [21] it is proved that, up to isomorphism, there exist at least p local nearrings on elementary abelian additive groups of order p^3, which are not nearfields.

The next natural step is to investigate groups of order p^4 as the additive groups of local nearrings. In this paper we consider groups of nilpotency class 2 of order p^4 which are the additive group of local nearrings. It was shown that, for odd p, out of 6 of such groups 4 of them are the additive groups of local nearrings.

1. Preliminaries

We will give the basic definitions.

Definition 1. *A non-empty set R with two binary operations "$+$" and "$\cdot$" is a* ***nearring*** *if:*

1) $(R, +)$ *is a group with neutral element* 0;

2) $(R, \cdot)$ *is a semigroup*;

3) $x \cdot (y + z) = x \cdot y + x \cdot z$ *for all* x, y, $z \in R$.

Such a nearring is called a left nearring. If axiom 3) is replaced by an axiom $(x + y) \cdot z = x \cdot z + y \cdot z$ for all x, y, $z \in R$, then we get a right nearring.

The group $(R, +)$ of a nearring R is denoted by R^+ and called the *additive group* of R. It is easy to see that for each subgroup M of R^+ and for each element $x \in R$ the set $xM = \{x \cdot y | y \in M\}$ is a subgroup of R^+ and in particular $x \cdot 0 = 0$. If in addition $0 \cdot x = 0$ for all $x \in R$, then the nearring R is called *zero-symmetric*. Furthermore, R is a *nearring with an identity* i if the semigroup $(R, \cdot)$ is a monoid with identity element i. In the latter case the group of all invertible elements of the monoid $(R, \cdot)$ is denoted by R^* and called the *multiplicative group* of R. A subgroup M of R^+ is called R^*-*invariant*, if $rM \leq M$ for each $r \in R^*$, and (R, R)-*subgroup*, if $xMy \subseteq M$ for arbitrary x, $y \in R$.

The following assertion is well-known (see, for instance, [8], Theorem 3).

Lemma 1. *The exponent of the additive group of a finite nearring R with identity i is equal to the additive order of i which coincides with the additive order of every invertible element of R.*

Definition 2. *A nearring R with identity is called **local** if the set L of all non-invertible elements of R forms a subgroup of the additive group R^+ and a **nearfield**, if $L = 0$.*

Throughout this paper L will denote the subgroup of non-invertible elements of R.

The following lemma characterizes the main properties of finite local nearrings (see [3], Lemma 3.2).

Lemma 2. *Let R be a local nearring with identity i. Then the following statements hold:*

1) *L is an (R, R)-subgroup of R^+;*

2) *each proper R^*-invariant subgroup of R^+ is contained in L;*

3) *the set $i + L$ forms a subgroup of the multiplicative group R^*.*

Finite local nearrings with a cyclic subgroup of non-invertible elements are described in [20, Theorem 1].

Theorem 1. *Let R be a local nearring of order p^n with $n > 1$ whose subgroup L is cyclic and non-trivial. Then the additive group R^+ is either cyclic or is an elementary abelian group of order p^2. In the first case, R is a commutative local ring, which is isomorphic to residual ring $\mathbb{Z}/p^n\mathbb{Z}$ with $n \geq 2$, in the other case there exist p non-isomorphic such nearrings R with $|L| = p$, from which $p-1$ are zero-symmetric nearrings and their multiplicative groups R^* are isomorphic to a semidirect product of two cyclic subgroups of orders p and $p - 1$.*

The following theorem was proved by Maxson in [12, Theorem 2.1].

Theorem 2. *If R is a finite local nearring which is not a nearfield, then $|R| < |L|^2$.*

As a consequence of Theorems 1 and 2 we have the following result.

Corollary 1. *Let R be a local nearring of order p^4 with non-abelian additive group and is not a nearfield. Then the subgroup of non-invertible elements L is a non-cyclic group of order p^3 or p^2.*

Due to Onyshchuck, Sysak [15], let G be a group and $\operatorname{End} G$ be the set of all its endomorphisms, which can be considered as a semigroup with respect to the composition operation of endomorphisms. For each $g \in G$ we denote by $g^{\operatorname{End} G}$ the set $\{g^\alpha | \alpha \in \operatorname{End} G\}$ of all images of the element g with respect to endomorphisms of $\operatorname{End} G$.

Definition 3 ([15]). *A group G is called* **endocyclic** *if it contains an element g with $G = g^{\operatorname{End} G}$.*

It is clear that in this case g is an element of maximum order in G.

We recall the following definition.

Definition 4. *A finite non-abelian group whose proper subgroups are abelian is called a **Miller–Moreno group**.*

2. Groups of nilpotency class 2 of order p^4

We will consider groups of nilpotency class 2 of order p^4.

Let $[n, i]$ be the i-th group of order n in the SmallGroups library in the computer system algebra GAP. We denote by C_n the cyclic group of order n.

It is an easy exercise for example in GAP to get the following assertion.

Remark 1. There are 6 groups of nilpotency class 2 of order $2^4 = 16$, which are:

1. $(C_4 \times C_2) \rtimes C_2$ $[16, 3]$;
2. $C_4 \rtimes C_4$ $[16, 4]$;
3. $C_8 \rtimes C_2$ $[16, 6]$;
4. $C_2 \times D_8$ $[16, 11]$;
5. $C_2 \times Q_8$ $[16, 12]$;
6. $(C_4 \times C_2) \rtimes C_2$ $[16, 13]$.

The following theorem contains the classification of groups of nilpotency class 2 of order p^4, where p is an odd prime (see, [5] and, for example, [2]).

Theorem 3. *There are 6 groups of nilpotency class 2 of order p^4, where p is an odd prime, which are:*

- $G_1 = \langle a, b \colon a^{p^2} = b^p = [a,[a,b]] = [b,[a,b]] = [a,b]^p = e\rangle = (C_{p^2} \times C_p) \rtimes C_p$;
- $G_2 = \langle a, b \colon a^{p^2} = b^{p^2} = e, [b,a] = b^p\rangle = C_{p^2} \rtimes C_{p^2}$;
- $G_3 = \langle a, b \colon a^{p^3} = b^p = e, [b,a] = a^{p^2}\rangle = C_{p^3} \rtimes C_p$;
- $G_4 = \langle a, b, d \colon a^p = b^p = c^p = d^p = [a,c] = [b,c] = [a,d] = [b,d] = [c,d] = e\rangle = C_p \times ((C_p \times C_p) \rtimes C_p)$, *where* $c = [a,b]$;
- $G_5 = \langle a, b, c \colon a^{p^2} = b^p = c^p = [a,c] = [b,c] = e, [b,a] = a^p\rangle = C_p \times (C_{p^2} \rtimes C_p)$;
- $G_6 = \langle a, b, c \colon a^{p^2} = b^p = c^p = [a,b] = [a,c] = e, [c,b] = a^p\rangle = (C_{p^2} \times C_p) \rtimes C_p$.

3. Groups of nilpotency class 2 of order 16 and local nearrings

As was mentioned above a library of all non-isomorphic nearrings with identity of order up to 31 are contained in the package SONATA, and so all non-isomorphic local nearrings of order 16 (see [4]).

Lemma 3. *The following groups of nilpotency class* 2 *and only they are the additive groups of local nearrings of order* 16:

1. $(C_4 \times C_2) \rtimes C_2$ $[16, 3]$;
2. $C_4 \rtimes C_4$ $[16, 4]$;
3. $C_8 \rtimes C_2$ $[16, 6]$;
4. $C_2 \times Q_8$ $[16, 12]$.

Let $n(G)$ be the number of all non-isomorphic local nearrings R whose additive group R^+ is isomorphic to the group G.

$StructureDescription(R^+)$	$n(R^+)$
$(C_4 \times C_2) \rtimes C_2$	37
$C_4 \rtimes C_4$	24
$C_8 \rtimes C_2$	33
$C_2 \times Q_8$	2

3.1. The groups G_1, G_2 and G_3 and local nearrings

The groups G_1, G_2 and G_3 from Theorem 3 are Miller–Moreno groups.

Due to [23] G_2 and G_3 are the groups $G(p^2, p^2)$ and $G(p^3, p)$, respectively (see, for example, Lemma 2 [23]). Therefore, by Theorem 2 [23] there exists a local nearring R whose additive group R^+ is isomorphic to G_3. As a consequence, there does not exist a local nearring on the additive group G_2.

Let R be a local nearring whose additive group of R^+ is isomorphic to G_3. Then $R^+ = \langle a\rangle + \langle b\rangle$ for some elements a and b of R satisfying the relations $ap^3 = 0$, $bp = 0$ and $-b + a + b = a(1 - p^2)$. In particular, each element $x \in R$ is uniquely written in the form $x = ax_1 + bx_2$ with coefficients $0 \le x_1 < p^3$ and $0 \le x_2 < p$.

The formula for multiplying elements of local nearrings on Miller–Moreno metacyclic groups is defined in [23]. The multiplication formula for arbitrary elements of a zero-symmetric local nearring on G_3 is given in the proving of [23, Theorem 2], namely:

$$x \cdot y = a(x_1y_1 + p^2x_1x_2\binom{y_1}{2}) + b(x_2y_1 + \beta(x)y_2),$$

where $\beta(x) = \begin{cases} 1, & if\ x_1 \not\equiv 0 \pmod p; \\ 0, & if\ x_1 \equiv 0 \pmod p. \end{cases}$

Example 1. Let $G \cong C_{27} \rtimes C_3$. If $x = ax_1 + bx_2$ and $y = ay_1 + by_2 \in G$ and $(G, +, \cdot)$ is a local nearring, then as above " $\cdot$ " can be the following multiplication:

$$x \cdot y = a(x_1y_1 + 9x_1x_2\binom{y_1}{2}) + b(x_2y_1 + \beta(x)y_2),$$

where $\beta(x) = \begin{cases} 1, & if\ x_1 \not\equiv 0 \pmod 3; \\ 0, & if\ x_1 \equiv 0 \pmod 3. \end{cases}$

A computer program verified that the nearring obtained in Example 1 is indeed a local nearring, is deposited on GitHub:

`https://github.com/raemarina/Examples/blob/main/LNR_81-6.txt.`

From the package LocalNR and [24] we have the following number of all non-isomorphic zero-symmetric local nearrings on G_3 of orders 81 and 625.

$StructureDescription(R^+)$	$n(R^+)$
$C_{27} \rtimes C_3$	10
$C_{125} \rtimes C_5$	5

Analogously, G_1 is the group $G(p^2, p, p)$ according to [18]. Hence, by Theorem 3 [18] there exist a local nearring whose additive group is isomorphic to G_1. Since G_1 is a Miller–Moreno non-metacyclic group, using [18, Theorem 3], for arbitrary elements $x = ax_1 + bx_2 + cx_3$ and $y = ay_1 + by_2 + cy_3$ of G_1 we obtain the following multiplication formula:

$$x \cdot y = a(x_1y_1 + p^k x_2y_2) + b(x_2y_1 + x_1y_2) + c(-x_1x_2\binom{y_1}{2} + x_3y_1 + x_1^2y_3),$$

where $k = 1,\ 2$.

It is easy to see that $R = (G_1, +, \cdot)$ is a non-zero-symmetric local nearring.

Example 2. Let $G \cong (C_9 \times C_3) \rtimes C_3$. If $x = ax_1 + bx_2 + cx_3$ and $y = ay_1 + by_2 + cy_3 \in G$ and $(G, +, \cdot)$ is a local nearring, then as above "$\cdot$" can be one of the following multiplications:

(1) $x \cdot y = a(x_1y_1 + 3x_2y_2) + b(x_2y_1 + x_1y_2) + c(-x_1x_2\binom{y_1}{2} + x_3y_1 + x_1^2y_3)$;

(2) $x \cdot y = a(x_1y_1) + b(x_2y_1 + x_1y_2) + c(-x_1x_2\binom{y_1}{2} + x_3y_1 + x_1^2y_3)$.

A computer program verified that the nearring obtained in Example 2 is indeed a local nearring, is deposited on GitHub:

`https://github.com/raemarina/Examples/blob/main/LNR_81-3.txt`.

From the package LocalNR and [24] we have the following number of all non-isomorphic zero-symmetric local nearrings on G_1 of orders 81 and 625.

$StructureDescription(R^+)$	$n(R^+)$
$(C_9 \times C_3) \rtimes C_3$	46
$(C_{25} \times C_5) \rtimes C_5$	154

3.2. The group G_4

Let G_4 be additively written group from Theorem 3. Then $G_4 = \langle a\rangle + \langle b\rangle + \langle c\rangle + \langle d\rangle$ for some elements a, b, c and d of R satisfying the relations $ap = 0$, $bp = 0$, $cp = 0$, $dp = 0$, $a + b = b + a + c$, $a + c = c + a$, $b + c = c + b$, $a + d = d + a$, $b + d = d + b$ and $c + d = d + c$.

Lemma 4. *For arbitrary integers k and l in the group G_4 the equalities $-ak-bl+ak+bl=c(kl)$ and $bl+ak=-c(kl)+ak+bl$ hold.*

Proof. Since $-a-b+a+b=c$, we get $-b+a+b=a+c$. Then

$$-bl+ak+bl=(a+cl)k=ak+c(kl).$$

Therefore, $-ak-bl+ak+bl=c(kl)$ and, so $bl+ak=-c(kl)+ak+bl$. □

Lemma 5. *For any natural numbers k, l, n, m and r in the group G_4 the equality $(ak+bl+cm+dn)r=a(kr)+b(lr)+c(mr-kl\binom{r}{2})+d(nr)$ holds.*

Proof. The proof will be carried out by induction on r. For $r=1$ the equality is valid. Let for r the equality hold, i.e.

$$(ak+bl+cm+dn)r=a(kr)+b(lr)+c(mr-kl\binom{r}{2})+d(nr).$$

Let us prove the equality for $r+1$:

$$\begin{aligned}&(ak+bl+cm+dn)(r+1)=\\&\quad=a(kr)+b(lr)+ak+bl+c(kl\tbinom{r}{2})+c(m(r+1))+\\&\quad+d(n(r+1))=a(k(r+1))+b(l(r+1))+c(-klr)+\\&\quad+c(-kl\tbinom{r}{2})+c(m(r+1))+d(n(r+1))=a(k(r+1))+\\&\quad+b(l(r+1))+c(m(r+1)-kl(r+\tbinom{r}{2}))+d(n(r+1))=\\&\quad=a(k(r+1))+b(l(r+1))+c(m(r+1)-kl\tbinom{r+1}{2})+dn(r+1).\end{aligned}$$

Therefore, the equality is valid for any r. □

3.3. Nearrings with identity whose additive groups are isomorphic to G_4

Let R be a nearring with identity whose additive group R^+ is isomorphic to G_4. Then $R^+=\langle a\rangle+\langle b\rangle+\langle c\rangle+\langle d\rangle$ for some elements a, b, c and d of R satisfying the relations $ap=0$, $bp=0$, $cp=0$, $dp=0$, $a+b=b+a+c$, $a+c=c+a$, $b+c=c+b$, $a+d=d+a$, $b+d=d+b$ and $c+d=d+c$. In particular, each element $x\in R$ is uniquely written in the form $x=ax_1+bx_2+cx_3+dx_4$ with coefficients $0\le x_1<p$, $0\le x_2<p$, $0\le x_3<p$ and $0\le x_4<p$.

Since the order of the element a is equal to the exponent of group G, then by Lemma 1 we can assume that a is an identity of R, i.e.

$ax = xa = x$ for each $x \in R$. Furthermore, for each $x \in R$ there exist coefficients $\alpha(x)$, $\beta(x)$, $\gamma(x)$, $\varphi(x)$, $\lambda(x)$, $\mu(x)$, $\nu(x)$ and $\psi(x)$ such that $xb = a\alpha(x) + b\beta(x) + c\gamma(x) + d\varphi(x)$ and $xd = a\lambda(x) + b\mu(x) + c\nu(x) + d\psi(x)$. It is clear that they are uniquely defined modulo p, so that some mappings $\alpha\colon R \to \mathbb{Z}_p$, $\beta\colon R \to \mathbb{Z}_p$, $\gamma\colon R \to \mathbb{Z}_p$, $\varphi\colon R \to \mathbb{Z}_p$, $\lambda\colon R \to \mathbb{Z}_p$, $\mu\colon R \to \mathbb{Z}_p$, $\nu\colon R \to \mathbb{Z}_p$ and $\psi\colon R \to \mathbb{Z}_p$ are determined.

Lemma 6. *Let R be a nearring with identity whose additive group R^+ is isomorphic to G_4. If a coincides with identity element of R, $x = ax_1 + bx_2 + cx_3 + dx_4$, $y = ay_1 + by_2 + cy_3 + dy_4 \in R$, $xb = a\alpha(x) + b\beta(x) + c\gamma(x) + d\varphi(x)$ and $xd = a\lambda(x) + b\mu(x) + c\nu(x) + d\psi(x)$, then*

$$\begin{gathered} xy = a(x_1y_1 + \alpha(x)y_2 + \lambda(x)y_4) + b(x_2y_1 + \beta(x)y_2 + \mu(x)y_4) + \\ +c(-x_1x_2\tbinom{y_1}{2} - \alpha(x)\beta(x)\tbinom{y_2}{2} - x_2\alpha(x)y_1y_2 - \lambda(x)\mu(x)\tbinom{y_4}{2} - \\ -x_2\alpha(x)y_3 + x_3y_1 + \gamma(x)y_2 + x_1\beta(x)y_3 + \nu(x)y_4) + d(x_4y_1 + \\ +\varphi(x)y_2 + \psi(x)y_4). \end{gathered}$$

Moreover, for the mappings
$\alpha\colon R \to \mathbb{Z}_p, \beta\colon R \to \mathbb{Z}_p, \gamma\colon R \to \mathbb{Z}_p, \varphi\colon R \to \mathbb{Z}_p, \lambda\colon R \to \mathbb{Z}_p,$
$\mu\colon R \to \mathbb{Z}_p, \nu\colon R \to \mathbb{Z}_p$ *and* $\psi\colon R \to \mathbb{Z}_p$ *the following statements hold*:

(0) $\alpha(0) \equiv 0 \pmod p$, $\beta(0) \equiv 0 \pmod p$, $\gamma(0) \equiv 0 \pmod p$; $\varphi(0) \equiv 0 \pmod p$, $\lambda(0) \equiv 0 \pmod p$, $\mu(0) \equiv 0 \pmod p$, $\nu(0) \equiv 0 \pmod p$ *and* $\psi(0) \equiv 0 \pmod p$ *if and only if the nearring R is zero-symmetric*;

(1) $\alpha(xy) \equiv x_1\alpha(y) + \alpha(x)\beta(y) + \lambda(x)\varphi(y) \pmod p$;

(2) $\beta(xy) \equiv x_2\alpha(y) + \beta(x)\beta(y) + \mu(x)\varphi(y) \pmod p$;

(3) $\gamma(xy) \equiv -x_1x_2\binom{\alpha(y)}{2} - \alpha(x)\beta(x)\binom{\beta(y)}{2} - x_2\alpha(x)\alpha(y)\beta(y) - \lambda(x)\mu(x)\binom{\varphi(y)}{2} - x_2\alpha(x)\gamma(y) + x_3\alpha(y) + \gamma(x)\beta(y) + x_1\beta(x)\gamma(y) + \nu(x)\varphi(y) \pmod p$;

(4) $\varphi(xy) \equiv x_4\alpha(y) + \varphi(x)\beta(y) + \psi(x)\varphi(y) \pmod p$;

(5) $\lambda(xy) \equiv x_1\lambda(y) + \alpha(x)\mu(y) + \lambda(x)\psi(y) \pmod p$;

(6) $\mu(xy) \equiv x_2\lambda(y) + \beta(x)\mu(y) + \mu(x)\psi(y) \pmod p$;

(7) $\nu(xy) \equiv -x_1x_2\binom{\lambda(y)}{2} - \alpha(x)\beta(x)\binom{\mu(y)}{2} - x_2\alpha(x)\lambda(y)\mu(y) - \lambda(x)\mu(x)\binom{\psi(y)}{2} - x_2\alpha(x)\nu(y) + x_3\lambda(y) + \gamma(x)\mu(y) + x_1\beta(x)\nu(y) + \nu(x)\psi(y) \pmod p$;

(8) $\psi(xy) \equiv x_4\lambda(y) + \varphi(x)\mu(y) + \psi(x)\psi(y) \pmod{p}$.

Proof. Since $0 \cdot a = a \cdot 0 = 0$, it follows that R is a zero-symmetric nearring if and only if

$$0 = 0 \cdot b = a\alpha(0) + b\beta(0) + c\gamma(0) + d\varphi(0)$$

and

$$0 = 0 \cdot d = a\lambda(0) + b\mu(0) + c\nu(0) + d\psi(0).$$

Equivalently we have

$\alpha(0) \equiv 0 \pmod{p}$, $\beta(0) \equiv 0 \pmod{p}$, $\gamma(0) \equiv 0 \pmod{p}$,
$\varphi(0) \equiv 0 \pmod{p}$, $\lambda(0) \equiv 0 \pmod{p}$, $\mu(0) \equiv 0 \pmod{p}$,
$\nu(0) \equiv 0 \pmod{p}$ and $\psi(0) \equiv 0 \pmod{p}$.

Moreover, since $c = -a - b + a + b$ and the left distributive law we have $0 \cdot c = -0 \cdot a - 0 \cdot b + 0 \cdot a + 0 \cdot b = 0$, whence

$$0 \cdot x = 0 \cdot (ax_1 + bx_2 + cx_3 + dx_4) = (0 \cdot a)x_1 + (0 \cdot b)x_2 + (0 \cdot c)x_3 + (0 \cdot d)x_4 = 0.$$

So that statement (0) holds.

Further, using Lemma 4, we derive

$$\begin{gathered} xc = -xa - xb + xa + xb = -cx_3 - bx_2 - ax_1 - c\gamma(x) - \\ -b\beta(x) - a\alpha(x) + ax_1 + bx_2 + cx_3 + a\alpha(x) + b\beta(x) + c\gamma(x) = \\ = -bx_2 - ax_1 - b\beta(x) - a\alpha(x) + ax_1 + bx_2 + a\alpha(x) + b\beta(x) = \\ = -bx_2 + cx_1\beta(x) - b\beta(x) - ax_1 - a(\alpha(x) - x_1) + bx_2 + a\alpha(x) + \\ +b\beta(x) = cx_1\beta(x) - b(x_2 + \beta(x)) - a\alpha(x) + bx_2 + a\alpha(x) + b\beta(x) = \\ = cx_1\beta(x) - b(x_2 + \beta(x)) - a\alpha(x) - cx_2\alpha(x) + a\alpha(x) + bx_2 + \\ +b\beta(x) = c(x_1\beta(x) - x_2\alpha(x)) - b(x_2 + \beta(x)) + bx_2 + b\beta(x) = \\ = c(x_1\beta(x) - x_2\alpha(x)). \end{gathered}$$

Further, using the left distributive law, we obtain

$$\begin{gathered} xy = (ax_1 + bx_2 + cx_3 + dx_4)y_1 + (a\alpha(x) + b\beta(x) + c\gamma(x) + \\ d\varphi(x))y_2 + (c(x_1\beta(x) - x_2\alpha(x)))y_3 + \\ +(a\lambda(x) + b\mu(x) + c\nu(x) + d\psi(x))y_4. \end{gathered}$$

By Lemma 5, we get

$$\begin{gathered} (ax_1 + bx_2 + cx_3 + dx_4)y_1 = ax_1y_1 + bx_2y_1 + \\ +c(x_3y_1 - x_1x_2\tbinom{y_1}{2}) + dx_4y_1, \\ (a\alpha(x) + b\beta(x) + c\gamma(x) + d\varphi(x))y_2 = a\alpha(x)y_2 + b\beta(x)y_2 + \\ +c(\gamma(x)y_2 - \alpha(x)\beta(x)\tbinom{y_2}{2}) + d\varphi(x)y_2, \\ (a\lambda(x) + b\mu(x) + c\nu(x) + d\psi(x))y_4 = a\lambda(x)y_4 + b\mu(x)y_4 + \\ +c(\nu(x)y_4 - \lambda(x)\mu(x)\tbinom{y_4}{2}) + d\psi(x)y_4. \end{gathered}$$

By Lemma 5, we have

$$bx_2y_1 + a\alpha(x)y_2 = a\alpha(x)y_2 + bx_2y_1 - cx_2\alpha(x)y_1y_2,$$

and

$$b\beta(x)y_2 + a\lambda(x)y_4 = a\lambda(x)y_4 + b\beta(x)y_2 - c\lambda(x)\beta(x)y_2y_4.$$

Hence and using the left distributive law, we have

$$\begin{gathered} xy = a(x_1y_1 + \alpha(x)y_2 + \lambda(x)y_4) + b(x_2y_1 + \beta(x)y_2 + \mu(x)y_4)+ \\ +c(-x_1x_2\tbinom{y_1}{2} - \alpha(x)\beta(x)\tbinom{y_2}{2} - x_2\alpha(x)y_1y_2 - \lambda(x)\mu(x)\tbinom{y_4}{2}- \\ -x_2\alpha(x)y_3 + x_3y_1 + \gamma(x)y_2 + x_1\beta(x)y_3 + \nu(x)y_4) + d(x_4y_1+ \\ +\varphi(x)y_2 + \psi(x)y_4). \end{gathered}$$

The associativity of multiplication in R implies that for all x, $y \in R$

1) $$(xy)b = x(yb)$$

and

2) $$(xy)d = x(yd).$$

According to $xb = a\alpha(x) + b\beta(x) + c\gamma(x) + d\varphi(x)$, we obtain

3) $$(xy)b = a\alpha(xy) + b\beta(xy) + c\gamma(xy) + d\varphi(xy)$$

and $yb = a\alpha(y) + b\beta(y) + c\gamma(y) + d\varphi(y)$. Substituting the last equation to the right part of equality 1), we also have

4) $$\begin{gathered} x(yb) = a(x_1\alpha(y) + \alpha(x)\beta(y) + \lambda(x)\varphi(y)) + b(x_2\alpha(y) + \beta(x)\beta(y)+ \\ +\mu(x)\varphi(y)) + c(-x_1x_2\tbinom{\alpha(y)}{2} - \alpha(x)\beta(x)\tbinom{\beta(y)}{2} - x_2\alpha(x)\alpha(y)\beta(y)- \\ -\lambda(x)\mu(x)\tbinom{\varphi(y)}{2} - x_2\alpha(x)\gamma(y) + x_3\alpha(y) + \gamma(x)\beta(y) + x_1\beta(x)\gamma(y)+ \\ +\nu(x)\varphi(y)) + d(x_4\alpha(y) + \varphi(x)\beta(y) + \psi(x)\varphi(y)). \end{gathered}$$

Since equality 1) implies the congruence of the corresponding coefficients in formulas 3) and 4), we obtain statements (1)–(4).

Next, according to $xd = a\lambda(x) + b\mu(x) + c\nu(x) + d\psi(x)$ instead of y in equality 2), we get

5) $$(xy)d = a\lambda(xy) + b\mu(xy) + c\nu(xy) + d\psi(xy)$$

and $yd = a\lambda(y) + b\mu(y) + c\nu(y) + d\psi(y)$. Substituting the last equation to the right part of equality 2), we also have

6) $x(yd) = a(x_1\lambda(y) + \alpha(x)\mu(y) + \lambda(x)\psi(y)) + b(x_2\lambda(y) + \beta(x)\mu(y) +$
$+\mu(x)\psi(y)) + c(-x_1x_2\binom{\lambda(y)}{2} - \alpha(x)\beta(x)\binom{\mu(y)}{2} - x_2\alpha(x)\lambda(y)\mu(y) -$
$-\lambda(x)\mu(x)\binom{\psi(y)}{2} - x_2\alpha(x)\nu(y) + x_3\lambda(y) + \gamma(x)\mu(y) + x_1\beta(x)\nu(y) +$
$+\nu(x)\psi(y)) + d(x_4\lambda(y) + \varphi(x)\mu(y) + \psi(x)\psi(y)).$

Finally, comparing the coefficients under a, b, c and d in formulas 5) and 6), we derive statements (5)–(8) of the lemma. □

3.4. Local nearrings whose additive groups are isomorphic to G_4

Let R be a local nearring whose additive group R^+ is isomorphic to G_4. Then $R^+ = \langle a\rangle + \langle b\rangle + \langle c\rangle + \langle d\rangle$ for some elements a, b, c and d of R satisfying the relations $ap = 0$, $bp = 0$, $cp = 0$, $dp = 0$, $a + b = b + a + c$, $a + c = c + a$, $b + c = c + b$, $d + c = c + d$, $a + d = d + a$ and $b + d = d + b$. In particular, each element $x \in R$ is uniquely written in the form $x = ax_1 + bx_2 + cx_3 + dx_4$ with coefficients $0 \le x_1 < p$, $0 \le x_2 < p$, $0 \le x_3 < p$ and $0 \le x_4 < p$.

Since order of the element a is equal to the exponent of group G, i.e. p, it follows that by Lemma 1 we can assume that a is an identity of R, i.e. $ax = xa = x$ for each $x \in R$. Furthermore, for each $x \in R$ there exist coefficients $\alpha(x)$, $\beta(x)$, $\gamma(x)$, $\varphi(x)$, $\lambda(x)$, $\mu(x)$, $\nu(x)$ and $\psi(x)$ such that $xb = a\alpha(x) + b\beta(x) + c\gamma(x) + d\varphi(x)$ and $xd = a\lambda(x) + b\mu(x) + c\nu(x) + d\psi(x)$. It is clear that they are uniquely defined modulo p, so that some mappings $\alpha\colon R \to \mathbb{Z}_p$, $\beta\colon R \to \mathbb{Z}_p$, $\gamma\colon R \to \mathbb{Z}_p$, $\varphi\colon R \to \mathbb{Z}_p$, $\lambda\colon R \to \mathbb{Z}_p$, $\mu\colon R \to \mathbb{Z}_p$, $\nu\colon R \to \mathbb{Z}_p$ and $\psi\colon R \to \mathbb{Z}_p$ are determined.

By Corollary 1, L is the normal subgroup of order p^3 or p^2 in R. Since L consists the derived subgroup of R^+ it follows that the generators b and c we can choose such that $c = -a - b + a + b$. If $|L| = p^3$ then $L = \langle b\rangle + \langle c\rangle + \langle d\rangle$. Since $R^* = R \setminus L$ it follows

$$R^* = \{ax_1 + bx_2 + cx_3 + dx_4 \mid x_1 \not\equiv 0 \pmod p\}$$

and $x = ax_1 + bx_2 + cx_3 + dx_4$ is invertible if and only if $x_1 \not\equiv 0 \pmod p$.

Throughout this section let R be a local nearring with $|R : L| = p$.

Lemma 7. *If a coincides with identity element of R, $x = ax_1 + bx_2 + cx_3 + dx_4$, $y = ay_1 + by_2 + cy_3 + dy_4 \in R$, $|R : L| = p$, $xb = a\alpha(x) + b\beta(x) + c\gamma(x) + d\varphi(x)$ and $xd = a\lambda(x) + b\mu(x) + c\nu(x) + d\psi(x)$, then*

$$xy = a(x_1y_1) + b(x_2y_1 + \beta(x)y_2 + \mu(x)y_4) + c(-x_1x_2\tbinom{y_1}{2} + x_3y_1 + \gamma(x)y_2 + x_1\beta(x)y_3 + \nu(x)y_4) + d(x_4y_1 + \varphi(x)y_2 + \psi(x)y_4).\ (*)$$

Moreover, for the mappings
$\alpha\colon R \to \mathbb{Z}_p, \beta\colon R \to \mathbb{Z}_p, \gamma\colon R \to \mathbb{Z}_p, \varphi\colon R \to \mathbb{Z}_p, \lambda\colon R \to \mathbb{Z}_p,$
$\mu\colon R \to \mathbb{Z}_p, \nu\colon R \to \mathbb{Z}_p$ *and* $\psi\colon R \to \mathbb{Z}_p$ *the following statements hold*:

(0) $\beta(0) \equiv 0 \pmod p$, $\gamma(0) \equiv 0 \pmod p$, $\varphi(0) \equiv 0 \pmod p$, $\lambda(0) \equiv 0 \pmod p$, $\mu(0) \equiv 0 \pmod p$, $\nu(0) \equiv 0 \pmod p$ *and* $\psi(0) \equiv 0 \pmod p$ *if and only if the nearring R is zero-symmetric;*

(1) $\alpha(x) \equiv 0 \pmod p$ *and* $\lambda(x) \equiv 0 \pmod p$;

(2) *if* $\beta(x) \equiv 0 \pmod p$, *then* $x_1 \equiv 0 \pmod p$;

(3) $\beta(xy) \equiv \beta(x)\beta(y) + \mu(x)\varphi(y) \pmod p$;

(4) $\gamma(xy) \equiv \gamma(x)\beta(y) + x_1\beta(x)\gamma(y) + \nu(x)\varphi(y) \pmod p$;

(5) $\varphi(xy) \equiv \varphi(x)\beta(y) + \psi(x)\varphi(y) \pmod p$;

(6) $\mu(xy) \equiv \beta(x)\mu(y) + \mu(x)\psi(y) \pmod p$;

(7) $\nu(xy) \equiv \gamma(x)\mu(y) + x_1\beta(x)\gamma(y) + \nu(x)\psi(y) \pmod p$;

(8) $\psi(xy) \equiv \varphi(x)\mu(y) + \psi(x)\psi(y) \pmod p$.

Proof. If $|L| = p^3$, then $L = \langle b\rangle + \langle c\rangle + \langle d\rangle$. Since L is the (R,R)-subgroup in R^+ by statement 1) of Lemma 2 it follows that $xb \in L$ and $xd \in L$, hence $a\alpha(x) \in L$ and $a\lambda(x) \in L$ for each $x \in R$. Thus $\alpha(x) \equiv 0 \pmod p$ and $\lambda(x) \equiv 0 \pmod p$, so we get statement (1). Substituting the obtained value of $\alpha(x) \equiv 0 \pmod p$ and $\lambda(x) \equiv 0 \pmod p$ in statements (2)–(4) and (6)–(8) from Lemma 6, we obtain statement (3)–(8) of the lemma and the formula for multiplication $(*)$. Putting $y = c$, we get $xc = c(x_1\beta(x))$. Hence, if $\beta(x) \equiv 0 \pmod p$, then $xc = 0$, and so $x \in L$. Therefore, $x_1 \equiv 0 \pmod p$, as claimed in statement (2). Indeed, statement (0) repeats the statement (0) of Lemma 6. □

Next, we give examples of local nearrings.

Lemma 8. *Let R be a local nearring whose additive group of R^+ is isomorphic to G_4 and $|R : L| = p$. If $x = ax_1 + bx_2 + cx_3 + dx_4$, $y = ay_1 + by_2 + cy_3 + dy_4 \in R$, then the mappings $\beta\colon R \to \mathbb{Z}_p$, $\gamma\colon R \to \mathbb{Z}_p$, $\varphi\colon R \to \mathbb{Z}_p$, $\mu\colon R \to \mathbb{Z}_p$, $\nu\colon R \to \mathbb{Z}_p$ and $\psi\colon R \to \mathbb{Z}_p$ from multiplication $(*)$ can be one of the following*:

1) $\beta(x) = {x_1}^i$ *and* $\psi(x) = {x_1}^i$ *(*$0 < i < p$*)*, $\gamma(x) = \varphi(x) = \mu(x) = \nu(x) = 0$;

2) $\beta(x) = 1$ *and* $\psi(x) = 1$, $\gamma(x) = \varphi(x) = \mu(x) = \nu(x) = 0$.

Proof. It is easy to check that the functions from statements 1) and 2) satisfy conditions 2)–8) of Theorem 7. □

As a consequence of Lemma 8 we have the following result.

Theorem 4. *For each odd prime p there exists a local nearring R whose additive group* R^+ *is isomorphic to* G_4*.*

Example 3. Let $G \cong C_3 \times ((C_3 \times C_3) \rtimes C_3)$. If $x = ax_1 + bx_2 + cx_3 + dx_4$ and $y = ay_1 + by_2 + cy_3 + dy_4 \in G$ and $(G, +, \cdot)$ is a local nearring, then by Lemma 8 " $\cdot$ " can be one of the following multiplications:

(1) $x \cdot y = ax_1y_1 + b(x_2y_1 + x_1y_2) + c(-x_1x_2\binom{y_1}{2} + x_3y_1 + x_1^2y_3) + d(x_4y_1 + x_1y_4)$;

(2) $x \cdot y = ax_1y_1 + b(x_2y_1 + x_1^2y_2) + c(-x_1x_2\binom{y_1}{2} + x_3y_1 + y_3) + d(x_4y_1 + x_1^2y_4)$;

(3) $x \cdot y = ax_1y_1 + b(x_2y_1 + y_2) + c(-x_1x_2\binom{y_1}{2} + x_3y_1 + x_1y_3) + d(x_4y_1 + y_4)$.

A computer program verified that for $p = 3$ the nearring obtained in Lemma 8 is indeed a local nearring (see Example 3), is deposited on GitHub:
`https://github.com/raemarina/Examples/blob/main/LNR_81-12.txt`.

From the package LocalNR and [24] we have the following number of all non-isomorphic zero-symmetric local nearrings on G_4 of orders 81 and 625.

$StructureDescription(R^+)$	$n(R^+)$
$C_3 \times ((C_3 \times C_3) \rtimes C_3)$	**794**
$\lvert R : L\rvert = 3$	782
$\lvert R : L\rvert = 9$	12
$C_5 \times ((C_5 \times C_5) \rtimes C_5)$	**2090**
$\lvert R : L\rvert = 5$	2078
$\lvert R : L\rvert = 25$	12

3.5. The group G_5

Let G_5 be additively written group from Theorem 3. Then $G_5 = \langle a \rangle + \langle b \rangle + \langle c \rangle$ for some elements a, b and c of R satisfying the relations $ap^2 = bp = cp = 0$, $a + b = b + a(1 - p)$, $a + c = c + a$ and $b + c = c + b$.

Recall that the exponent of a finite p-group is the maximal order of its elements. The following assertion is easily verified.

Lemma 9. *If x is an element of maximal order in G_5, then there exist generators a, b and c of this group such that $a = x$ and the relations $ap^2 = bp = cp = 0$, $-b + a + b = a(1 - p)$, $a + c = c + a$, $b + c = c + b$ hold.*

Lemma 10. *For any natural numbers k, r, s and t in the group G_5 the equalities $ck+bs+ar = ar(1 + sp) + bs + ck$ and $(ar+bs+ck)t = ar(t+s\binom{t}{2}p)+bst+ckt$ hold.*

Proof. Let $q = 1+p$. Since $-b+a+b = a(1-p)$, $a+c = c+a$ and $b+c = c+b$ it follows $c + b + a = aq + b + c$, so $ck + bs + ar = arq^s + bs + ck$ for arbitrary integers $k \geq 0$, $r \geq 0$ and $s \geq 0$. Taking into consideration, that

$$q^s = (1 + p)^s \equiv 1 + sp \pmod{p^2}$$

by binomial's formula, giving $ck + bs + ar = ar(1 + sp) + bs + ck$. Next, $(ar + bs + ck)t = ar(1 + q^s + \cdots + q^{s(t-1)}) + bst + ckt$ by induction on t. Therefore, $1+q^s+\cdots+q^{s(t-1)} \equiv 1+(1+sp)+\cdots+(1+s(t-1)p) = t+s\binom{t}{2}p \pmod{p^2}$, thus $(ar + bs + ckt)t = ar(t + s\binom{t}{2}p) + bst + ckt$. □

3.6. Nearrings with identity whose additive groups are isomorphic to G_5

Let R be a nearring with identity whose additive group R^+ is isomorphic to G_5. Then $R^+ = \langle a\rangle + \langle b\rangle + \langle c\rangle$ with elements a, b and c, where a coincides with identity element of R and the relations $ap^2 = bp = cp = 0$, $a + b = b + a(1-p)$, $a + c = c + a$, $b + c = c + b$ are valid. Moreover, each element $x \in R$ is uniquely written in the form $x = ax_1+bx_2+cx_3$ with coefficients $0 \leq x_1 < p^2$, $0 \leq x_2 < p$ and $0 \leq x_3 < p$.

Consider a coincides with identity element of R, so that $xa = ax = x$ for each $x \in R$. Furthermore, for each $x \in R$ there exist integers $\alpha(x)$, $\beta(x)$, $\gamma(x)$, $\nu(x)$, $\mu(x)$ and $\phi(x)$ such that $xb = a\alpha(x) + b\beta(x) + c\gamma(x)$ and $xc = a\nu(x)+b\mu(x)+c\phi(x)$. It is clear that modulo p^2, p, p and p^2, p, p, respectively, these integers are uniquely determined by x and so some mappings $\alpha\colon R \to \mathbb{Z}_{p^2}$, $\beta\colon R \to \mathbb{Z}_p$, $\gamma\colon R \to \mathbb{Z}_p$, $\nu\colon R \to \mathbb{Z}_{p^2}$, $\mu\colon R \to \mathbb{Z}_p$ and $\phi\colon R \to \mathbb{Z}_p$ are determined.

Lemma 11. *Let $x = ax_1 + bx_2 + cx_3$ and $y = ay_1 + by_2 + cy_3$ be elements of R. If a coincides with identity element of R, then*

$$\begin{aligned} xy = a(x_1y_1 + \alpha(x)y_2 + x_1x_2\tbinom{y_1}{2}p + \nu(x)y_3)+ \\ + b(x_2y_1 + \beta(x)y_2 + \mu(x)y_3) + c(x_3y_1 + \gamma(x)y_2 + \phi(x)y_3). \quad (**) \end{aligned}$$

Moreover, for the mappings

$$\alpha\colon R \to \mathbb{Z}_{p^2}, \beta\colon R \to \mathbb{Z}_p, \gamma\colon R \to \mathbb{Z}_p,\ \nu\colon R \to \mathbb{Z}_{p^2},$$
$\mu\colon R \to \mathbb{Z}_p$ *and* $\phi\colon R \to \mathbb{Z}_p$ *the following statements hold:*

(0) $\alpha(0) = \beta(0) = \gamma(0) = \nu(0) = \mu(0) = \phi(0) = 0$ *if and only if the nearring R is zero-symmetric*;

(1) $\alpha(a) = 0$, $\beta(a) = 1$, $\gamma(a) = 0$, $\nu(c) = 0$, $\mu(c) = 0$ *and* $\phi(c) = 1$;

(2) $\alpha(x) \equiv 0 \pmod p$ *and* $\nu(x) \equiv 0 \pmod p$;

(3) $\alpha(xy) = x_1\alpha(y) + \alpha(x)\beta(y) + x_1x_2\binom{\alpha(y)}{2}p + \nu(x)\gamma(y) \pmod{p^2}$;

(4) $\beta(xy) = x_2\alpha(y) + \beta(x)\beta(y) + \mu(x)\gamma(y) \pmod p$;

(5) $\gamma(xy) = x_3\alpha(y) + \gamma(x)\beta(y) + \phi(x)\gamma(y) \pmod p$;

(6) $\nu(xy) = x_1\nu(y) + \alpha(x)\mu(y) + x_1x_2\binom{\mu(y)}{2}p + \nu(x)\phi(y) \pmod{p^2}$;

(7) $\mu(xy) = x_2\nu(y) + \beta(x)\mu(y) + \mu(x)\phi(y) \pmod p$;

(8) $\phi(xy) = x_3\nu(y) + \gamma(x)\mu(y) + \phi(x)\phi(y) \pmod p$.

Proof. By the left distributive law, we have

$$xy = (xa)y_1 + (xb)y_2 + (xc)y_3 = (ax_1 + bx_2 + cx_3)y_1 + (a\alpha(x)+ \\ +b\beta(x) + c\gamma(x))y_2 + (a\nu(x) + b\mu(x) + c\phi(x))y_3.$$

Furthermore, Lemma 10 implies that

$$(ax_1 + bx_2 + cx_3)y_1 = ax_1(y_1 + x_2\binom{y_1}{2}p) + bx_2y_1 + cx_3y_1,$$

$$(a\alpha(x) + b\beta(x) + c\gamma(x))y_2 = \\ = a\alpha(x)(y_2 + \beta(x)\binom{y_2}{2}p) + b\beta(x)y_2 + c\gamma(x)y_2,$$

and

$$(a\nu(x) + b\mu(x) + c\phi(x))y_3 = \\ = a\nu(x)(y_3 + \mu(x)\binom{y_3}{2}p) + b\mu(x)y_3 + c\phi(x)y_3.$$

By Lemma 10, we have

$$bx_2y_1 + a\alpha(x)(y_2 + \beta(x)\binom{y_2}{2}p) \\ = a\alpha(x)(y_2 - \beta(x)\binom{y_2}{2}p)(1 - x_2y_1p) + bx_2y_1$$

and

$$b\beta(x)y_2 + a\nu(x)(y_3 + \mu(x)\binom{y_3}{2}p) \\ = a\nu(x)(y_2 - \mu(x)\binom{y_2}{2}p)(1 - \beta(x)y_2) + b\beta(x)y_2.$$

Thus we get

$$xy = a((x_1y_1 + \alpha(x)y_2) + (x_1x_2\binom{y_1}{2} + \alpha(x)y_2 + \alpha(x)x_2y_1y_2p+ \\ + \alpha(x)\beta(x)\binom{y_2}{2})p + \nu(x)y_3 + \nu(x)x_2y_1y_2p + \nu(x)\beta(x)y_2y_3p+ \\ + \nu(x)\mu(x)\binom{y_3}{2}p) + b(x_2y_1 + \beta(x)y_2 + \mu(x)y_3)+ \\ + c(x_3y_1 + \gamma(x)y_2 + \phi(x)y_3).$$

As $0 \cdot a = a \cdot 0 = 0$, the nearring R is zero-symmetric if and only if $0 = 0 \cdot b = a\alpha(0) + b\beta(0) + c\gamma(0)$ and $0 = 0 \cdot c = a\nu(0) + b\mu(0) + c\phi(0)$ whence

$\alpha(0) = \beta(0) = \gamma(0) = \nu(0) = \mu(0) = \phi(0) = 0$. Similarly, from the equalities $b = ab = a\alpha(a) + b\beta(a)$ and $c = ac = a\nu(c) + b\mu(c) + c\phi(c)$ it follows that $\alpha(a) = 0$, $\beta(a) = 1$, $\gamma(a) = 0$, $\nu(c) = 0$, $\mu(c) = 0$ and $\phi(c) = 1$, we obtain statement (1). Since $(xb)p = x(bp) = 0$ and $xb = a\alpha(x)+b\beta(x)+c\gamma(x)$, we have $0 = (a\alpha(x)+b\beta(x)+c\gamma(x))p = a\alpha(x)(p+\beta(x)\binom{p}{2}p) = a\alpha(x)p$ by Lemma 10 and hence $\alpha(x) \equiv 0 \pmod p$. Moreover, $(xc)p = x(cp) = 0$ and $xc = a\nu(x)+b\mu(x)+c\phi(x)$, we have $0 = (a\nu(x) + b\mu(x) + c\phi(x))p = a\nu(x)(p + \mu(x)\binom{p}{2}p) = a\nu(x)p$ by Lemma 10 and hence $\nu(x) \equiv 0 \pmod p$, and so statement (2). Therefore we obtain

$$xy = a(x_1y_1 + \alpha(x)y_2 + x_1x_2\tbinom{y_1}{2}p + \nu(x)y_3)+ \\ + b(x_2y_1 + \beta(x)y_2 + \mu(x)y_3) + c(x_3y_1 + \gamma(x)y_2 + \phi(x)y_3),$$

as desired in $(**)$.

Finally, the associativity of multiplication in R implies that $x(yb) = (xy)b = a\alpha(xy) + b\beta(xy) + c\gamma(xy)$ and $x(yc) = (xy)c = a\nu(xy) + b\mu(xy) + c\phi(xy)$. Furthermore, substituting $yb = a\alpha(y) + b\beta(y) + c\gamma(y)$ instead of y in formula $(**)$, we also have

$$x(yb) = a(x_1\alpha(y) + \alpha(x)\beta(y) + x_1x_2\tbinom{\alpha(y)}{2}p + \nu(x)\gamma(y)) + b(x_2\alpha(y)+ \\ + \beta(x)\beta(y) + \mu(x)\gamma(y)) + c(x_3\alpha(y) + \gamma(x)\beta(y) + \phi(x)\gamma(y)).$$

Comparing the coefficients under a and b in two expressions obtained for $x(yb)$, we derive statements (3)–(5) of the lemma.

Next, substituting $yc = a\nu(y) + b\mu(y) + c\phi(y)$ instead of y in formula $(**)$, we get

$$x(yc) = a(x_1\nu(y) + \alpha(x)\mu(y) + x_1x_2\tbinom{\mu(y)}{2}p + \nu(x)\phi(y)) + b(x_2\nu(y)+ \\ + \beta(x)\mu(y) + \mu(x)\phi(y)) + c(x_3\nu(y) + \gamma(x)\mu(y) + \phi(x)\phi(y)).$$

Finally, comparing the coefficients under a and b in two expressions obtained for $x(yc)$, we derive statements (6)–(8) of the lemma. □

3.7. Local nearrings whose additive groups are isomorphic to G_5

Let R be a local nearring whose additive group R^+ is isomorphic to G_5. Then $R^+ = \langle a\rangle + \langle b\rangle + \langle c\rangle$ with elements a, b and c, where a coincides with identity element of R and the relations $ap^2 = bp = cp = 0$, $a + b = b + a(1 - p)$, $a + c = c + a$ and $b + c = c + b$ are valid. Moreover, each element $x \in R$ is uniquely written in the form $x = ax_1 + bx_2 + cx_3$ with coefficients $0 \le x_1 < p^2$, $0 \le x_2 < p$ and $0 \le x_3 < p$.

Consider a coincides with identity element of R, so that $xa = ax = x$ for each $x \in R$. Furthermore, for each $x \in R$ there exist integers $\alpha(x)$, $\beta(x)$, $\gamma(x)$, $\nu(x)$, $\mu(x)$ and $\phi(x)$ such that $xb = a\alpha(x) + b\beta(x) + c\gamma(x)$ and $xc =$

$a\nu(x) + b\mu(x) + c\phi(x)$. It is clear that modulo p^2, p, p and p^2, p, p, respectively, these integers are uniquely determined by x and so some mappings $\alpha\colon R \to \mathbb{Z}_{p^2}$, $\beta\colon R \to \mathbb{Z}_p$, $\gamma\colon R \to \mathbb{Z}_p$, $\nu\colon R \to \mathbb{Z}_{p^2}$, $\mu\colon R \to \mathbb{Z}_p$ and $\phi\colon R \to \mathbb{Z}_p$ are determined.

By Corollary 1, L is the normal subgroup of order p^3 or p^2 in R. Through this section let R be a local nearring with $|R : L| = p$.

If $|L| = p^3$, then $L = \langle ap\rangle + \langle b\rangle + \langle c\rangle$. Since $R^* = R \setminus L$ it follows that

$$R^* = \{ax_1 + bx_2 + cx_3 \mid x_1 \not\equiv 0 \pmod p\}$$

and $x = ax_1 + bx_2 + cx_3$ is invertible if and only if $x_1 \not\equiv 0 \pmod p$. Since L is the (R, R)-subgroup in R^+ by statement 1) of Lemma 2 it follows that $xb \in L$ and $xc \in L$, hence $a\alpha(x) \in L$ and $a\nu(x) \in L$ for each $x \in R$. Thus $\alpha(x) \equiv 0 \pmod p$ and $\nu(x) \equiv 0 \pmod p$, as in statement (2) of Theorem 12. Therefore, for local nearrings R we have the same multiplication as for nearrings with identity, i.e. multiplication $(**)$.

Lemma 12. *Let $x = ax_1 + bx_2 + cx_3$ and $y = ay_1 + by_2 + cy_3$ be elements of R and $|R : L| = p$. If a coincides with identity element of R, then multiplication $(**)$ holds for the mappings from Theorem 12.*

Next, we will give examples of local nearrings.

Lemma 13. *Let R be a local nearring whose additive group of R^+ is isomorphic to G_5 and $|R : L| = p$. If $x = ax_1 + bx_2 + cx_3$, $y = ay_1 + by_2 + cy_3 \in R$, then the mappings $\alpha : R \to \mathbb{Z}_{p^2}$, $\beta : R \to \mathbb{Z}_p$, $\gamma : R \to \mathbb{Z}_p$, $\nu : R \to \mathbb{Z}_{p^2}$, $\mu : R \to \mathbb{Z}_p$ and $\phi : R \to \mathbb{Z}_p$ can be one of the following:*

1) $\beta(x) = \phi(x) = \begin{cases} 1, & if\ x_1 \not\equiv 0 \pmod p; \\ 0, & if\ x_1 \equiv 0 \pmod p, \end{cases}$ $\alpha(x) = \gamma(x) = \mu(x) = \nu(x) = 0;$

2) $\beta(x) = \phi(x) = 1$, $\alpha(x) = \gamma(x) = \mu(x) = \nu(x) = 0$.

Proof. It is easy to check that the functions from statements 1) and 2) satisfy conditions 1)–8) of Theorem 7. ☐

As a consequence of Lemma 13 we have the following result.

Theorem 5. *For each odd prime p there exists a local nearring R whose additive group R^+ is isomorphic to G_5.*

Example 4. Let $G \cong (C_9 \rtimes C_3) \times C_3$. If $x = ax_1 + bx_2 + cx_3$ and $y = ay_1 + by_2 + cy_3 \in G$ and $(G, +, \cdot)$ is a local nearring, then by Lemma 8 "$\cdot$" can be one of the following multiplications.

(1) $x \cdot y = a(x_1y_1 + 3x_1x_2\binom{y_1}{2}) + b(x_2y_1 + \beta(x)y_2) + c(x_3y_1 + \phi(x)y_3)$, where

$$\beta(x) = \phi(x) = \begin{cases} 1, & if\ x_1 \not\equiv 0 \pmod 3; \\ 0, & if\ x_1 \equiv 0 \pmod 3, \end{cases}$$

(2) $x \cdot y = a(x_1y_1 + 3x_1x_2\binom{y_1}{2}) + b(x_2y_1 + y_2) + c(x_3y_1 + y_3)$.

A computer program verified that for $p = 3$ the nearring obtained in Lemma 13 is indeed a local nearring (see Example 4), is deposited on GitHub:
`https://github.com/raemarina/Examples/blob/main/LNR_81-13.txt`.

From the package LocalNR and [24] we have the following number of all non-isomorphic local nearrings on groups G_5 of orders 81 and 625.

$StructureDescription(R^+)$	$n(R^+)$
$(C_9 \times C_3) \rtimes C_3$	337
$(C_{25} \times C_5) \rtimes C_5$	630

3.8. The groups G_6

Let G_6 be additively written group from Theorem 3. Then $G_6 = \langle a \rangle + \langle b \rangle + \langle c \rangle$ for some elements a, b and c of R satisfying the relations $ap^2 = 0$, $bp = 0$, $cp = 0$, $a + b = b + a$, $a + c = c + a$, $c + b = b + c + ap$.

Lemma 14. *G_6 is not endocyclic.*

Proof. Consider the group $G_6 = \langle a, b, c \colon ap^2 = bp = cp = 0, a+b = b+a, a+c = c+a, c+b = b+c+ap \rangle$. Since a subgroup $\langle ap \rangle$ is a unique subgroup of order p^2 it follows that $\langle ap \rangle$ is fixed under the action of $\operatorname{End} G_6$. Hence G_6 is not endocyclic by Definition 3. □

Conjecture 1. *There does not exist a local nearring whose additive group is isomorphic to G_6.*

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