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Low-dimensional nilpotent Leibniz algebras and their automorphism groups

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Abstract. Let L be an algebra over a field F with the binary operations $+$ and $[,]$. Then L is called a Leibniz algebra if it satisfies the Leibniz identity: $[a,[b,c]] = [[a,b],c] + [b,[a,c]]$ for all $a, b, c \in L$. A linear transformation f of L is called an endomorphism of L, if $f([a,b]) = [f(a), f(b)]$ for all $a, b \in L$. A bijective endomorphism of L is called an automorphism of L. The set of all automorphisms of Leibniz algebra is a group with respect to the operation of multiplication of automorphisms. The main goal of this article is to describe the structure of the automorphism group of a certain type of nilpotent three-dimensional Leibniz algebras over arbitrary field F.

Introduction

Let L be an algebra over a field F with the binary operations $+$ and $[,]$. Then L is called a *left Leibniz algebra* if it satisfies the *left Leibniz identity*:

$$[a,[b,c]] = [[a,b],c] + [b,[a,c]]$$

for all $a, b, c \in L$ [2].

By analogy, we can define the right Leibniz algebra. We can make the transition from left Leibniz algebra to right Leibniz algebra and vice versa. All results that are valid for left Leibniz algebras are also valid

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for right Leibniz algebras. For some reasons, we prefer to work with left Leibniz algebras. Therefore, in this article, the term "Leibniz algebra" means "left Leibniz algebra". The term "Leibniz algebra" first appeared in the book [13] and the article [14]. The theory of Leibniz algebras is very actively developing in various directions. A significant part of modern results, which are devoted to the properties, structure and classification of Leibniz algebras, is contained in the book [1].

It is important to note that Leibniz algebras are a very wide generalization of Lie algebras. On the other hand, if L is a Leibniz algebra, in which $[a, a] = 0$ for every $a \in L$, then L is a Lie algebra. Therefore, Lie algebras can be characterized as anticommutative Leibniz algebras. Many results from the theory of Lie algebras are very often transferred to Leibniz algebras. In the same time, there is a very significant difference between Lie and Leibniz algebras (see, for example, survey articles [3, 4, 11, 15]).

Let L be a Leibniz algebra over a field F. A linear transformation f of L is called an *endomorphism* of L, if

$$f([a, b]) = [f(a), f(b)]$$

for all $a, b \in L$. A bijective endomorphism of L is called an *automorphism* of L. It is easy to prove that the set $Aut_{[,]}(L)$ of all automorphisms of L is a group by a multiplication of automorphisms (see, for example, [9]).

The description of automorphism groups of any algebraic structure is one of the most important and natural problems of any branch of algebra. Obviously, Leibniz algebras are no exception. It is natural to start searching for the structure of automorphism groups for those Leibniz algebras whose structure is well studied.

The first natural step is to consider one-generated Leibniz algebras. The description of automorphism groups of infinite-dimensional one-generated Leibniz algebras was obtained in the article [12]. A description of automorphism groups of finite-dimensional one-generated Leibniz algebras was obtained in the article [9].

The second natural step is to consider low-dimensional Leibniz algebras. The case of dimension 2 is quite simple. There are only two types of two-dimensional non-isomorphic non-Lie Leibniz algebras (see [4]):

- $Lei_1(2, F) = Fa_1 \oplus Fa_2,\ [a_1, a_1] = a_2,$
  $[a_1, a_2] = [a_2, a_1] = [a_2, a_2] = 0;$
- $Lei_2(2, F) = Fa_1 \oplus Fa_2,\ [a_1, a_1] = [a_1, a_2] = a_2,$
  $[a_2, a_1] = [a_2, a_2] = 0.$

The structures of automorphism groups of such Leibniz algebras were described in [10].

The situation with three-dimensional Leibniz algebras is much more complicated. The most detailed description of such algebras over arbitrary fields can be found in [8]. According to the results of this article, there are 40 types of three-dimensional non-isomorphic non-Lie Leibniz algebras. Thus, we have a rather big problem of describing automorphism groups of three-dimensional Leibniz algebras. Here is a list of three-dimensional Leibniz algebras $Fa_1 \oplus Fa_2 \oplus Fa_3$ for which the structure of automorphism groups has already been found (here we give only non-zero products):

- $Lei_1(3, F):\ [a_1, a_1] = a_2,\ [a_1, a_2] = a_3$ [10];
- $Lei_2(3, F):\ [a_1, a_1] = a_3$ [10];
- $Lei_3(3, F):\ [a_1, a_1] = [a_1, a_2] = a_3$ [6];
- $Lei_4(3, F):\ [a_1, a_1] = a_3,\ [a_2, a_2] = \lambda a_3,\ \lambda \neq 0$ [7];
- $Lei_5(3, F):\ [a_1, a_1] = a_3,\ [a_1, a_2] = a_2 + \lambda a_3$ [5].

In this paper, we continue our research for the next type of three-dimensional Leibniz algebras, the structure of which will be illustrated in the next section.

1. Some information about Leibniz algebras of dimension 3 with a one-dimensional Leibniz kernel

We recall some necessary definitions.

Let L be a Leibniz algebra over a field F. Then L is called *abelian* if $[a, b] = 0$ for every $a, b \in L$. If A, B are subspaces of L, then $[A, B]$ will denote a subspace generated by the elements $[a, b]$, $a \in A$, $b \in B$. A subspace S of L is called a *subalgebra* of L, if $[a, b] \in S$ for all $a, b \in S$. A subalgebra I of L is called a *left* (respectively *right*) *ideal* of L, if $[b, a] \in I$ (respectively $[a, b] \in I$) for every $a \in I$, $b \in L$. A subalgebra I of L is called an *ideal* of L if it is both a left ideal and a right ideal.

Denote by $Leib(L)$ the subspace generated by the elements $[a, a]$, $a \in L$. It is easy to prove that $Leib(L)$ is an ideal of L. The ideal $Leib(L)$ is called the *Leibniz kernel* of L.

The *left* $\zeta^{\text{left}}(L)$ and the *right centers* $\zeta^{\text{right}}(L)$ of L are defined by the following rules:

$$\zeta^{\text{left}}(L) = \{a \in L|\ [a,b] = 0 \text{ for each } b \in L\},$$
$$\zeta^{\text{right}}(L) = \{a \in L|\ [b,a] = 0 \text{ for each } b \in L\}.$$

We note that the left center of L is an ideal, but that is not true for the right center. The right center is a subalgebra of L and, in general, the left and right centers are distinct.

The *center* $\zeta(L)$ of L is defined by the following rule:

$$\zeta(L) = \{a \in L|\ [a,b] = 0 = [b,a] \text{ for each } b \in L\}.$$

The center is an ideal of L.

Define the *lower central series* of L

$$L = \gamma_1(L) \geq \gamma_2(L) \geq \dots \gamma_\alpha(L) \geq \gamma_{\alpha+1}(L) \geq \dots \gamma_\tau(L)$$

by the rule: $\gamma_1(L) = L$, $\gamma_2(L) = [L,L]$, $\gamma_{\alpha+1}(L) = [L, \gamma_\alpha(L)]$ for all ordinals α and $\gamma_\lambda(L) = \bigcap_{\mu<\lambda} \gamma_\mu(L)$ for the limit ordinals λ. We say that L is *nilpotent*, if there exists a positive integer k such that $\gamma_k(L) = \langle 0 \rangle$.

Let L be a non-Lie Leibniz algebra over a field F of dimension 3. Suppose that the center of L includes the Leibniz kernel, $dim_F(Leib(L)) = 1$ and $L/Leib(L)$ is abelian.

We note that the center $\zeta(L)$ has dimension at most 2. Suppose that $\zeta(L)$ has dimension 1. In this case, $\zeta(L) = Leib(L)$. Since L is not a Lie algebra, there is an element a_1 such that $[a_1, a_1] = a_3 \neq 0$. We note that $a_3 \in \zeta(L)$. It follows that $[a_1, a_3] = [a_3, a_1] = [a_3, a_3] = 0$. Then $\zeta(L) = Fa_3$. Since $L/Leib(L)$ is abelian, for every element $x \in L$ we have $[a_1, x], [x, a_1] \in \zeta(L) \leq \langle a_1 \rangle = Fa_1 \oplus Fa_3$. It follows that $\langle a_1 \rangle$ is an ideal of L. Since $dim_F(\langle a_1 \rangle) = 2$, $\langle a_1 \rangle \neq L$.

Suppose that there exists an element b such that $b \notin \langle a_1 \rangle$, $[b,b] = 0$. We have $[b, a_1] = \gamma a_3$ for some $\gamma \in F$. The following two cases appear here: $\gamma = 0$ and $\gamma \neq 0$.

Let $\gamma = 0$. Then $[a_1, b] = \alpha a_3$ for some $\alpha \in F$. If we suppose that $\alpha = 0$, then $b \in \zeta(L)$. But in this case, $dim_F(\zeta(L)) = 2$, and we obtain a contradiction, which shows that $\alpha \neq 0$. Put $a_2 = \alpha^{-1} b$, then $[a_2, a_2] = [a_2, a_1] = 0$, $[a_1, a_2] = a_3$, and we come to Leibniz algebra of type $Lei_3(3, F)$.

Let $\gamma \neq 0$. Put $a_2 = \gamma^{-1} b$, then $[a_2, a_2] = 0$, $[a_2, a_1] = a_3$. We have $[a_1, a_2] = \alpha a_3$ for some $\alpha \in F$. If $\alpha = 0$, then $a_2 \in \zeta^{\text{right}}(L)$, and we come to the following type of nilpotent Leibniz algebras:

$$Lei_6(3,F) = L_6 = Fa_1 \oplus Fa_2 \oplus Fa_3 \text{ where } [a_1,a_1] = [a_2,a_1] = a_3,$$
$$[a_1,a_2] = [a_1,a_3] = [a_2,a_2] = [a_2,a_3] = [a_3,a_1] = [a_3,a_2] = [a_3,a_3] = 0.$$

In other words, L_6 is a direct sum of ideal $A = Fa_1 \oplus Fa_3$ and subalgebra $B = Fa_2$. Moreover, A is a nilpotent one-generated Leibniz algebra of dimension 2, $[A,B] = \langle 0\rangle$, $[B,A] = Fa_3$, $\zeta^{\text{right}}(L_6) = Fa_2 \oplus Fa_3$ and

$$Leib(L_6) = [L_6, L_6] = \zeta^{\text{left}}(L_6) = \zeta(L_6) = Fa_3.$$

Here are some general useful properties of automorphism groups of Leibniz algebras (see Lemmas 2.1, 2.4 in [10]).

Lemma 1. *Let L be a Leibniz algebra over a field F and f be an automorphism of L. Then $f(\zeta^{\text{left}}(L)) = \zeta^{\text{left}}(L)$, $f(\zeta^{\text{right}}(L)) = \zeta^{\text{right}}(L)$, $f(\zeta(L)) = \zeta(L)$, $f([L,L]) = [L,L]$.*

Let L be a Leibniz algebra over a field F, A be a subalgebra of L and $G = Aut_{[,]}(L)$. Put

$$C_G(A) = \{\alpha \in G |\ \alpha(a) = a \text{ for every } a \in A\}.$$

If A is an ideal of L, then put

$$C_G(L/A) = \{\alpha \in G |\ \alpha(a + A) = a + A \text{ for every } a \in L\}.$$

Lemma 2. *Let L be a Leibniz algebra over a field F and $G = Aut_{[,]}(L)$. If A is a G-invariant subalgebra, then $C_G(A)$ and $C_G(L/A)$ are normal subgroups of G.*

2. Automorphism group of $Lei_6(3,F)$.

Theorem 1. *Let G be an automorphism group of Leibniz algebra $Lei_6(3,F)$. Then G is isomorphic to a subgroup H of $GL_3(F)$, consisting of the matrices of the form*

$$\begin{pmatrix} \alpha_1 & 0 & 0 \\ \alpha_2 & \alpha_1 + \alpha_2 & 0 \\ \alpha_3 & \beta_3 & \alpha_1^2 + \alpha_1\alpha_2 \end{pmatrix}$$

where $\alpha_1 \neq 0$, $\alpha_1 + \alpha_2 \neq 0$. *A subgroup* H *is a semidirect product of normal subgroup* $S_3(L, F)$, *which is isomorphic to a subgroup of* $GL_3(F)$, *consisting of the matrices of the form*

$$\begin{pmatrix} 1 & 0 & 0 \\ \alpha_2 & 1+\alpha_2 & 0 \\ \alpha_3 & \beta_3 & 1+\alpha_2 \end{pmatrix}$$

and a subgroup $D_3(L, F)$, *consisting of the matrices of the form*

$$\begin{pmatrix} \sigma & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & \sigma^2 \end{pmatrix}.$$

In particular, $D_3(L, F)$ *is isomorphic to multiplicative group of a field* F. *Furthermore,* $S_3(L, F)$ *is a semidirect product of subgroup* $T_3(L, F)$, *which is normal in* G *and isomorphic to a subgroup of* $GL_3(F)$, *consisting of the matrices of the form*

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \alpha_3 & \beta_3 & 1 \end{pmatrix},$$

and a subgroup $J_3(L, F)$, *which is isomorphic to a subgroup of* $GL_3(F)$, *consisting of the matrices of the form*

$$\begin{pmatrix} 1 & 0 & 0 \\ \lambda & 1+\lambda & 0 \\ 0 & 0 & 1+\lambda \end{pmatrix}.$$

A subgroup $T_3(L, F)$ *is isomorphic to direct product of two copy of additive group of a field* F, *and a subgroup* $J_3(L, F)$ *is isomorphic to multiplicative group of a field* F.

Proof. Let $L = Lei_6(3, F)$, $f \in Aut_{[,]}(L)$. By Lemma 1,

$$f(Fa_3) = Fa_3 = \zeta(L) = [L, L],$$
$$f(Fa_2 \oplus Fa_3) = Fa_2 \oplus Fa_3 = \zeta^{\text{right}}(L),$$

so that

$$f(a_1) = \alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3,$$
$$f(a_2) = \beta_2 a_2 + \beta_3 a_3,$$
$$f(a_3) = \gamma_3 a_3.$$

Then

$$\begin{aligned} f(a_3) &= f([a_1, a_1]) = [f(a_1), f(a_1)] \\ &= [\alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3, \alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3] \\ &= \alpha_1^2 [a_1, a_1] + \alpha_1 \alpha_2 [a_2, a_1] = \alpha_1^2 a_3 + \alpha_1 \alpha_2 a_3 \\ &= (\alpha_1^2 + \alpha_1 \alpha_2) a_3; \\ f(a_3) &= f([a_2, a_1]) = [f(a_2), f(a_1)] \\ &= [\beta_2 a_2 + \beta_3 a_3, \alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3] \\ &= \alpha_1 \beta_2 [a_2, a_1] = \alpha_1 \beta_2 a_3. \end{aligned}$$

Thus, we obtain an equality $\alpha_1(\alpha_1 + \alpha_2) = \alpha_1\beta_2$. Being an automorphism, f is a non-degenerate linear transformation, so that $\alpha_1 \neq 0$. It follows that $\alpha_1 + \alpha_2 = \beta_2$. Thus, an automorphism f has in basis $\{a_1, a_2, a_3\}$ the following matrix

$$\begin{pmatrix} \alpha_1 & 0 & 0 \\ \alpha_2 & \alpha_1 + \alpha_2 & 0 \\ \alpha_3 & \beta_3 & \alpha_1^2 + \alpha_1 \alpha_2 \end{pmatrix}.$$

Denote by Ξ the canonical monomorphism of $Aut_{[,]}(L)$ in $M_3(F)$.

We note that the right center $\zeta^{\text{right}}(Lei_6(3, F)) = Fa_2 \oplus Fa_3$ is an ideal of L. Put

$$\begin{aligned} S &= \{f|\ f \in End(L), f(a_1) \in a_1 + \zeta^{\text{right}}(L)\} \\ &= C_{End(L)}(L/\zeta^{\text{right}}(L)). \end{aligned}$$

If $f \in S \cap Aut_{[,]}(L)$, then

$$\begin{aligned} f(a_1) &= a_1 + \alpha_2 a_2 + \alpha_3 a_3, \\ f(a_2) &= (1 + \alpha_2) a_2 + \beta_3 a_3, \\ f(a_3) &= (1 + \alpha_2) a_3. \end{aligned}$$

If x is an arbitrary element of L, that is $x = \xi_1 a_1 + \xi_2 a_2 + \xi_3 a_3$, then

$$\begin{aligned} f(x) &= \xi_1 f(a_1) + \xi_2 f(a_2) + \xi_3 f(a_3) \\ &= \xi_1 a_1 + \xi_1 \alpha_2 a_2 + \xi_1 \alpha_3 a_3 + \xi_2((1 + \alpha_2) a_2 + \beta_3 a_3) + \xi_3 (1 + \alpha_2) a_3 \\ &= \xi_1 a_1 + (\xi_1 \alpha_2 + \xi_2 + \xi_2 \alpha_2) a_2 + (\xi_1 \alpha_3 + \xi_2 \beta_3 + \xi_3 (1 + \alpha_2)) a_3. \end{aligned}$$

Conversely, let λ, μ, ν be the elements of F, v be a linear transformation of L, defined by the rule: if $x = \xi_1 a_1 + \xi_2 a_2 + \xi_3 a_3$, then

$$v(x) = \xi_1 a_1 + (\xi_1 \lambda + \xi_2 + \xi_2 \lambda) a_2 + (\xi_1 \mu + \xi_2 \nu + \xi_3 (1 + \lambda)) a_3.$$

Let x, y be the arbitrary elements of algebra L, $x = \xi_1 a_1 + \xi_2 a_2 + \xi_3 a_3$, $y = \eta_1 a_1 + \eta_2 a_2 + \eta_3 a_3$, where $\xi_1, \xi_2, \xi_3, \eta_1, \eta_2, \eta_3 \in F$. Then

$$\begin{aligned}
[x, y] &= [\xi_1 a_1 + \xi_2 a_2 + \xi_3 a_3, \eta_1 a_1 + \eta_2 a_2 + \eta_3 a_3] \\
&= \xi_1 \eta_1 [a_1, a_1] + \xi_2 \eta_1 [a_2, a_1] \\
&= (\xi_1 \eta_1 + \xi_2 \eta_1) a_3; \\
v([x, y]) &= v((\xi_1 \eta_1 + \xi_2 \eta_1) a_3) = (\xi_1 \eta_1 + \xi_2 \eta_1) v(a_3) \\
&= (\xi_1 \eta_1 + \xi_2 \eta_1)(1 + \lambda) a_3; \\
[v(x), v(y)] &= [\xi_1 a_1 + (\xi_1 \lambda + \xi_2 + \xi_2 \lambda) a_2 + (\xi_1 \mu + \xi_2 \nu + \xi_3 (1 + \lambda)) a_3, \\
&\quad \eta_1 a_1 + (\eta_1 \lambda + \eta_2 + \eta_2 \lambda) a_2 + (\eta_1 \mu + \eta_2 \nu + \eta_3 (1 + \lambda)) a_3] \\
&= \xi_1 \eta_1 [a_1, a_1] + \eta_1 (\xi_1 \lambda + \xi_2 + \xi_2 \lambda)[a_2, a_1] \\
&= (\xi_1 \eta_1 + \eta_1 (\xi_1 \lambda + \xi_2 + \xi_2 \lambda)) a_3 \\
&= (\xi_1 \eta_1 + \xi_1 \eta_1 \lambda + \xi_2 \eta_1 + \xi_2 \eta_1 \lambda) a_3 \\
&= (\xi_1 \eta_1 + \xi_2 \eta_1)(1 + \lambda) a_3,
\end{aligned}$$

so that $v([x, y]) = [v(x), v(y)]$. It shows that $S \leqslant Aut_{[,]}(L)$. Moreover, by Lemma 2, S is a normal subgroup of $Aut_{[,]}(L)$.

Put $S_3(L, F) = \Xi(S)$. Then $S_3(L, F)$ is a subgroup of a group $T_3(F)$, which consist of the matrices, having the following form:

$$\begin{pmatrix} 1 & 0 & 0 \\ \alpha_2 & 1 + \alpha_2 & 0 \\ \alpha_3 & \beta_3 & 1 + \alpha_2 \end{pmatrix}.$$

Let

$$\begin{aligned}
T &= \{f \mid f \in End(L), f(a_1) \in a_1 + [L, L], f(a_2) \in a_2 + [L, L]\} \\
&= C_{End(L)}(L/[L, L]).
\end{aligned}$$

If $f \in T \cap Aut_{[,]}(L)$, then

$$\begin{aligned}
f(a_1) &= a_1 + \alpha_3 a_3, \\
f(a_2) &= a_2 + \beta_3 a_3, \\
f(a_3) &= a_3.
\end{aligned}$$

If x is an arbitrary element of L, that is $x = \xi_1 a_1 + \xi_2 a_2 + \xi_3 a_3$, then

$$\begin{aligned}
f(x) &= \xi_1 f(a_1) + \xi_2 f(a_2) + \xi_3 f(a_3) \\
&= \xi_1 a_1 + \xi_1 \alpha_3 a_3 + \xi_2 a_2 + \xi_2 \beta_3 a_3 + \xi_3 a_3 \\
&= \xi_1 a_1 + \xi_2 a_2 + (\xi_1 \alpha_3 + \xi_2 \beta_3 + \xi_3) a_3.
\end{aligned}$$

Conversely, let λ, μ be the elements of F, z be a linear transformation of L, defined by the rule: if $x = \xi_1 a_1 + \xi_2 a_2 + \xi_3 a_3$, then

$$z(x) = \xi_1 a_1 + \xi_2 a_2 + (\xi_1\lambda + \xi_2\mu + \xi_3)a_3.$$

Let x, y be the arbitrary elements of algebra L, $x = \xi_1 a_1 + \xi_2 a_2 + \xi_3 a_3$, $y = \eta_1 a_1 + \eta_2 a_2 + \eta_3 a_3$, where $\xi_1, \xi_2, \xi_3, \eta_1, \eta_2, \eta_3 \in F$. Then

$$\begin{aligned}
[x, y] &= [\xi_1 a_1 + \xi_2 a_2 + \xi_3 a_3, \eta_1 a_1 + \eta_2 a_2 + \eta_3 a_3] \\
&= \eta_1(\xi_1 + \xi_2)a_3; \\
z([x, y]) &= z(\eta_1(\xi_1 + \xi_2)a_3) = \eta_1(\xi_1 + \xi_2)z(a_3) \\
&= \eta_1(\xi_1 + \xi_2)a_3; \\
[z(x), z(y)] &= [\xi_1 a_1 + \xi_2 a_2 + (\xi_1\lambda + \xi_2\mu + \xi_3)a_3, \\
&\quad \eta_1 a_1 + \eta_2 a_2 + (\eta_1\lambda + \eta_2\mu + \eta_3)a_3] \\
&= \xi_1\eta_1[a_1, a_1] + \xi_2\eta_1[a_1, a_2] = \eta_1(\xi_1 + \xi_2)a_3,
\end{aligned}$$

so that $z([x, y]) = [z(x), z(y)]$. It shows that T is a subgroup of $Aut_{[,]}(L)$.

Put $T_3(L, F) = \Xi(T)$. Then $T_3(L, F)$ is a subgroup of a group $UT_3(F)$ of all unitriangular matrices over a field F, which consist of the matrices, having the following form:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \alpha_3 & \beta_3 & 1 \end{pmatrix}.$$

It is not hard to see, that $T_3(L, F)$ is abelian and it is isomorphic to direct product of two copy of additive group of a field F. Clearly,

$$\Xi(Aut_{[,]}(L)) \cap UT_3(F) = T_3(L, F),$$

so it follows that a subgroup T is normal in $Aut_{[,]}(L)$.

Let $\lambda \in F$. Put

$$J = \{f |\ f \in S, f(a_1) = a_1 + \lambda a_2, f(a_2) = (1 + \lambda)a_2, f(a_3) = (1 + \lambda)a_3\}.$$

Put $J_3(L, F) = \Xi(J)$. Then $J_3(L, F)$ is a subset of $T_3(F)$, which consist of the matrices, having the following form:

$$\begin{pmatrix} 1 & 0 & 0 \\ \lambda & 1 + \lambda & 0 \\ 0 & 0 & 1 + \lambda \end{pmatrix}.$$

An equality

$$\begin{pmatrix} 1 & 0 & 0 \\ \lambda & 1+\lambda & 0 \\ 0 & 0 & 1+\lambda \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ \mu & 1+\mu & 0 \\ 0 & 0 & 1+\mu \end{pmatrix} =$$
$$\begin{pmatrix} 1 & 0 & 0 \\ (1+\lambda)(1+\mu)-1 & (1+\lambda)(1+\mu) & 0 \\ 0 & 0 & (1+\lambda)(1+\mu) \end{pmatrix}$$

shows that $J_3(L, F)$ is a subgroup of $S_3(L, F)$. Moreover, it is not hard to see that $J_3(L, F)$ is isomorphic to multiplicative group of a field F. Furthermore, it is not hard to see that the matrix equation

$$\begin{pmatrix} 1 & 0 & 0 \\ \alpha_2 & 1+\alpha_2 & 0 \\ \alpha_3 & \beta_3 & 1+\alpha_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ x & y & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ z & 1+z & 0 \\ 0 & 0 & 1+z \end{pmatrix} =$$
$$\begin{pmatrix} 1 & 0 & 0 \\ z & 1+z & 0 \\ x+yz & y+yz & 1+z \end{pmatrix}$$

has the solutions. It proves that $S_3(L, F) = T_3(L, F)J_3(L, F)$ and therefore $S = TJ$. Clearly, the intersection $T \cap J$ is trivial.

Let

$$D = \{f |\ f \in Aut_{[,]}(L), f(a_1) = \sigma a_1, f(a_2) = \sigma a_2, \sigma \in F\}.$$

By proved above, $f(a_3) = \sigma^2 a_3$.

Put $D_3(L, F) = \Xi(D)$. Then $D_3(L, F)$ is a subset of $T_3(F)$, which consist of the matrices, having the following form:

$$\begin{pmatrix} \sigma & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & \sigma^2 \end{pmatrix}.$$

An equality

$$\begin{pmatrix} \sigma & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & \sigma^2 \end{pmatrix} \begin{pmatrix} \nu & 0 & 0 \\ 0 & \nu & 0 \\ 0 & 0 & \nu^2 \end{pmatrix} = \begin{pmatrix} \sigma\nu & 0 & 0 \\ 0 & \sigma\nu & 0 \\ 0 & 0 & \sigma^2\nu^2 \end{pmatrix}$$

shows that $D_3(L, F)$ is a subgroup of $\Xi(Aut_{[,]}(L))$. Moreover, it is not hard to see that $D_3(L, F)$ is isomorphic to multiplicative group of a field F. Furthermore, an equality

$$\begin{pmatrix} \alpha_1 & 0 & 0 \\ \alpha_2 & \alpha_1+\alpha_2 & 0 \\ \alpha_3 & \beta_3 & \alpha_1^2+\alpha_1\alpha_2 \end{pmatrix} =$$
$$\begin{pmatrix} 1 & 0 & 0 \\ \alpha_2\alpha_1^{-1} & 1+\alpha_2\alpha_1^{-1} & 0 \\ \alpha_3\alpha_1^{-1} & \beta_3\alpha_1^{-1} & 1+\alpha_2\alpha_1^{-1} \end{pmatrix} \begin{pmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_1 & 0 \\ 0 & 0 & \alpha_1^2 \end{pmatrix}$$

proves that $\Xi(Aut_{[,]}(L)) = S_3(L,F)D_3(L,F)$, that is $Aut_{[,]}(L) = SD$. Clearly, the intersection $S \cap D$ is trivial. Thus, we obtain that

$$Aut_{[,]}(L) = S \leftthreetimes D,$$

$D \cong F^{\times}$, $S = T \leftthreetimes J$, T is normal in $Aut_{[,]}(L)$, $T \cong F_+ \times F_+$, $J \cong F^{\times}$. $\square$

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