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# Extended binary Golay codes by a group algebra

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Abstract. We study the construction of extended binary Golay codes with respect to the principle (left) ideals of the group algebra $\mathbb{F}_2G$ of a group $G$ of order 24 over a field of two elements $\mathbb{F}_2$. All elements $v \in \mathbb{F}_2G$, which generate the principle ideals that define extended binary Golay codes, have been found programmatically.

## Introduction

Extended Golay codes have been first introduced by Marcel J. E. Golay in the paper [1] in 1949. These codes are examples of extreme binary self-dual codes of Type II (linear binary self-dual codes with Hamming distance between arbitrary codewords to be multiples of 4 that has the highest possible minimum Hamming distance among such codes with a fixed dimension of the codeword space and their length). The mathematical significance of the Golay code goes far beyond its error-correcting properties. The extended Golay binary code is associated with the Mathieu group $M_{24}$. In addition, this code is the main element in the construction of the 24-dimensional Leach lattice.

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Extended binary Golay codes have been studied for a long time and many different constructions have been established to build these codes. We consider the construction of linear binary codes proposed by T. Hurley in [2]. The method implements a pioneering approach, proposed by C. D. Berman, later by K. H. Zimmerman [14], which considers one-sided ideals in group algebras of finite groups over finite fields as codes above the same fields.

First we are going to describe the used construction and define the extended Golay binary codes.

Let $\mathbb{F}_2$ be a field with two elements, $G = \{g_1, g_2, \ldots, g_n\}$ be a finite group of order $n$ and $v = \alpha_{g_1}g_1 + \alpha_{g_2}g_2 + \ldots + \alpha_{g_n}g_n$ be an element of the group algebra $\mathbb{F}_2G$ ($\alpha_i \in \mathbb{F}_2$). Let us define $\sigma(v) \in M(n, \mathbb{F}_2)$ as the following matrix

$$\sigma(v) = \begin{pmatrix} \alpha_{g_1^{-1}g_1} & \alpha_{g_1^{-1}g_2} & \cdots & \alpha_{g_1^{-1}g_n} \\ \alpha_{g_2^{-1}g_1} & \alpha_{g_2^{-1}g_2} & \cdots & \alpha_{g_2^{-1}g_n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{g_n^{-1}g_1} & \alpha_{g_n^{-1}g_2} & \cdots & \alpha_{g_n^{-1}g_n} \end{pmatrix}.$$

The map $v \to \sigma(v)$ is considered in the representation theory of finite groups over a field as a regular representation of the group algebra $\mathbb{F}_2G$, which corresponds to the arrangement $g_1^{-1}, g_2^{-1}, \ldots, g_n^{-1}$ of the elements of $G$. Let us define for a given element $v \in \mathbb{F}_2G$ a binary code as follows: $C(v)$ is the subspace of the space $\mathbb{F}_2^n$ generated by the rows of the matrix $\sigma(v)$. In the space $\mathbb{F}_2^n$ it is introduced the scalar product $[(v_1, \ldots, v_1), (w_1, \ldots, w_1)] = \sum_{i=1}^{n} v_i w_i$ as well as the corresponding orthogonal complement $C^{\perp} = \{v \in \mathbb{F}_2^n | [v, w] = 0, w \in C\}$. The binary code $C$ is called self-orthogonal if $C \subset C^{\perp}$ and self-dual — if $C = C^{\perp}$. It is clear that the code $C(v)$ is self-orthogonal if $\sigma(v)\sigma(v)^T = 0$. For an element $v = \alpha_{g_1}g_1 + \alpha_{g_2}g_2 + \ldots + \alpha_{g_n}g_n \in \mathbb{F}_2G$ we denote $v^* = \alpha_{g_1}g_1^{-1} + \alpha_{g_2}g_2^{-1} + \ldots + \alpha_{g_n}g_n^{-1} \in \mathbb{F}_2G$. It's obvious that $\sigma(v)^T = \sigma(v^*)$.

In [3], an extended binary Golay code was constructed in the form $C(v)$ for some element $v$ of the group algebra $F_2S_4$, where $S_4$ is the symmetric group of order 24. In [11], a similar result was obtained for another group of order 24, namely the dihedral group $D_{24}$. The following theorem was used in the construction.

**Theorem 1** ([11]). *Let $G$ be a finite group of order 24 with an element $v$ of the group algebra $\mathbb{F}_2G$. If*

*1)* $v = v^*$,

*2)* $v^2 = 0$,

*3)* $\mathrm{rank}(\sigma(v)) = 12$,

*then the code* $C(v)$ *is self-dual.*

It was shown in [4] that from 15 non-isomorphic groups of order 24, it is possible to construct a code also for the groups $C_3 \times D_8$, $(C_6 \times C_2) \rtimes C_2$, $C_2 \times A_4$.

**Theorem 2** ([5])**.** *Let* $G$ *be a finite group of order* 24 *with an element* $v$ *of the group algebra* $\mathbb{F}_2G$*. The code* $C(v)$ *is self-dual if and only if*

*1)* $vv^* = 0$,

*2)* $\mathrm{rank}(\sigma(v)) = 12$.

It was established in [5–7] that a binary Golay code can be constructed in this way for the groups $(C_6 \times C_2) \rtimes C_2$, $D_{24}$, $C_3 \times D_8$. Theorem 2 gives a sufficient condition for the code $C(v)$ to be an extended binary Golay code for the elements of the group algebra $F_2G$ of the group $G$ of order 24. In the following section we construct a code by group algebras for the groups $C_2 \times A_4$ and $S_4$. It is shown in [4] that an extended binary Golay code cannot be constructed for the rest of the group of order 24.

## 1. The construction of codes by the groups $C_2 \times A_4$ and $S_4$

**Lemma 1.** *Let* $G = \langle x, y, z, w | x^2 = y^2 = z^2 = w^3 = 1, xy = yx, xz = zx, xw = wx, yz = zy, yw = wyz, zw = wy\rangle$ *be the group* $C_2 \times A_4$*,* $v = \alpha_1 + \alpha_2 y + \alpha_3 z + \alpha_4 yz + \alpha_5 w + \alpha_6 yw + \alpha_7 zw + \alpha_8 yzw + \alpha_9 w^2 + \alpha_{10} yw^2 + \alpha_{11} zw^2 + \alpha_{12} yzw^2 + \alpha_{13} x + \alpha_{14} xy + \alpha_{15} xz + \alpha_{16} xyz + \alpha_{17} xw + \alpha_{18} xyw + \alpha_{19} xzw + \alpha_{20} xyzw + \alpha_{21} xw^2 + \alpha_{22} xyw^2 + \alpha_{23} xzw^2 + \alpha_{24} xyzw^2$*. If the code* $C(v)$ *is self-dual, then*

*1)* $\sum\limits_{i=1}^{24} \alpha_i = 0$;

*2)* $(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)(\alpha_{17} + \alpha_{18} + \alpha_{19} + \alpha_{20} + \alpha_{21} + \alpha_{22} + \alpha_{23} + \alpha_{24}) + (\alpha_5 + \alpha_6 + \alpha_7 + \alpha_8)(\alpha_{13} + \alpha_{14} + \alpha_{15} + \alpha_{16} + \alpha_{21} + \alpha_{22} + \alpha_{23} + \alpha_{24}) + (\alpha_9 + \alpha_{10} + \alpha_{11} + \alpha_{12})(\alpha_{13} + \alpha_{14} + \alpha_{15} + \alpha_{16} + \alpha_{17} + \alpha_{18} + \alpha_{19} + \alpha_{20}) = 0$;

*3)* $(\alpha_1+\alpha_2+\alpha_3+\alpha_4)(\alpha_5+\alpha_6+\alpha_7+\alpha_8+\alpha_9+\alpha_{10}+\alpha_{11}+\alpha_{12})+(\alpha_5+\alpha_6+\alpha_7+\alpha_8)(\alpha_9+\alpha_{10}+\alpha_{11}+\alpha_{12})+(\alpha_{13}+\alpha_{14}+\alpha_{15}+\alpha_{16})(\alpha_{17}+\alpha_{18}+\alpha_{19}+\alpha_{20}+\alpha_{21}+\alpha_{22}+\alpha_{23}+\alpha_{24})+(\alpha_{17}+\alpha_{18}+\alpha_{19}+\alpha_{20})(\alpha_{21}+\alpha_{22}+\alpha_{23}+\alpha_{24})=0.$

Calculations in the group show that the matrix $\sigma(v)\sigma(v)^T=\sigma(vv^*)$ has the following form:

$$\begin{pmatrix}
\gamma_1 & 0 & 0 & 0 & \gamma_2 & \gamma_3 & \gamma_4 & \gamma_5 & \gamma_2 & \gamma_4 & \gamma_5 & \gamma_3 & 0 & 0 & 0 & 0 & \gamma_6 & \gamma_7 & \gamma_8 & \gamma_9 & \gamma_6 & \gamma_8 & \gamma_9 & \gamma_7 \\
0 & \gamma_1 & 0 & 0 & \gamma_3 & \gamma_2 & \gamma_5 & \gamma_4 & \gamma_4 & \gamma_2 & \gamma_3 & \gamma_5 & 0 & 0 & 0 & 0 & \gamma_7 & \gamma_6 & \gamma_9 & \gamma_8 & \gamma_8 & \gamma_6 & \gamma_7 & \gamma_9 \\
0 & 0 & \gamma_1 & 0 & \gamma_4 & \gamma_5 & \gamma_2 & \gamma_3 & \gamma_5 & \gamma_3 & \gamma_2 & \gamma_4 & 0 & 0 & 0 & 0 & \gamma_8 & \gamma_9 & \gamma_6 & \gamma_7 & \gamma_9 & \gamma_7 & \gamma_6 & \gamma_8 \\
0 & 0 & 0 & \gamma_1 & \gamma_5 & \gamma_4 & \gamma_3 & \gamma_2 & \gamma_3 & \gamma_5 & \gamma_4 & \gamma_2 & 0 & 0 & 0 & 0 & \gamma_9 & \gamma_8 & \gamma_7 & \gamma_6 & \gamma_7 & \gamma_9 & \gamma_8 & \gamma_6 \\
\gamma_2 & \gamma_3 & \gamma_4 & \gamma_5 & \gamma_1 & 0 & 0 & 0 & \gamma_2 & \gamma_5 & \gamma_3 & \gamma_4 & \gamma_6 & \gamma_7 & \gamma_8 & \gamma_9 & 0 & 0 & 0 & 0 & \gamma_6 & \gamma_9 & \gamma_7 & \gamma_8 \\
\gamma_3 & \gamma_2 & \gamma_5 & \gamma_4 & 0 & \gamma_1 & 0 & 0 & \gamma_5 & \gamma_2 & \gamma_4 & \gamma_3 & \gamma_7 & \gamma_6 & \gamma_9 & \gamma_8 & 0 & 0 & 0 & 0 & \gamma_9 & \gamma_6 & \gamma_8 & \gamma_7 \\
\gamma_4 & \gamma_5 & \gamma_2 & \gamma_3 & 0 & 0 & \gamma_1 & 0 & \gamma_3 & \gamma_4 & \gamma_2 & \gamma_5 & \gamma_8 & \gamma_9 & \gamma_6 & \gamma_7 & 0 & 0 & 0 & 0 & \gamma_7 & \gamma_8 & \gamma_6 & \gamma_9 \\
\gamma_5 & \gamma_4 & \gamma_3 & \gamma_2 & 0 & 0 & 0 & \gamma_1 & \gamma_4 & \gamma_3 & \gamma_5 & \gamma_2 & \gamma_9 & \gamma_8 & \gamma_7 & \gamma_6 & 0 & 0 & 0 & 0 & \gamma_8 & \gamma_7 & \gamma_9 & \gamma_6 \\
\gamma_2 & \gamma_4 & \gamma_5 & \gamma_3 & \gamma_2 & \gamma_5 & \gamma_3 & \gamma_4 & \gamma_1 & 0 & 0 & 0 & \gamma_6 & \gamma_8 & \gamma_9 & \gamma_7 & \gamma_6 & \gamma_9 & \gamma_7 & \gamma_8 & 0 & 0 & 0 & 0 \\
\gamma_4 & \gamma_2 & \gamma_3 & \gamma_5 & \gamma_5 & \gamma_2 & \gamma_4 & \gamma_3 & 0 & \gamma_1 & 0 & 0 & \gamma_8 & \gamma_6 & \gamma_7 & \gamma_9 & \gamma_9 & \gamma_6 & \gamma_8 & \gamma_7 & 0 & 0 & 0 & 0 \\
\gamma_5 & \gamma_3 & \gamma_2 & \gamma_4 & \gamma_3 & \gamma_4 & \gamma_2 & \gamma_5 & 0 & 0 & \gamma_1 & 0 & \gamma_9 & \gamma_7 & \gamma_6 & \gamma_8 & \gamma_7 & \gamma_8 & \gamma_6 & \gamma_9 & 0 & 0 & 0 & 0 \\
\gamma_3 & \gamma_5 & \gamma_4 & \gamma_2 & \gamma_4 & \gamma_3 & \gamma_5 & \gamma_2 & 0 & 0 & 0 & \gamma_1 & \gamma_7 & \gamma_9 & \gamma_8 & \gamma_6 & \gamma_8 & \gamma_7 & \gamma_9 & \gamma_6 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \gamma_6 & \gamma_7 & \gamma_8 & \gamma_9 & \gamma_6 & \gamma_8 & \gamma_9 & \gamma_7 & \gamma_1 & 0 & 0 & 0 & \gamma_2 & \gamma_3 & \gamma_4 & \gamma_5 & \gamma_2 & \gamma_4 & \gamma_5 & \gamma_3 \\
0 & 0 & 0 & 0 & \gamma_7 & \gamma_6 & \gamma_9 & \gamma_8 & \gamma_8 & \gamma_6 & \gamma_7 & \gamma_9 & 0 & \gamma_1 & 0 & 0 & \gamma_3 & \gamma_2 & \gamma_5 & \gamma_4 & \gamma_4 & \gamma_2 & \gamma_3 & \gamma_5 \\
0 & 0 & 0 & 0 & \gamma_8 & \gamma_9 & \gamma_6 & \gamma_7 & \gamma_9 & \gamma_7 & \gamma_6 & \gamma_8 & 0 & 0 & \gamma_1 & 0 & \gamma_4 & \gamma_5 & \gamma_2 & \gamma_3 & \gamma_5 & \gamma_3 & \gamma_2 & \gamma_4 \\
0 & 0 & 0 & 0 & \gamma_9 & \gamma_8 & \gamma_7 & \gamma_6 & \gamma_7 & \gamma_9 & \gamma_8 & \gamma_6 & 0 & 0 & 0 & \gamma_1 & \gamma_5 & \gamma_4 & \gamma_3 & \gamma_2 & \gamma_3 & \gamma_5 & \gamma_4 & \gamma_2 \\
\gamma_6 & \gamma_7 & \gamma_8 & \gamma_9 & 0 & 0 & 0 & 0 & \gamma_6 & \gamma_9 & \gamma_7 & \gamma_8 & \gamma_2 & \gamma_3 & \gamma_4 & \gamma_5 & \gamma_1 & 0 & 0 & 0 & \gamma_2 & \gamma_5 & \gamma_3 & \gamma_4 \\
\gamma_7 & \gamma_6 & \gamma_9 & \gamma_8 & 0 & 0 & 0 & 0 & \gamma_9 & \gamma_6 & \gamma_8 & \gamma_7 & \gamma_3 & \gamma_2 & \gamma_5 & \gamma_4 & 0 & \gamma_1 & 0 & 0 & \gamma_5 & \gamma_2 & \gamma_4 & \gamma_3 \\
\gamma_8 & \gamma_9 & \gamma_6 & \gamma_7 & 0 & 0 & 0 & 0 & \gamma_7 & \gamma_8 & \gamma_6 & \gamma_9 & \gamma_4 & \gamma_5 & \gamma_2 & \gamma_3 & 0 & 0 & \gamma_1 & 0 & \gamma_3 & \gamma_4 & \gamma_2 & \gamma_5 \\
\gamma_9 & \gamma_8 & \gamma_7 & \gamma_6 & 0 & 0 & 0 & 0 & \gamma_8 & \gamma_7 & \gamma_9 & \gamma_6 & \gamma_5 & \gamma_4 & \gamma_3 & \gamma_2 & 0 & 0 & 0 & \gamma_1 & \gamma_4 & \gamma_3 & \gamma_5 & \gamma_2 \\
\gamma_6 & \gamma_8 & \gamma_9 & \gamma_7 & \gamma_6 & \gamma_9 & \gamma_7 & \gamma_8 & 0 & 0 & 0 & 0 & \gamma_2 & \gamma_4 & \gamma_5 & \gamma_3 & \gamma_2 & \gamma_5 & \gamma_3 & \gamma_4 & \gamma_1 & 0 & 0 & 0 \\
\gamma_8 & \gamma_6 & \gamma_7 & \gamma_9 & \gamma_9 & \gamma_6 & \gamma_8 & \gamma_7 & 0 & 0 & 0 & 0 & \gamma_4 & \gamma_2 & \gamma_3 & \gamma_5 & \gamma_5 & \gamma_2 & \gamma_4 & \gamma_3 & 0 & \gamma_1 & 0 & 0 \\
\gamma_9 & \gamma_7 & \gamma_6 & \gamma_8 & \gamma_7 & \gamma_8 & \gamma_6 & \gamma_9 & 0 & 0 & 0 & 0 & \gamma_5 & \gamma_3 & \gamma_2 & \gamma_4 & \gamma_3 & \gamma_4 & \gamma_2 & \gamma_5 & 0 & 0 & \gamma_1 & 0 \\
\gamma_7 & \gamma_9 & \gamma_8 & \gamma_6 & \gamma_8 & \gamma_7 & \gamma_9 & \gamma_6 & 0 & 0 & 0 & 0 & \gamma_3 & \gamma_5 & \gamma_4 & \gamma_2 & \gamma_4 & \gamma_3 & \gamma_5 & \gamma_2 & 0 & 0 & 0 & \gamma_1
\end{pmatrix},$$

where $\gamma_1=\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7+\alpha_8+\alpha_9+\alpha_{10}+\alpha_{11}+\alpha_{12}+\alpha_{13}+\alpha_{14}+\alpha_{15}+\alpha_{16}+\alpha_{17}+\alpha_{18}+\alpha_{19}+\alpha_{20}+\alpha_{21}+\alpha_{22}+\alpha_{23}+\alpha_{24}$;

$\gamma_2=a_1(a_9+a_5)+a_2(a_7+a_{12})+a_3(a_8+a_{10})+a_4(a_6+a_{11})+a_{13}(a_{17}+a_{21})+a_{14}(a_{24}+a_{19})+a_{15}(a_{22}+a_{20})+a_{16}(a_{23}+a_{18})+a_{10}a_8+a_9a_5+a_{12}a_7+a_{11}a_6+a_{17}a_{21}+a_{24}a_{19}+a_{22}a_{20}+a_{23}a_{18}$;

$\gamma_3=a_1(a_{12}+a_6)+a_2(a_9+a_8)+a_3(a_{11}+a_7)+a_4(a_{10}+a_5)+a_{13}(a_{24}+a_{18})+a_{14}(a_{21}+a_{20})+a_{15}(a_{23}+a_{19})+a_{16}(a_{22}+a_{17})+a_{10}a_5+a_{12}a_6+a_9a_8+a_{11}a_7+a_{17}a_{22}+a_{24}a_{18}+a_{21}a_{20}+a_{23}a_{19}$;

$\gamma_4=a_1(a_{10}+a_7)+a_2(a_{11}+a_5)+a_3(a_9+a_6)+a_4(a_{12}+a_8)+a_{13}(a_{22}+a_{19})+a_{14}(a_{23}+a_{17})+a_{15}(a_{21}+a_{18})+a_{16}(a_{24}+a_{20})+a_{10}a_7+a_9a_6+a_{12}a_8+a_{11}a_5+a_{17}a_{23}+a_{24}a_{20}+a_{22}a_{19}+a_{21}a_{18}$;

$\gamma_5=a_1(a_8+a_{11})+a_2(a_6+a_{10})+a_3(a_5+a_{12})+a_4(a_7+a_9)+a_{13}(a_{20}+a_{23})+a_{14}(a_{18}+a_{22})+a_{15}(a_{17}+a_{24})+a_{16}(a_{19}+a_{21})+a_{10}a_6+a_9a_7+a_{12}a_5+a_{11}a_8+a_{17}a_{24}+a_{20}a_{23}+a_{19}a_{21}+a_{22}a_{18}$.

From the last four equations we obtain that $\gamma_2+\gamma_3+\gamma_4+\gamma_5=(\alpha_1+\alpha_2+\alpha_3+\alpha_4)(\alpha_5+\alpha_6+\alpha_7+\alpha_8+\alpha_9+\alpha_{10}+\alpha_{11}+\alpha_{12})+(\alpha_5+\alpha_6+\alpha_7+\alpha_8)(\alpha_9+\alpha_{10}+\alpha_{11}+\alpha_{12})+(\alpha_{13}+\alpha_{14}+\alpha_{15}+\alpha_{16})(\alpha_{17}+\alpha_{18}+\alpha_{19}+\alpha_{20}+\alpha_{21}+\alpha_{22}+\alpha_{23}+\alpha_{24})+(\alpha_{17}+\alpha_{18}+\alpha_{19}+\alpha_{20})(\alpha_{21}+\alpha_{22}+\alpha_{23}+\alpha_{24})$.

If the code $C(v)$ is self-dual, then from Theorem 2 we have that the qualities $vv^*=0$ and $\sigma(v)\sigma(v)^T=\sigma(vv^*)=0$ are met. Therefore $\gamma_i=0$

$(i = 1, \dots, 9)$. Then $\gamma_1 = 0$, $\gamma_2 + \gamma_3 + \gamma_4 + \gamma_5 = 0$.

From the product of matrices $\sigma(v)$ and $\sigma(v)^T$ we also have the following elements:

$\gamma_6 = a_1(a_{17}+a_{21})+a_2(a_{24}+a_{19})+a_3(a_{22}+a_{20})+a_4(a_{23}+a_{18})+a_5(a_{13}+a_{21}) + a_6(a_{16} + a_{23}) + a_7(a_{14} + a_{24}) + a_{15}(a_8 + a_{10}) + a_{11}(a_{16} + a_{18}) + a_{12}(a_{14} + a_{19}) + a_9a_{17} + a_{10}a_{20} + a_{13}a_9 + a_{22}a_8$;

$\gamma_7 = a_1(a_{18}+a_{24})+a_2(a_{21}+a_{20})+a_3(a_{23}+a_{19})+a_4(a_{22}+a_{17})+a_5(a_{16}+a_{22})+a_6(a_{13}+a_{24})+a_{15}(a_7+a_{11})+a_{14}(a_8+a_9)+a_{10}(a_{17}+a_{16})+a_{12}(a_{18}+a_{13}) + a_9a_{20} + a_{11}a_{19} + a_{21}a_8 + a_{23}a_7$;

$\gamma_8 = a_1(a_{22}+a_{19})+a_2(a_{23}+a_{17})+a_3(a_{21}+a_{18})+a_4(a_{24}+a_{20})+a_5(a_{14}+a_{23})+a_6(a_{15}+a_{21})+a_7(a_{13}+a_{22})+a_8(a_{16}+a_{24})+a_9(a_{18}+a_{15})+a_{10}(a_{19}+a_{13}) + a_{11}(a_{14} + a_{17}) + a_{12}(a_{16} + a_{20})$;

$\gamma_9 = a_1(a_{23}+a_{20})+a_2(a_{22}+a_{18})+a_3(a_{24}+a_{17})+a_4(a_{21}+a_{19})+a_5(a_{15}+a_{24})+a_6(a_{14}+a_{22})+a_7(a_{16}+a_{21})+a_8(a_{13}+a_{23})+a_9(a_{19}+a_{16})+a_{10}(a_{18}+a_{14}) + a_{11}(a_{20} + a_{13}) + a_{12}(a_{17} + a_{15})$.

From these four equations we obtain that $\gamma_6 + \gamma_7 + \gamma_8 + \gamma_9 = (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)(\alpha_{17} + \alpha_{18} + \alpha_{19} + \alpha_{20} + \alpha_{21} + \alpha_{22} + \alpha_{23} + \alpha_{24}) + (\alpha_5 + \alpha_6 + \alpha_7 + \alpha_8)(\alpha_{13} + \alpha_{14} + \alpha_{15} + \alpha_{16} + \alpha_{21} + \alpha_{22} + \alpha_{23} + \alpha_{24}) + (\alpha_9 + \alpha_{10} + \alpha_{11} + \alpha_{12})(\alpha_{13} + \alpha_{14} + \alpha_{15} + \alpha_{16} + \alpha_{17} + \alpha_{18} + \alpha_{19} + \alpha_{20})$.

If $C(v)$ is self-dual, then $\gamma_6 + \gamma_7 + \gamma_8 + \gamma_9 = 0$.

Therefore, from these conditions we get the corresponding equations given in the conclusion of the lemma.

For example, one of the obtained elements is $v = 1+y+z+yz+w+yw + zw + yzw + w^2 + yw^2 + yzw^2 + xz + xyz + xyw + xzw + xyzw^2$. For this element we consider $v^* = 1 + y + z + yz + w^2 + yzw^2 + yw^2 + zw^2 + w + zw + yw + xz + xyz + xyzw^2 + xyw^2 + xyw \neq v$.

Thus, $vv^* = 0$. From the form of $v$ we have

$$\sigma(v) = \begin{pmatrix}
1&1&1&1&1&1&1&1&1&1&0&1&0&0&1&1&0&1&1&0&0&0&0&1\\
1&1&1&1&1&1&1&1&1&1&1&0&0&0&1&1&1&0&0&1&0&0&1&0\\
1&1&1&1&1&1&1&1&0&1&1&1&1&1&0&0&1&0&0&1&0&1&0&0\\
1&1&1&1&1&1&1&1&1&0&1&1&1&1&0&0&0&1&1&0&1&0&0&0\\
1&1&1&0&1&1&1&1&1&1&1&1&0&1&0&0&0&1&0&1&0&0&1&1\\
1&1&0&1&1&1&1&1&1&1&1&1&1&0&0&0&1&0&1&0&0&0&1&1\\
1&0&1&1&1&1&1&1&1&1&1&1&0&0&0&1&0&1&0&1&1&1&0&0\\
0&1&1&1&1&1&1&1&1&1&1&1&0&0&1&0&1&0&1&0&1&1&0&0\\
1&1&1&1&1&0&1&1&1&1&1&1&0&1&0&1&0&0&1&0&0&1&1&0\\
1&1&1&1&0&1&1&1&1&1&1&1&1&0&1&0&0&0&0&1&1&0&0&1\\
1&1&1&1&1&1&1&0&1&1&1&1&0&1&0&1&1&0&0&0&1&0&0&1\\
1&1&1&1&1&1&0&1&1&1&1&1&1&0&1&0&0&1&0&0&0&1&1&0\\
0&0&1&1&0&1&1&0&0&0&0&1&1&1&1&1&1&1&1&1&1&1&0&1\\
0&0&1&1&1&0&0&1&0&0&1&0&1&1&1&1&1&1&1&1&1&1&1&0\\
1&1&0&0&1&0&0&1&0&1&0&0&1&1&1&1&1&1&1&1&0&1&1&1\\
1&1&0&0&0&1&1&0&1&0&0&0&1&1&1&1&1&1&1&1&1&0&1&1\\
0&1&0&0&0&1&0&1&0&0&1&1&1&1&1&0&1&1&1&1&1&1&1&1\\
1&0&0&0&1&0&1&0&0&0&1&1&1&1&0&1&1&1&1&1&1&1&1&1\\
0&0&0&1&0&1&0&1&1&1&0&0&1&0&1&1&1&1&1&1&1&1&1&1\\
0&0&1&0&1&0&1&0&1&1&0&0&0&1&1&1&1&1&1&1&1&1&1&1\\
0&1&0&1&0&0&1&0&0&1&1&0&1&1&1&1&1&0&1&1&1&1&1&1\\
1&0&1&0&0&0&0&1&1&0&0&1&1&1&1&1&0&1&1&1&1&1&1&1\\
0&1&0&1&1&0&0&0&1&0&0&1&1&1&1&1&1&1&1&0&1&1&1&1\\
1&0&1&0&0&1&0&0&0&1&1&0&1&1&1&1&1&1&0&1&1&1&1&1
\end{pmatrix}.$$

As a result of calculations we obtain the number of elements $v \in \mathbb{F}_2(C_2 \times A_4)$ such that $C(v)$ is an extended binary Golay code. We present these results for comparison with the number of the same elements under the condition $v = v^*$.

Table 1.1: Number of elements of the group algebra $\mathbb{F}_2(C_2 \times A_4)$

| Minimum Hamming distance $C(v)$ | 2 | 4 | 6 | 8 |
|---|---|---|---|---|
| Number of elements $v$, $v = v^*$ | 640 | 8 704 | 768 | 768 |
| Number of elements $v$ | 10 752 | 96 768 | 18 432 | 18 432 |

**Lemma 2.** *Let* $G = S_4 = \langle a, b, c, d | a^2 = b^2 = c^3 = d^2 = 1, a^b = a, a^c = ab, b^c = a, a^d = a, b^d = ab, c^d = c^2 \rangle$ *be the symmetric group of degree four,* $v = \alpha_1 + \alpha_2 a + \alpha_3 b + \alpha_4 ab + \alpha_5 c + \alpha_6 ac + \alpha_7 bc + \alpha_8 abc + \alpha_9 c^2 + \alpha_{10} ac^2 + \alpha_{11} bc^2 + \alpha_{12} abc^2 + \alpha_{13} d + \alpha_{14} ad + \alpha_{15} bd + \alpha_{16} abd + \alpha_{17} cd + \alpha_{18} acd + \alpha_{19} bcd + \alpha_{20} abcd + \alpha_{21} c^2 d + \alpha_{22} ac^2 d + \alpha_{23} bc^2 d + \alpha_{24} abc^2 d$. *If the code* $C(v)$ *is self-dual, then*

*1)* $\sum_{i=1}^{24} \alpha_i = 0$;

*2)* $(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)(\alpha_5 + \alpha_6 + \alpha_7 + \alpha_8 + \alpha_9 + \alpha_{10} + \alpha_{11} + \alpha_{12}) + (\alpha_5 + \alpha_6 + \alpha_7 + \alpha_8)(\alpha_9 + \alpha_{10} + \alpha_{11} + \alpha_{12}) + (\alpha_{13} + \alpha_{14} + \alpha_{15} + \alpha_{16})(\alpha_{17} + \alpha_{18} + \alpha_{19} + \alpha_{20} + \alpha_{21} + \alpha_{22} + \alpha_{23} + \alpha_{24}) + (\alpha_{17} + \alpha_{18} + \alpha_{19} + \alpha_{20})(\alpha_{21} + \alpha_{22} + \alpha_{23} + \alpha_{24}) = 0$;

*3)* $(\alpha_1 + \alpha_2)(\alpha_{16} + \alpha_{15}) + (\alpha_3 + \alpha_4)(\alpha_{13} + \alpha_{14}) + (\alpha_5 + \alpha_6)(\alpha_{24} + \alpha_{23}) + (\alpha_7 + \alpha_8)(\alpha_{21} + \alpha_{22}) + (\alpha_9 + \alpha_{10})(\alpha_{20} + \alpha_{19}) + (\alpha_{11} + \alpha_{12})(\alpha_{17} + \alpha_{18}) = 0$;

*4)* $(\alpha_1 + \alpha_4)(\alpha_{19} + \alpha_{18}) + (\alpha_3 + \alpha_2)(\alpha_{20} + \alpha_{17}) + (\alpha_5 + \alpha_8)(\alpha_{15} + \alpha_{14}) + (\alpha_7 + \alpha_6)(\alpha_{16} + \alpha_{13}) + (\alpha_9 + \alpha_{12})(\alpha_{23} + \alpha_{22}) + (\alpha_{11} + \alpha_{10})(\alpha_{24} + \alpha_{21}) = 0$;

*5)* $(\alpha_1 + \alpha_3)(\alpha_{24} + \alpha_{22}) + (\alpha_2 + \alpha_4)(\alpha_{21} + \alpha_{23}) + (\alpha_5 + \alpha_7)(\alpha_{20} + \alpha_{18}) + (\alpha_6 + \alpha_8)(\alpha_{17} + \alpha_{19}) + (\alpha_9 + \alpha_{11})(\alpha_{16} + \alpha_{14}) + (\alpha_{10} + \alpha_{12})(\alpha_{13} + \alpha_{15}) = 0$.

Calculations in the group show that the matrix $\sigma(v)\sigma(v)^T = \sigma(vv^*)$ is equals to the following matrix with the elements $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ and $\gamma_5$ of the same form as in the proof of Lemma 1, i.e.
$\gamma_1 = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8 + \alpha_9 + \alpha_{10} + \alpha_{11} + \alpha_{12} + \alpha_{13} + \alpha_{14} + \alpha_{15} + \alpha_{16} + \alpha_{17} + \alpha_{18} + \alpha_{19} + \alpha_{20} + \alpha_{21} + \alpha_{22} + \alpha_{23} + \alpha_{24}$;

$\gamma_2 = a_1(a_9 + a_5) + a_2(a_7 + a_{12}) + a_3(a_8 + a_{10}) + a_4(a_6 + a_{11}) + a_{13}(a_{17} + a_{21}) + a_{14}(a_{24} + a_{19}) + a_{15}(a_{22} + a_{20}) + a_{16}(a_{23} + a_{18}) + a_{10}a_8 + a_9 a_5 + a_{12}a_7 + a_{11}a_6 + a_{17}a_{21} + a_{24}a_{19} + a_{22}a_{20} + a_{23}a_{18}$;

$\gamma_3 = a_1(a_{12} + a_6) + a_2(a_9 + a_8) + a_3(a_{11} + a_7) + a_4(a_{10} + a_5) + a_{13}(a_{24} + a_{18}) + a_{14}(a_{21} + a_{20}) + a_{15}(a_{23} + a_{19}) + a_{16}(a_{22} + a_{17}) + a_{10}a_5 + a_{12}a_6 + a_9 a_8 + a_{11}a_7 + a_{17}a_{22} + a_{24}a_{18} + a_{21}a_{20} + a_{23}a_{19}$;

$\gamma_4 = a_1(a_{10}+a_7)+a_2(a_{11}+a_5)+a_3(a_9+a_6)+a_4(a_{12}+a_8)+a_{13}(a_{22}+a_{19})+a_{14}(a_{23}+a_{17})+a_{15}(a_{21}+a_{18})+a_{16}(a_{24}+a_{20})+a_{10}a_7+a_9a_6+a_{12}a_8+a_{11}a_5+a_{17}a_{23}+a_{24}a_{20}+a_{22}a_{19}+a_{21}a_{18}$;

$\gamma_5 = a_1(a_8+a_{11})+a_2(a_6+a_{10})+a_3(a_5+a_{12})+a_4(a_7+a_9)+a_{13}(a_{20}+a_{23})+a_{14}(a_{18}+a_{22})+a_{15}(a_{17}+a_{24})+a_{16}(a_{19}+a_{21})+a_{10}a_6+a_9a_7+a_{12}a_5+a_{11}a_8+a_{17}a_{24}+a_{20}a_{23}+a_{19}a_{21}+a_{22}a_{18}$ :

$$\begin{pmatrix}
\gamma_1 & 0 & 0 & 0 & \gamma_2 & \gamma_3 & \gamma_4 & \gamma_5 & \gamma_2 & \gamma_4 & \gamma_5 & \gamma_3 & 0 & 0 & \gamma_6 & \gamma_6 & 0 & \gamma_7 & \gamma_7 & 0 & 0 & \gamma_8 & 0 & \gamma_8 \\
0 & \gamma_1 & 0 & 0 & \gamma_3 & \gamma_2 & \gamma_5 & \gamma_4 & \gamma_4 & \gamma_2 & \gamma_3 & \gamma_5 & 0 & 0 & \gamma_6 & \gamma_6 & \gamma_7 & 0 & 0 & \gamma_7 & \gamma_8 & 0 & \gamma_8 & 0 \\
0 & 0 & \gamma_1 & 0 & \gamma_4 & \gamma_5 & \gamma_2 & \gamma_3 & \gamma_5 & \gamma_3 & \gamma_2 & \gamma_4 & \gamma_6 & \gamma_6 & 0 & 0 & \gamma_7 & 0 & 0 & \gamma_7 & 0 & \gamma_8 & 0 & \gamma_8 \\
0 & 0 & 0 & \gamma_1 & \gamma_5 & \gamma_4 & \gamma_3 & \gamma_2 & \gamma_3 & \gamma_5 & \gamma_4 & \gamma_2 & \gamma_6 & \gamma_6 & 0 & 0 & 0 & \gamma_7 & \gamma_7 & 0 & \gamma_8 & 0 & \gamma_8 & 0 \\
\gamma_2 & \gamma_3 & \gamma_4 & \gamma_5 & \gamma_1 & 0 & 0 & 0 & \gamma_2 & \gamma_5 & \gamma_3 & \gamma_4 & 0 & \gamma_8 & \gamma_8 & 0 & 0 & \gamma_6 & 0 & \gamma_6 & 0 & 0 & \gamma_7 & \gamma_7 \\
\gamma_3 & \gamma_2 & \gamma_5 & \gamma_4 & 0 & \gamma_1 & 0 & 0 & \gamma_5 & \gamma_2 & \gamma_4 & \gamma_3 & \gamma_8 & 0 & 0 & \gamma_8 & \gamma_6 & 0 & \gamma_6 & 0 & 0 & 0 & \gamma_7 & \gamma_7 \\
\gamma_4 & \gamma_5 & \gamma_2 & \gamma_3 & 0 & 0 & \gamma_1 & 0 & \gamma_3 & \gamma_4 & \gamma_2 & \gamma_5 & \gamma_8 & 0 & 0 & \gamma_8 & 0 & \gamma_6 & 0 & \gamma_6 & \gamma_7 & \gamma_7 & 0 & 0 \\
\gamma_5 & \gamma_4 & \gamma_3 & \gamma_2 & 0 & 0 & 0 & \gamma_1 & \gamma_4 & \gamma_3 & \gamma_5 & \gamma_2 & 0 & \gamma_8 & \gamma_8 & 0 & \gamma_6 & 0 & \gamma_6 & 0 & \gamma_7 & \gamma_7 & 0 & 0 \\
\gamma_2 & \gamma_4 & \gamma_5 & \gamma_3 & \gamma_2 & \gamma_5 & \gamma_3 & \gamma_4 & \gamma_1 & 0 & 0 & 0 & 0 & \gamma_7 & 0 & \gamma_7 & 0 & 0 & \gamma_8 & \gamma_8 & 0 & \gamma_6 & \gamma_6 & 0 \\
\gamma_4 & \gamma_2 & \gamma_3 & \gamma_5 & \gamma_5 & \gamma_2 & \gamma_4 & \gamma_3 & 0 & \gamma_1 & 0 & 0 & \gamma_7 & 0 & \gamma_7 & 0 & 0 & 0 & \gamma_8 & \gamma_8 & \gamma_6 & 0 & 0 & \gamma_6 \\
\gamma_5 & \gamma_3 & \gamma_2 & \gamma_4 & \gamma_3 & \gamma_4 & \gamma_2 & \gamma_5 & 0 & 0 & \gamma_1 & 0 & 0 & \gamma_7 & 0 & \gamma_7 & \gamma_8 & \gamma_8 & 0 & 0 & \gamma_6 & 0 & 0 & \gamma_6 \\
\gamma_3 & \gamma_5 & \gamma_4 & \gamma_2 & \gamma_4 & \gamma_3 & \gamma_5 & \gamma_2 & 0 & 0 & 0 & \gamma_1 & \gamma_7 & 0 & \gamma_7 & 0 & \gamma_8 & \gamma_8 & 0 & 0 & 0 & \gamma_6 & \gamma_6 & 0 \\
0 & 0 & \gamma_6 & \gamma_6 & 0 & \gamma_8 & \gamma_8 & 0 & 0 & \gamma_7 & 0 & \gamma_7 & \gamma_1 & 0 & 0 & 0 & \gamma_2 & \gamma_4 & \gamma_3 & \gamma_5 & \gamma_2 & \gamma_3 & \gamma_5 & \gamma_4 \\
0 & 0 & \gamma_6 & \gamma_6 & \gamma_8 & 0 & 0 & \gamma_8 & \gamma_7 & 0 & \gamma_7 & 0 & 0 & \gamma_1 & 0 & 0 & \gamma_4 & \gamma_2 & \gamma_5 & \gamma_3 & \gamma_3 & \gamma_2 & \gamma_4 & \gamma_5 \\
\gamma_6 & \gamma_6 & 0 & 0 & \gamma_8 & 0 & 0 & \gamma_8 & 0 & \gamma_7 & 0 & \gamma_7 & 0 & 0 & \gamma_1 & 0 & \gamma_3 & \gamma_5 & \gamma_2 & \gamma_4 & \gamma_5 & \gamma_4 & \gamma_2 & \gamma_3 \\
\gamma_6 & \gamma_6 & 0 & 0 & 0 & \gamma_8 & \gamma_8 & 0 & \gamma_7 & 0 & \gamma_7 & 0 & 0 & 0 & 0 & \gamma_1 & \gamma_5 & \gamma_3 & \gamma_4 & \gamma_2 & \gamma_4 & \gamma_5 & \gamma_3 & \gamma_2 \\
0 & \gamma_7 & \gamma_7 & 0 & 0 & \gamma_6 & 0 & \gamma_6 & 0 & 0 & \gamma_8 & \gamma_8 & \gamma_2 & \gamma_4 & \gamma_3 & \gamma_5 & \gamma_1 & 0 & 0 & 0 & \gamma_2 & \gamma_5 & \gamma_4 & \gamma_3 \\
\gamma_7 & 0 & 0 & \gamma_7 & \gamma_6 & 0 & \gamma_6 & 0 & 0 & 0 & \gamma_8 & \gamma_8 & \gamma_4 & \gamma_2 & \gamma_5 & \gamma_3 & 0 & \gamma_1 & 0 & 0 & \gamma_5 & \gamma_2 & \gamma_3 & \gamma_4 \\
\gamma_7 & 0 & 0 & \gamma_7 & 0 & \gamma_6 & 0 & \gamma_6 & \gamma_8 & \gamma_8 & 0 & 0 & \gamma_3 & \gamma_5 & \gamma_2 & \gamma_4 & 0 & 0 & \gamma_1 & 0 & \gamma_4 & \gamma_3 & \gamma_2 & \gamma_5 \\
0 & \gamma_7 & \gamma_7 & 0 & \gamma_6 & 0 & \gamma_6 & 0 & \gamma_8 & \gamma_8 & 0 & 0 & \gamma_5 & \gamma_3 & \gamma_4 & \gamma_2 & 0 & 0 & 0 & \gamma_1 & \gamma_3 & \gamma_4 & \gamma_5 & \gamma_2 \\
0 & \gamma_8 & 0 & \gamma_8 & 0 & 0 & \gamma_7 & \gamma_7 & 0 & \gamma_6 & \gamma_6 & 0 & \gamma_2 & \gamma_3 & \gamma_5 & \gamma_4 & \gamma_2 & \gamma_5 & \gamma_4 & \gamma_3 & \gamma_1 & 0 & 0 & 0 \\
\gamma_8 & 0 & \gamma_8 & 0 & 0 & 0 & \gamma_7 & \gamma_7 & \gamma_6 & 0 & 0 & \gamma_6 & \gamma_3 & \gamma_2 & \gamma_4 & \gamma_5 & \gamma_5 & \gamma_2 & \gamma_3 & \gamma_4 & 0 & \gamma_1 & 0 & 0 \\
0 & \gamma_8 & 0 & \gamma_8 & \gamma_7 & \gamma_7 & 0 & 0 & \gamma_6 & 0 & 0 & \gamma_6 & \gamma_5 & \gamma_4 & \gamma_2 & \gamma_3 & \gamma_4 & \gamma_3 & \gamma_2 & \gamma_5 & 0 & 0 & \gamma_1 & 0 \\
\gamma_8 & 0 & \gamma_8 & 0 & \gamma_7 & \gamma_7 & 0 & 0 & 0 & \gamma_6 & \gamma_6 & 0 & \gamma_4 & \gamma_5 & \gamma_3 & \gamma_2 & \gamma_3 & \gamma_4 & \gamma_5 & \gamma_2 & 0 & 0 & 0 & \gamma_1
\end{pmatrix}.$$

From the last four equations (before the matrix) we obtain that $\gamma_2+\gamma_3+\gamma_4+\gamma_5 = (\alpha_1+\alpha_2+\alpha_3+\alpha_4)(\alpha_5+\alpha_6+\alpha_7+\alpha_8+\alpha_9+\alpha_{10}+\alpha_{11}+\alpha_{12})+(\alpha_5+\alpha_6+\alpha_7+\alpha_8)(\alpha_9+\alpha_{10}+\alpha_{11}+\alpha_{12})+(\alpha_{13}+\alpha_{14}+\alpha_{15}+\alpha_{16})(\alpha_{17}+\alpha_{18}+\alpha_{19}+\alpha_{20}+\alpha_{21}+\alpha_{22}+\alpha_{23}+\alpha_{24})+(\alpha_{17}+\alpha_{18}+\alpha_{19}+\alpha_{20})(\alpha_{21}+\alpha_{22}+\alpha_{23}+\alpha_{24})$.

If the code $C(v)$ is self-dual, then from Theorem 2 we have the conditions $vv^* = 0$ and $\sigma(v)\sigma(v)^T = \sigma(vv^*) = 0$ are met. Therefore $\gamma_i = 0$ $(i = 1, \ldots, 8)$. Then $\gamma_1 = 0$, $\gamma_2+\gamma_3+\gamma_4+\gamma_5 = 0$.

From the product of matrices $\sigma(v)$ and $\sigma(v)^T$ we also have the following elements:

$\gamma_6 = (\alpha_1+\alpha_2)(\alpha_{16}+\alpha_{15})+(\alpha_3+\alpha_4)(\alpha_{13}+\alpha_{14})+(\alpha_5+\alpha_6)(\alpha_{24}+\alpha_{23})+(\alpha_7+\alpha_8)(\alpha_{21}+\alpha_{22})+(\alpha_9+\alpha_{10})(\alpha_{20}+\alpha_{19})+(\alpha_{11}+\alpha_{12})(\alpha_{17}+\alpha_{18})$,

$\gamma_7 = (\alpha_1+\alpha_4)(\alpha_{19}+\alpha_{18})+(\alpha_3+\alpha_2)(\alpha_{20}+\alpha_{17})+(\alpha_5+\alpha_8)(\alpha_{15}+\alpha_{14})+(\alpha_7+\alpha_6)(\alpha_{16}+\alpha_{13})+(\alpha_9+\alpha_{12})(\alpha_{23}+\alpha_{22})+(\alpha_{11}+\alpha_{10})(\alpha_{24}+\alpha_{21})$,

$\gamma_8 = (\alpha_1+\alpha_3)(\alpha_{24}+\alpha_{22})+(\alpha_2+\alpha_4)(\alpha_{21}+\alpha_{23})+(\alpha_5+\alpha_7)(\alpha_{20}+\alpha_{18})+(\alpha_6+\alpha_8)(\alpha_{17}+\alpha_{19})+(\alpha_9+\alpha_{11})(\alpha_{16}+\alpha_{14})+(\alpha_{10}+\alpha_{12})(\alpha_{13}+\alpha_{15})$.

If $C(v)$ is self-dual, then $\gamma_6 = 0$, $\gamma_7 = 0$, and $\gamma_8 = 0$.

Therefore, from these conditions we get the corresponding equations given in the conclusion of the lemma, and, thus, Lemma 2 is proved.

For example, one of the obtained elements is $v = 1+a+b+ab+c+ac+bc+abc+c^2+ac^2+bc^2+d+bd+bcd+abcd+c^2d$. For this element we consider $v^* = 1+a+b+ab+c^2+abc^2+ac^2+bc^2+c+bc+abc+d+abd+acd+abcd+c^2d \neq v$.

Thus, $vv^* = 0$. From the form of $v$ we have

$$\sigma(v) = \begin{pmatrix}
1&1&1&1&1&1&1&1&1&1&1&0&1&0&1&0&0&0&1&1&1&0&0&0\\
1&1&1&1&1&1&1&1&1&1&0&1&0&1&0&1&0&0&1&1&0&1&0&0\\
1&1&1&1&1&1&1&1&1&0&1&1&1&0&1&0&1&1&0&0&0&0&1&0\\
1&1&1&1&1&1&1&1&0&1&1&1&0&1&0&1&1&1&0&0&0&0&0&1\\
1&0&1&1&1&1&1&1&1&1&1&1&1&0&0&0&1&0&0&1&0&1&0&1\\
0&1&1&1&1&1&1&1&1&1&1&1&0&1&0&0&0&1&1&0&1&0&1&0\\
1&1&1&0&1&1&1&1&1&1&1&1&0&0&1&0&0&1&1&0&0&1&0&1\\
1&1&0&1&1&1&1&1&1&1&1&1&0&0&0&1&1&0&0&1&1&0&1&0\\
1&1&1&1&1&1&0&1&1&1&1&1&0&1&1&0&1&0&0&0&1&1&0&0\\
1&1&1&1&1&1&1&0&1&1&1&1&1&0&0&1&0&1&0&0&1&1&0&0\\
1&1&1&1&0&1&1&1&1&1&1&1&1&0&0&1&0&0&1&0&0&0&1&1\\
1&1&1&1&1&0&1&1&1&1&1&1&0&1&1&0&0&0&0&1&0&0&1&1\\
1&0&0&1&1&0&0&0&0&0&1&1&1&1&1&1&1&1&0&1&1&1&1&1\\
0&1&1&0&0&1&0&0&0&0&1&1&1&1&1&1&1&1&1&0&1&1&1&1\\
0&1&1&0&0&0&1&0&1&1&0&0&1&1&1&1&0&1&1&1&1&1&1&1\\
1&0&0&1&0&0&0&1&1&1&0&0&1&1&1&1&1&0&1&1&1&1&1&1\\
0&1&0&1&1&1&0&0&1&0&0&0&1&1&1&1&1&1&1&1&1&1&1&0\\
1&0&1&0&1&1&0&0&0&1&0&0&1&1&1&1&1&1&1&1&1&1&0&1\\
0&1&0&1&0&0&1&1&0&0&1&0&1&1&1&1&1&1&1&1&0&1&1&1\\
1&0&1&0&0&0&1&1&0&0&0&1&1&1&1&1&1&1&1&0&1&1&1&1\\
1&0&0&0&0&1&1&0&1&0&1&0&1&0&1&1&1&1&1&1&1&1&1&1\\
0&1&0&0&1&0&0&1&0&1&0&1&0&1&1&1&1&1&1&1&1&1&1&1\\
0&0&1&0&1&0&0&1&1&0&1&0&1&1&1&0&1&1&1&1&1&1&1&1\\
0&0&0&1&0&1&1&0&0&1&0&1&1&1&0&1&1&1&1&1&1&1&1&1
\end{pmatrix}.$$

As a result of calculations we have obtained the number of elements $v \in \mathbb{F}_2 S_4$ such that $C(v)$ is an extended binary Golay code. There are 55 296 elements. The number of the same elements under the condition $v = v^*$ is 192.

## 2. Main result

We consider the groups $G_1 = (C_6 \times C_2) \rtimes C_2$, $G_2 = D_{24}$, $G_3 = C_3 \times D_8$, $G_4 = C_2 \times A_4$, $G_5 = S_4$. Let us define the element $v \in \mathbb{F}_2((C_6 \times C_2) \rtimes C_2)$ for the group $G_1$:

$$v = \sum_{i=0}^{3} \left( (\alpha_{i+1} + \alpha_{i+5}x + \alpha_{i+9}x^2)y^i + (\alpha_{i+13} + \alpha_{i+17}x + \alpha_{i+21}x^2)y^i z \right).$$

For $G_2$ the element $v \in \mathbb{F}_2 D_{24}$ is the following:

$$v = \sum_{i=0}^{11} \left( \alpha_{i+1}x^i + \alpha_{i+13}x^i y \right).$$

For $G_3$ the element $v \in \mathbb{F}_2(C_3 \times D_8)$ we define as

$$v = \sum_{i=0}^{3} \left( (\alpha_{i+1} + \alpha_{i+5}x + \alpha_{i+9}x^2)y^i + (\alpha_{i+13} + \alpha_{i+17}x + \alpha_{i+21}x^2)y^i z \right).$$

For $G_4$ the element $v \in \mathbb{F}_2(C_2 \times A_4)$ is the following:

$$v = \alpha_1+\alpha_2 y+\alpha_3 z+\alpha_4 yz+\alpha_5 w+\alpha_6 yw+\alpha_7 zw+\alpha_8 yzw+\alpha_9 w^2+\alpha_{10} yw^2+ \alpha_{11} zw^2+\alpha_{12} yzw^2+\alpha_{13} x+\alpha_{14} xy+\alpha_{15} xz+\alpha_{16} xyz+\alpha_{17} xw+\alpha_{18} xyw+ \alpha_{19} xzw+\alpha_{20} xyzw+\alpha_{21} xw^2+\alpha_{22} xyw^2+\alpha_{23} xzw^2+\alpha_{24} xyzw^2.$$

Finally, for $G_5$ the element $v \in \mathbb{F}_2 S_4$ we define as

$$v = \alpha_1 1+\alpha_2 a+\alpha_3 b+\alpha_4 ab+\alpha_5 c+\alpha_6 ac+\alpha_7 bc+\alpha_8 abc+\alpha_9 c^2+\alpha_{10} ac^2+ \alpha_{11} bc^2+\alpha_{12} abc^2+\alpha_{13} d+\alpha_{14} ad+\alpha_{15} bd+\alpha_{16} abd+\alpha_{17} cd+\alpha_{18} acd+\alpha_{19} bcd+ \alpha_{20} abcd+\alpha_{21} c^2 d+\alpha_{22} ac^2 d+\alpha_{23} bc^2 d+\alpha_{24} abc^2 d.$$

**Theorem 3.** *Let $G$ be one of the groups $G_1, G_2, G_3, G_4, G_5$ and $v = \sum_{g\in G} \alpha_{g_i} g_i \in F_2 G$. If the code $C(v)$ is self-dual, then $\sum_{i=1}^{24} \alpha_i = 0$ for all the five groups,*

$$(\alpha_1+\alpha_3+\alpha_5+\alpha_7+\alpha_9+\alpha_{11})(\alpha_2+\alpha_4+\alpha_6+\alpha_8+\alpha_{10}+\alpha_{12})+ (\alpha_{13}+\alpha_{15}+\alpha_{17}+\alpha_{19}+\alpha_{21}+\alpha_{23})(\alpha_{14}+\alpha_{16}+\alpha_{18}+\alpha_{20}+\alpha_{22}+\alpha_{24}) = 0,$$

$$\alpha_1+(\alpha_1+\alpha_5)(\alpha_1+\alpha_9)+\alpha_2+(\alpha_2+\alpha_6)(\alpha_2+\alpha_{10})+\alpha_3+(\alpha_3+\alpha_7)(\alpha_3+\alpha_{11})+ \alpha_4+(\alpha_4+\alpha_8)(\alpha_4+\alpha_{12})+\alpha_{13}+(\alpha_{13}+\alpha_{17})(\alpha_{13}+\alpha_{21})+\alpha_{14}+(\alpha_{14}+\alpha_{18}) (\alpha_{14}+\alpha_{22})+\alpha_{15}+(\alpha_{15}+\alpha_{19})(\alpha_{15}+\alpha_{23})+\alpha_{16}+(\alpha_{16}+\alpha_{20})(\alpha_{16}+\alpha_{24}) = 0$$

*for the groups $G_1, G_2, G_3$,*

$$(\alpha_1+\alpha_2+\alpha_3+\alpha_4)(\alpha_{17}+\alpha_{18}+\alpha_{19}+\alpha_{20}+\alpha_{21}+\alpha_{22}+\alpha_{23}+\alpha_{24})+ (\alpha_5+\alpha_6+\alpha_7+\alpha_8)(\alpha_{13}+\alpha_{14}+\alpha_{15}+\alpha_{16}+\alpha_{21}+\alpha_{22}+\alpha_{23}+\alpha_{24})+ (\alpha_9+\alpha_{10}+\alpha_{11}+\alpha_{12})(\alpha_{13}+\alpha_{14}+\alpha_{15}+\alpha_{16}+\alpha_{17}+\alpha_{18}+\alpha_{19}+\alpha_{20}) = 0$$

*for the groups $G_3, G_4$ and*

$$(\alpha_1+\alpha_2+\alpha_3+\alpha_4)(\alpha_5+\alpha_6+\alpha_7+\alpha_8+\alpha_9+\alpha_{10}+\alpha_{11}+\alpha_{12})+ (\alpha_5+\alpha_6+\alpha_7+\alpha_8)(\alpha_9+\alpha_{10}+\alpha_{11}+\alpha_{12})+ (\alpha_{13}+\alpha_{14}+\alpha_{15}+\alpha_{16})(\alpha_{17}+\alpha_{18}+\alpha_{19}+\alpha_{20}+\alpha_{21}+\alpha_{22}+\alpha_{23}+\alpha_{24})+ (\alpha_{17}+\alpha_{18}+\alpha_{19}+\alpha_{20})(\alpha_{21}+\alpha_{22}+\alpha_{23}+\alpha_{24}) = 0$$

*for the groups $G_4, G_5$.*

Theorem 3 follows from the results obtained in [5–7] and two lemmas of Section 1.

As a result of calculations we obtain the number of elements $v \in \mathbb{F}_2G$ such that $C(v)$ is an extended binary Golay code. We present these results for comparison with the number of the same elements under the condition $v = v^*$.

Table 1.2: Number of elements from the group algebra $\mathbb{F}_2G$

| $G$ | Number of $v$ with $v = v^*$ | Number of all $v$ |
|---|---|---|
| $D_{24}$ | 768 | 36 864 |
| $(C_6 \times C_2) \rtimes C_2$ | 576 | 27 648 |
| $C_3 \times D_8$ | 128 | 12 288 |
| $C_2 \times A_4$ | 384 | 18 432 |
| $S_4$ | 192 | 55 296 |

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