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RESEARCH ARTICLE

On *k*-graceful labeling of pendant edge extension of complete bipartite graphs

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Abstract. For any two positive integers m, n, we denote the graph $K_{m,n} \odot K_1$ by G. Ma Ke-Jie proposed a conjecture [9] that pendant edge extension of a complete bipartite graph is a k-graceful graph for $k \geqslant 2$. In this paper we prove his conjecture for $n \leqslant m < n^2 + \lfloor \frac{k}{n} \rfloor + r$.

Introduction

Let $G = (V(G), E(G))$ be a finite simple connected graph with vertex set $V(G)$ and edge set $E(G)$. In this work $K_{m,n}$ denotes a regular complete bipartite graph. For all other terminology and notations we follow [2]. A function f is called a graceful labeling of a graph G with m edges if $f : V(G) \to \{0, 1, 2, \dots, m\}$ is injective and the induced function $F : E(G) \to \{1, 2, \dots, m\}$ defined as $F(e = uv) = |f(u) - f(v)|$ is bijective. This type of graph labeling, first introduced by Rosa in 1967 as a β-valuation [12], was used as a tool for decomposing a complete graph into isomorphic subgraphs. Both Rosa [12] and Golomb [5] proved that complete bipartite graphs are graceful. Also it is known that K_n is graceful if and only if $n \leqslant 4$ [4]. The k-graceful graphs, which is a natural generalization of graceful graphs was introduced independently by Slater [14] in 1982 and by Maheo and Thuillier [10] in 1982.

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Definition 1. A graph G is k-graceful, if there exists a mapping $f : V(G) \to \{0, 1, 2, \ldots |E(G)| + k - 1\}$ such that $f(x) \neq f(y)$ for distinct $x, y \in V(G)$ and an induced mapping is defined as $f^* : E(G) \to \{k, k + 1, \ldots, |E| + k - 1\}$, where $f^*(uv) = |f(u) - f(v)|$ is a bijection for all edges $uv \in E(G)$.

If a graph is k-graceful for any integer k, then it also called *arbitrarily graceful*. Obviously G is graceful when $k = 1$. In 2011, Li, Li, and Yan proved that $K_{m,n}$ is k-graceful [8].

The *corona* $G_1 \odot G_2$ of two graphs is the graph obtained by taking one copy of G_1, and p_1 copies of G_2 (where $|V(G_1)| = p_1$), and then joining the i^{th} vertex of G_1 by an edge to every vertex in the i^{th} copy of G_2. For positive integer m, and n, we define the graph $K_{m,n} \odot K_1$ by G, and in this paper we show that G is k-graceful. In [9], Ma ke-jie proposed the following conjecture,

Conjecture 1. 1-crown (pendant edge extension) of complete bipartite graph $K_{m,n} (m \leqslant n)$ is k-graceful graph for $k \geqslant 2$.

This conjecture has not been proved or disproved until today. Jirimutu [7] has showed that this conjecture is true when $m = 1$. In [1,11,13], it has been shown that pendant edge extension of a complete bipartite graph, i.e. $K_{m,n} \odot K_1$ is 1-graceful. In our paper we had a different approach to verify this conjecture. First in Theorem 1 we show that it holds for $k \geqslant mn + m + n$. Later we also show that our approach works when $k < mn + m + n$ with some restrictions (Theorem 2). Throughout the paper we assume that $m < n^2 + \lfloor \frac{k}{n} \rfloor + r$, where r is the smallest non-negative integer such that $k + r = nq$, for some $q \in \mathbb{Z}$. Note that since $k \geqslant 1$ as per assumption, q must be a positive integer. Finally in Lemma 2 we explain the reason behind the restriction of $m < n^2 + \lfloor \frac{k}{n} \rfloor + r$.

1. Main results

In this paper, we freely use the notation and graph theoretic terminologies introduced in [3] and [5]. Let us first specify the notations used in this section. We consider that the bipartite graph $K_{m,n}$ (where $m > n$) is composed of two sets (partitions) - one consisting of m vertices on the left side and the other consisting of n vertices on the right side. In the graph $G = K_{m,n} \odot K_1$, let $X = \{x_{1t}, x_{2t}, \ldots, x_{mt}\}$ and $Y = \{y_{1t}, y_{2t}, \ldots, y_{nt}\}$ define the vertices on the left and right partitions respectively. Let us assume that $t = 0$ denote the vertices on $K_{m,n}$, and $t = 1$ denote the vertices on the leaves extended from $K_{m,n}$.

Based on the above notations we define the vertex labeling using the following two functions.

$$f(x_{it}) = n(i-1) + te_{it}$$

and

$$f(y_{jt}) = mn + m + n + k - j - te_{jt},$$

where

$$e_{it} = \begin{cases} k + i - 1, & \text{if } 1 \leqslant i \leqslant r \\ i + p + k, & \text{if } p(n-1) + r + 1 \leqslant i \\ & \leqslant \min\{(p+1)(n-1) + r, m\} \\ i + n + k - 1, & \text{if } (s+1)(n-1) < i - r \leqslant m \end{cases}$$

and

$$e_{jt} = \begin{cases} k + r + n(j-1), & \text{if } 1 \leqslant j \leqslant \min\{\ell + 1, n\} \\ k + m + j - 1, & \text{if } \ell + 2 \leqslant j \leqslant n \end{cases}$$

where ℓ is the largest integer less than $\frac{m-r}{n-1}$, $s = \min\{\ell, n\}$, and $p \in \{0, 1, \ldots, s\}$.

Lemma 1. *For the above vertex labeling f, the induced edge labeling function f^* is bijective.*

Proof. We show that the induced function $f^* : E(G) \to \{k, k+1, \ldots, |E(G)| + k - 1\}$ is bijective, where $f^*(u \sim v) = |f(u) - f(v)|$, for all $u, v \in V(G)$. To prove the bijection we need to show that there is a one-to-one correspondence between the two sets $\{f^*(x_{i0} \sim y_{i0}) \cup f^*(x_{i1} \sim x_{i0}) \cup f^*(y_{i1} \sim y_{i0})\}$ and $\{k, k+1, \ldots, k + mn + m + n - 1\}$ (note that $|E(G)| = mn + m + n$). Now, from the definition of the function f, it is easy to observe that

$$f^*(x_{i0} \sim y_{i0}) = \{k + m + n, k + m + n + 1, \ldots, k + mn + m + n - 1\}.$$

Therefore, it remains to show that there is also a one-to-one correspondence between the two sets $\{f^*(x_{i1} \sim x_{i0}) \cup f^*(y_{i1} \sim y_{i0})\}$ and $\{k, k+1, \ldots, k + m + n - 1\}$. To proceed further, depending on the values of m we consider the following three cases and in each case we show that $f^*(x_{i1} \sim x_{i0}) \cup f^*(y_{i1} \sim y_{i0}) = \{k, k+1, \ldots, k + m + n - 1\}$.

Case 1: $m \leqslant (n-1)^2 + r$ (for any r).

Note that since $\ell < \frac{m-r}{n-1} \leqslant n-1$ which implies $\ell \leqslant n-2$, hence $s = \min\{\ell, n-1\} = \ell$.

$$f^*(x_{i1} \sim x_{i0}) = \begin{cases} k+i-1, & \text{if } 1 \leqslant i \leqslant r \\ i+p+k, & \text{if } p(n-1)+r+1 \leqslant i \\ & \leqslant \min\{(p+1)(n-1)+r, m\} \end{cases}$$

$$f^*(y_{j1} \sim y_{j0}) = \begin{cases} k+r+n(j-1), & \text{if } 1 \leqslant j \leqslant s+1 \\ k+m+j-1, & \text{if } s+2 \leqslant j \leqslant n \end{cases}$$

It is easy to observe that $f^*(y_{j1} \sim y_{j0}) = \{k+r, k+r+n, k+r+2n, \ldots, k+r+sn, k+m+s+1, k+m+s+2, \ldots, k+m+n-1\}$. On the other hand $f^*(x_{i1} \sim x_{i0}) = \{k, k+1, \ldots, k+r-1\} \bigcup \left(\bigcup_{p=0}^{s} A_p \right)$, where

$$\begin{aligned} A_0 &= \{k+r+1, k+r+2, \ldots, k+r+n-1\} \\ A_1 &= \{k+r+n+1, k+r+n+2, \ldots, k+r+2n-1\} \\ &\ \vdots \\ A_s &= \{k+r+ns+1, k+r+ns+2, \ldots, m+k+s\}. \end{aligned}$$

Hence we obtain that $f^*(x_{i1} \sim x_{i0}) \cup f^*(y_{i1} \sim y_{i0}) = \{k, k+1, \ldots, k+m+n-1\}$. Also, the uniqueness of all the elements, of the form $f^*(u \sim v)$, is evident from the pattern of the sets mentioned above. This is enough to ensure the bijection. Therefore, it follows that f^* maps the set of pendant extension edges uniquely to the set $\{k, k+1, \ldots, k+m+n-1\}$, when $m \leqslant (n-1)^2 + r$.

Case 2: $(n-1)^2 < m - r \leqslant n^2 - n$.

Observe that $\ell < \frac{m-r}{n-1} \leqslant \frac{n^2-n}{n-1} = n$. On the other hand, the facts that ℓ is the greatest integer less than $(m-r)/(n-1)$ and $m-r > (n-1)^2$, together imply that ℓ must be greater than $n-1$. Hence $s = \min\{\ell, n-1\} = n-1$, and consequently $\min\{(s+1)(n-1)+r, m\} = m$.

$$f^*(y_{j1} \sim y_{j0}) = k+r+n(j-1), \quad \text{if } 1 \leqslant j \leqslant n$$

$$f^*(x_{i1} \sim x_{i0}) = \begin{cases} k+i-1, & \text{if } 1 \leqslant i \leqslant r \\ i+p+k, & \text{if } s(n-1)+r+1 \leqslant i \\ & \leqslant \min\{(s+1)(n-1)+r, m\} \end{cases}$$

It is easy to observe that $f^*(y_{j1} \sim y_{j0}) = \{k+r, k+r+n, k+r+2n, \ldots, k+r+n^2-n\}$. On the other hand

$$f^*(x_{i1} \sim x_{i0}) = \{k, k+1, \ldots, k+r-1\} \bigcup \Big(\bigcup_{p=0}^{s} B_p \Big),$$

where

$$\begin{aligned} B_0 &= \{k+r+1, k+r+2, \ldots, k+r+n-1\} \\ B_1 &= \{k+r+n+1, k+r+n+2, \ldots, k+r+2n-1\} \\ &\vdots \\ B_{s-1} &= \{k+r+ns-n+1, k+r+ns-n+2, \ldots, k+r+ns-1\} \\ B_s &= \{k+r+ns+1, k+r+ns+2, \ldots, m+k+s\}. \end{aligned}$$

In this case also we observe that $f^*(x_{i1} \sim x_{i0}) \cup f^*(y_{i1} \sim y_{i0}) = \{k, k+1, \ldots, k+m+s\}$, where $s = n-1$. Also, the uniqueness of all the elements $f^*(u \sim v)$ is evident from the pattern of the sets mentioned above. Similar arguments as Case 1 indicate the bijection of f^* in this case. Therefore, it follows that f^* maps the set of pendant extension edges uniquely to the set $\{k, k+1, \ldots, k+m+n-1\}$, when $(n-1)^2 < m-r \leqslant n^2-n$.

Case 3: $n^2 - n < m - r < n^2 + \lfloor \frac{k}{n} \rfloor$.

As $n^2 - n < m - r$, it is clear that in this case $s = \min\{\ell, n-1\} = n-1$. Also $\min\{(s+1)(n-1)+r, m\} = \min\{n^2-n+r, m\} = (s+1)(n-1)+r$.

$$f^*(y_{j1} \sim y_{j0}) = k + r + n(n-j), \quad \text{if } 1 \leqslant j \leqslant n$$

$$f^*(x_{i1} \sim x_{i0}) = \begin{cases} k+i-1, & \text{if } 1 \leqslant i \leqslant r \\ i+p+k, & \text{if } p(n-1)+r+1 \leqslant i \\ & \leqslant \min\{(p+1)(n-1)+r, m\} \\ i+k+n-1, & \text{if } (s+1)(n-1)+r+1 \\ & \leqslant i \leqslant m \end{cases}$$

where $p \in \{0, 1, 2, \ldots, n-1\}$. It is easy to observe that $f^*(y_{j1} \sim y_{j0}) = \{k+r+n^2-n, k+r+n^2-2n, \ldots, k+r\}$. On the other hand

$$\begin{aligned} f^*(x_{i1} \sim x_{i0}) &= \{k, k+1, \ldots, k+r-1\} \\ &\bigcup \Big(\bigcup_{p=0}^{s} C_p \Big) \bigcup \{k+r+n^2, k+r+n^2+1, \ldots, m+k+n-1\}, \end{aligned}$$

where

$$\begin{aligned}
C_0 &= \{k+r+1, k+r+2, \ldots, k+r+n-1\},\\
C_1 &= \{k+r+n+1, k+r+n+2, \ldots, k+r+2n-1\},\\
&\vdots\\
C_{n-1} &= \{k+r+n^2-n+1, k+r+n^2-n+2, \ldots, k+r+n^2-1\}.
\end{aligned}$$

Proceeding similarly as the previous two cases we obtain that

$$f^*(x_{i1} \sim x_{i0}) \cup f^*(y_{i1} \sim y_{i0}) = \{k, k+1, \ldots, k+m+n-1\}.$$

Therefore, similar arguments as the previous two cases leads us to the fact that that f^* maps the set of pendant extension edges uniquely to the set $\{k, k+1, \ldots, k+m+n-1\}$, when $n^2 - n < m - r < n^2 + \lfloor \frac{k}{n} \rfloor$.

This completes the proof. □

Lemma 1 proves that all the edge labels obtained by f^* are distinct. Now, to prove the k-gracefulness of the graph, it remains to show that the the vertex labels defined by f are distinct as well. In the following section, we either prove the uniqueness of the vertex labels and when there are repetitions in the vertex labels under certain condition, we rearrange the assignments to ensure there is no repetition and also the uniqueness of the edge labels remains unaltered.

2. Vertex labeling

2.1. $k \geqslant mn + m + n$

In the following theorem, we see when the function f labels the graph k-gracefully, i.e., the vertex labels are unique. (In other words, there is no repetition in the vertex labels).

Theorem 1. *$K_{m,n} \odot K_1$ is k-graceful for any positive integer m, n when $m \leqslant n^2 + n$, and $k \geqslant mn + m + n$.*

Proof. We have already shown that the induced function f^* is bijective. It is easy to observe that the vertex labeling function f is well-defined. Hence to prove that $K_{m,n} \odot K_1$ is k-graceful, we just need to show that f is injective, when $k \geqslant mn + m + n$. Our approach is to show that f is distributing vertex labels in four mutually exclusive subsets. In other words, all of the vertex labels are distinct. Note that

$$f(x_{i0}) = n(i-1)$$

where $i \in \{1, 2, \ldots, m\}$, therefore $\bigcup_{i=1}^{m} f(x_{i0}) = \{0, n, \ldots, mn - n\}$. Next

$$f(y_{j0}) = mn + m + n + k - j$$

where $j \in \{1, 2, \ldots, n\}$, i.e.

$$\bigcup_{j=1}^{n} f(y_{j0}) = \{mn + m + n + k - 1, mn + m + n + k - 2, \ldots, mn + m + k\}.$$

Now we need to consider two cases based on the value r.

Case 1: $r = 0$. This occurs when k is a multiple of n

$$f(x_{i1}) = \begin{cases} n(i-1) & \text{if } p(n-1) + r + 1 \leqslant i \\ \quad + i + p + k, & \leqslant \min\{(p+1)(n-1) + r, m\} \\ n(i-1) & \text{if } (s+1)(n-1) < i - r \leqslant m \\ \quad + i + n + k - 1, & \end{cases}$$

Hence $\bigcup_{i=1}^{m} f(x_{i1}) = \{k + 1, k + 2, \ldots, mn + m + k - 1\}$. Next

$$f(y_{j1}) = \begin{cases} mn + m + 2n - j(n+1), & \text{if } 1 \leqslant j \leqslant \ell + 1; \\ & m \leqslant n^2 - n \\ mn + n - 2j + 1, & \text{if } \ell + 2 \leqslant j \leqslant n; \\ & m \leqslant n^2 - n \\ mn + m + 2n - j(n+1), & \text{if } m > n^2 - n \end{cases}$$

Hence if $m \leqslant n^2 - n$, then

$$\bigcup_{j=1}^{n} f(y_{j1}) = \{mn - n + 1, mn - n + 3, \ldots, mn + m + n - 1\}.$$

If $n^2 - n < m \leqslant n^2 + n$, then

$$\bigcup_{j=1}^{n} f(y_{j1}) = \{mn + m + n - 1, mn + m - 2, \ldots, mn + m - n^2\}.$$

Case 2: $r \neq 0$.

$$f(x_{i1}) = \begin{cases} n(i-1) + k + i - 1, & \text{if } 1 \leqslant i \leqslant r \\ n(i-1) & \text{if } p(n-1) + r + 1 \leqslant i \\ \quad + i + p + k, & \leqslant \min\{(p+1)(n-1) + r, m\} \\ n(i-1) & \text{if } (s+1)(n-1) < i - r \leqslant m \\ \quad + i + n + k - 1, & \end{cases}$$

Hence $\bigcup_{i=1}^{m} f(x_{i1}) = \{k, k+1, \dots, mn+m+k-1\}$. Next

$$f(y_{j1}) = \begin{cases} mn+m+2n-j(n+1)-r, & \text{if } 1 \leqslant j \leqslant \ell+1; \\ & m \leqslant n^2-n \\ mn+n-2j+1, & \text{if } \ell+2 \leqslant j \leqslant n; \\ & m \leqslant n^2-n \\ mn+m+2n-j(n+1)-r, & \text{if } m > n^2-n \end{cases}$$

Hence if $m \leqslant n^2 - n$, then

$$\bigcup_{j=1}^{n} f(y_{j1}) = \{mn-n+1, mn-n+3, \dots, mn+m+n-r-1\}.$$

If $n^2 - n < m \leqslant n^2 + n$, then

$$\bigcup_{j=1}^{n} f(y_{j1}) = \{mn+m+n-r-1, mn+m-r-2, \dots, mn+m-r-n^2\}.$$

For the sake of the proof we define the four sets as follows:

$$\begin{aligned} \mathsf{A} &= \{f(x_{i0}) \in \mathbb{Z} : i = 1, 2, \dots, m\} \\ \mathsf{B} &= \{f(x_{i1}) \in \mathbb{Z} : i = 1, 2, \dots, m\} \\ \mathsf{C} &= \{f(x_{j0}) \in \mathbb{Z} : i = 1, 2, \dots, n\} \\ \mathsf{D} &= \{f(x_{j1}) \in \mathbb{Z} : i = 1, 2, \dots, n\} \end{aligned}$$

From the pattern of the vertex labels obtained above, we see that all the elements in set X (for each $X \in \mathsf{A}, \mathsf{B}, \mathsf{C}, \mathsf{D}$) can be arranged in an increasing/decreasing order and this can also be observed that all the elements of the set X (for each $X \in \mathsf{A}, \mathsf{B}, \mathsf{C}, \mathsf{D}$) is distinct. Further, we assume that $\max_X$ and $\min_X$ denote the maximum and minimum element respectively, contained in set X (for each $X \in \mathsf{A}, \mathsf{B}, \mathsf{C}, \mathsf{D}$).

From the above discussion it can be easily verified (for the both the cases) that

$$\begin{aligned} &\min_{\mathsf{A}} = 0, && \max_{\mathsf{A}} = mn-n, \\ &\min_{\mathsf{B}} = k+1, && \max_{\mathsf{B}} = mn+m+k-1, \\ &\min_{\mathsf{C}} = k+mn+m, && \max_{\mathsf{C}} = mn+m+n+k-1, \\ &\min_{\mathsf{D}} = mn-n-r+1, && \max_{\mathsf{D}} = mn+m+n-1. \end{aligned}$$

Since $k \geqslant mn + m + n$, it can be easily noticed that

$$\min_{\mathsf{A}} < \max_{\mathsf{A}} < \min_{\mathsf{D}} < \max_{\mathsf{D}} < \min_{\mathsf{B}} < \max_{\mathsf{B}} < \min_{\mathsf{C}} < \max_{\mathsf{C}}$$

This implies $\mathsf{A} < \mathsf{B} < \mathsf{C} < \mathsf{D}$, i.e. we have

$$\begin{aligned} &\mathsf{A} \cap \mathsf{B} \cap \mathsf{C} \cap \mathsf{D} \\ &\quad = \Big(\bigcup_{i=1}^{m} f(x_{i0})\Big) \bigcap \Big(\bigcup_{i=1}^{m} f(x_{i1})\Big) \bigcap \Big(\bigcup_{j=1}^{n} f(j_{j0})\Big) \bigcap \Big(\bigcup_{j=1}^{n} f(y_{j0})\Big) = \varnothing. \end{aligned}$$

This completes the proof. □

Theorem 1 proves the uniqueness of the vertex labels when $k > mn + m + n$. Therefore, Lemma 1 and Theorem 1 together prove that $G = K_{m,n} \odot K_1$ is k-graceful when $k > mn + m + n$. In the following subsection we shall observe the case when $k < mn + m + n$.

2.2. $k < mn + m + n$

In this section we consider the case $k < mn + m + n$. We assume that $k + r = nq$, where q is any positive integer. In this section we first define few sets of integers.

- $A_1 = \{z \in \mathbb{Z} | z = (m + 1 - j)/n + m + 2 - i - j,$ where $1 \leqslant i \leqslant r, 1 \leqslant j \leqslant \min\{\ell + 1, n\}\}$
- $A_2 = \{z \in \mathbb{Z} | z = (m - j - p)/n + m + 2 - i - j,$ where $p(n-1) + r + 1 \leqslant i \leqslant \min\{(p+1)(n-1) + r, m\},$
  $1 \leqslant j \leqslant \min\{\ell + 1, n\}\}$
- $A_3 = \{z \in \mathbb{Z} | z = (m + 1 - j)/n + m + 1 - i - j,$ where $(s+1)(n-1) < i - r \leqslant m, 1 \leqslant j \leqslant n\}$
- $A_4 = \{z \in \mathbb{Z} | z = n(m + 1 - i) - 2j + 2,$ where $1 \leqslant i \leqslant r, \ell + 2 \leqslant j \leqslant n\}$
- $A_5 = \{z \in \mathbb{Z} | z = n(m + 1 - i) - 2j - p + 1,$ where $p(n-1) + r + 1 \leqslant i \leqslant \min\{(p+1)(n-1) + r, m\}, \ell + 2 \leqslant j \leqslant n\}$

Now we have the following theorem which describes that the graph $G = K_{m,n} \odot K_1$ is k-graceful, if k is not in this following sets of integers.

Theorem 2. *Let $G = K_{m,n} \odot K_1$, then G is k-graceful if each of the followings statements are true.*

1) *If $m \leqslant n^2 - n + r$, then q is not in A_1, A_2, and/or k is not in A_4, and A_5.*
2) *If $n^2 - n < m - r \leqslant n^2 + \lfloor \frac{k}{n} \rfloor$, then q is not in A_1, A_2, and/or A_3.*

Proof. First assume that $m \leqslant n^2 - n$. Then if $L_i = R_j$ for some i, j, then we have this following cases to consider, depending on the values of i and j.

Case 1: If $1 \leqslant i \leqslant r$, and $1 \leqslant j \leqslant \min\{\ell + 1, n\}$, then we have $(i-1)n + k + i - 1 = mn + m + n + k - j - (k + r + n(j-1))$, which simplifies to $n(k + r + i + j - m - 2) = m + 1 - j$. As we know $k + r = nq$, we easily arrive at $q = (m + 1 - j)/n + m + 2 - i - j \in A_1$.

Case 2: If $p(n-1) + r + 1 \leqslant i \leqslant \min\{(p+1)(n-1) + r, m\}$, and $1 \leqslant j \leqslant \min\{\ell+1, n\}$, then we have $(i-1)n+k+i+p = mn+m+n+k-j-(k+r+n(j-1))$, which simplifies to $n(k+r+i+j-m-2) = m-p-j$. As we know $k+r = nq$, we easily arrive at $q = (m-j-p)/n+m+2-i-j \in A_2$.

Case 3: If $1 \leqslant i \leqslant r$, and $\ell + 1 \leqslant j \leqslant n$, then we have $(i-1)n + k + i - 1 = mn + m + n + k - j - (k + m + j - 1)$, which implies to $k = n(m + 1 - i) - 2j + 2 \in A_4$.

Case 4: If $p(n-1)+r+1 \leqslant i \leqslant \min\{(p+1)(n-1)+r, m\}$, and $\ell+1 \leqslant j \leqslant n$, then we have $(i-1)n + k + i + p = mn + m + n + k - j - (k + m + j - 1)$, which implies to $k = n(m + 1 - i) - 2j - p + 1 \in A_5$.

Next we consider $n^2 - n \leqslant m \leqslant n^2 + n$. In this case it is easy to observe that $\min\{\ell + 1, n\} = n$. Then if $L_i = R_j$ for some i, j, then we again have some more cases to consider, depending on the values of i, and j. When $1 \leqslant i \leqslant r$, or $p(n-1) + r + 1 \leqslant i \leqslant \min\{(p+1)(n-1) + r, m\}$, then similar to Case(1), and Case(2), we can easily observe that q is not in A_1, A_2. The remaining case that we need to consider is $(s+1)(n-1) < i - r \leqslant m$ and $1 \leqslant j \leqslant n$. In this case we have $(i-1)n + k + i + n - 1 = mn + m + n + k - j - (k + r + n(j-1))$, which simplifies to $n(k + r + i + j - m - 1) = m + 1 - j$. As we know $k + r = nq$, we easily arrive at $q = (m + 1 - j)/n + m + 1 - i - j \in A_3$. This completes the proof. $\square$

Throughout the paper we have followed the restriction on m that $m \leqslant n^2 + \lfloor \frac{k}{n} \rfloor + r + 1$. In the following lemma we explain the reason behind this restriction. Actually, we observe that $\left(\bigcup_{i=1}^{m} f(x_{i0})\right) \bigcap \left(\bigcup_{i=1}^{m} f(x_{i1}) \neq \varnothing\right.$ when $m > n^2 + \lfloor \frac{k}{n} \rfloor + r + 1$. As a consequence of that we get a repetition in the vertex labels and hence f fails to remain injective anymore.

Lemma 2. *f is not injective when $m > n^2 + \lfloor \frac{k}{n} \rfloor + r + 1$.*

Proof. In this theorem we consider the case when $m > n^2 + \lfloor \frac{k}{n} \rfloor + r + 1$. There is no restriction on any other variable in this proof. For our convenience, throughout this proof we express $f(x_{ic})$ as $f(x_{i,c})$ (where $c = 0, 1$).

If possible we assume that f is injective i.e., $f(x_{i_1,j_1}) \neq f(x_{i_2,j_2})$ for any $i_1, i_2 \in \{1, 2, \ldots, m\}$ and $j_1, j_2 \in \{0, 1\}$.

First, note that $k = nq - r$, which implies $\lfloor k/n \rfloor = q - 1$.

Now, in this proof we must have $m > n^2 + \lfloor \frac{k}{n} \rfloor + r + 1$. So, we start with the assumption $m \geqslant n^2 + \lfloor \frac{k}{n} \rfloor + r + 2 = n^2 + q + r + 1$.

Without loss of any generality, we consider a particular case where $m = n^2+q+r+1$. Then we have from the definition, $f(x_{m,0}) = n(m-1) = n(n^2 + q + r)$.

On the other hand, since $m = n^2 + q + r + 1 > (n-1)^2$, then $s = \min\{\ell, n-1\} = n-1$ in this case. So, for $i = n(n-1) + r + 1$, we get $f(x_{i,1}) = \{n(n-1)+r+1-1\}+n(n-1)+r+1+n+k-1 = n(n^2+q+r)$. Therefore, we get a contradiction $f(x_{i_1,j_1}) = f(x_{i_2,j_2}) = n(n^2+q+r)$ when $i_1 = m, j_1 = 0$ and $i_2 = n(n-1) + r + 1, j_2 = 1$. Hence, this completes the proof. □

Conclusion

In this paper we have shown that $K_{m,n} \odot K_1$ is k-graceful for any integer $n \geqslant 2$ and $m < n^2 + \lfloor \frac{k}{n} \rfloor + r$. As our future work we would like to investigate whether is possible to label $K_{m,n} \odot K_1$ k-gracefully, for any m, and k.

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