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RESEARCH ARTICLE

On finite groups with K-$\mathfrak{N}_\sigma$-subnormal Schmidt subgroups

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Abstract. Let G be a finite group and $\sigma = \{\sigma_i | i \in I\}$ be some partition of the set of all primes. A subgroup A of G is said to K-$\mathfrak{N}_\sigma$-*subnormal* in G if there is a subgroup chain $A = A_o \leq A_1 \leq \cdots \leq A_n = G$ such that either $A_{i-1} \trianglelefteq A_i$ or $A_i/(A_{i-1})_{A_i} \in \mathfrak{N}_\sigma$ for all $i = 1, \ldots, n$, where $\mathfrak{N}_\sigma$ is a hereditary K-lattice saturated formation of all σ-nilpotent groups. The formation $\mathfrak{N}_\sigma$ is called K-*lattice* if in every finite group G the set $\mathcal{L}_{K\mathfrak{N}_\sigma}(G)$, of all K-$\mathfrak{N}_\sigma$-subnormal subgroup of G, is a sublattice of the lattice $\mathcal{L}(G)$ of all subgroups of G. In this paper we prove that if every Schmidt subgroup of G is K-$\mathfrak{N}_\sigma$-subnormal subgroup of G, then the commutator subgroup G' of G belongs to hereditary K-lattice saturated formation $\mathfrak{N}_\sigma$.

Introduction

Throughout this paper, all groups are finite and G always denotes a finite group. Moreover, $\mathbb{P}$ is the set of all primes, $\pi = \{p_1, \ldots, p_n\} \subseteq \mathbb{P}$ and $\pi' = \mathbb{P} \backslash \pi$. If n is an integer, the symbol $\pi(n)$ denotes the set of all primes dividing n; as usual, $\pi(G) = \pi(|G|)$, the set of all primes dividing the order of G.

Following [13, 7, 9, 10], a set $\mathcal{H}$ of subgroups of G is said to be a *complete Hall σ-set* of G if every non-identity member of $\mathcal{H}$ is a Hall

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σ_i-subgroup of G for some $\sigma_i \in \sigma(G)$ and $\mathcal{H}$ contains exactly one Hall σ_i-subgroup for every $\sigma_i \in \sigma(G)$. G is said to be; *σ-full* if G possesses a complete Hall σ-set; *σ-primary* if G is a σ_i-group for some i; *σ-nilpotent* if $G = G_1 \times G_2 \cdots \times G_n$ for some σ-primary groups $G_1, \ldots, G_n$; *σ-soluble* if every chief factor of G is σ-primary. A subgroup A of a G is said to *σ-subnormal* in G if there is a subgroup chain $A = A_o \leq A_1 \leq \cdots \leq A_n = G$ such that either $A_{i-1} \trianglelefteq A_i$ or $A_i/(A_{i-1})_{A_i}$ is σ-primary for all $i = 1, \ldots, n$.

The symbol $\mathfrak{N}_\sigma$ denotes the class of all σ-nilpotent groups, then $\mathfrak{N}_\sigma$ is a formation [13]. $F_\sigma(G)$ is the *σ-Fitting subgroup* of G, that is, the product of all normal σ-nilpotent subgroups of G.

Recall that a class of groups $\mathfrak{F}$ is called a *formation* if: (i) $G/N \in \mathfrak{F}$ whenever $G \in \mathfrak{F}$, and (ii) $G/(N \cap R) \in \mathfrak{F}$ whenever $G/N \in \mathfrak{F}$ and $G/R \in \mathfrak{F}$. The formation $\mathfrak{F}$ is called: *saturated or local* if $G \in \mathfrak{F}$ whenever $G/\Phi(G) \in \mathfrak{F}$; *hereditary* if $A \in \mathfrak{F}$ whenever $A \leq G \in \mathfrak{F}$. The $G^{\mathfrak{N}_\sigma}$ denotes the *$\mathfrak{N}_\sigma$-residual* of G [1], that is, the intersection of all normal subgroups N of G with $G/N \in \mathfrak{N}_\sigma$.

A subgroup A of G is said to be *K-$\mathfrak{N}_\sigma$-subnormal* [11] in G if there is a chain of subgroups $A = A_0 \leq A_1 \leq \cdots \leq A_t = G$ such that either $A_{i-1} \trianglelefteq A_i$ or $A_i/(A_{i-1})_{A_i} \in \mathfrak{N}_\sigma$ for all $i = 1, \ldots, t$.

The set $\mathcal{L}_{K\mathfrak{N}_\sigma}(G)$, of all $\mathfrak{N}_\sigma$-subnormal subgroups of G, is partially ordered with respect to set inclusion. Moreover, in the case where $\mathfrak{N}_\sigma$ is a hereditary formation the set $\mathcal{L}_{K\mathfrak{N}_\sigma}(G)$ is a lattice since $G \in \mathcal{L}_{K\mathfrak{N}_\sigma}(G)$ and, by [1, 6.1.7], for any $A_1, \ldots, A_n \in \mathcal{L}_{K\mathfrak{N}_\sigma}(G)$ the subgroup $A_1 \cap \cdots \cap A_n \in \mathcal{L}_{K\mathfrak{N}_\sigma}(G)$, so this intersection is the greatest lower bound for $\{A_1, \ldots, A_n\}$ in $\mathcal{L}_{K\mathfrak{N}_\sigma}(G)$.

The formation $\mathfrak{N}_\sigma$ is called *K-lattice* [1] if in every group G the lattice $\mathcal{L}_{K\mathfrak{N}_\sigma}(G)$ is a sublattice of the lattice $\mathcal{L}(G)$ of all subgroups G, that is, $A \cap B$, $\langle A, B\rangle \in \mathcal{L}_{K\mathfrak{N}_\sigma}(G)$ for all $A, B \in \mathcal{L}_{K\mathfrak{N}_\sigma}(G)$.

The full classification of K-lattice hereditary saturated formations were given in the remarkable papers [1, 2, 13, 18], and the authors proved useful in the study of many problems in the theory of finite groups. In the given article, we obtain further applications of the theory of K-lattice formations.

Let $\mathfrak{F}$ be a class of groups. If $G \notin \mathfrak{F}$ but every proper subgroup $H \in \mathfrak{F}$ of G, then G is called *$\mathfrak{F}$-critical* [4, p. 517]. An $\mathfrak{N}$-critical group is called *Schmidt group*, where $\mathfrak{N}$ is the class of all nilpotent groups.

A large number of publications are related to the study of influence on the structure of the group of its critical subgroups, in particular, Schmidt subgroups. Our main goal here is to prove the following result.

Theorem 1. *Let $\mathfrak{N}_\sigma$ be a hereditary K-lattice saturated formation of all σ-nilpotent groups. If every $\mathfrak{N}_\sigma$-critical subgroup of G is K-$\mathfrak{N}_\sigma$-subnormal in G, then $G/F_\sigma(G) \in \mathfrak{N}_\sigma$. Moreover, if all Schmidt subgroup of G is K-$\mathfrak{N}_\sigma$-subnormal in G, then $G/F_\sigma(G)$ is abelian.*

Note: It is notable that every σ-subnormal subgroup of G is K-$\mathfrak{N}_\sigma$-subnormal in G. But the converse is not true.

Corollary 1 ([14, Theorem 1.2])**.** *If every Schmidt subgroup of G is σ-subnormal in G, then $G/F_\sigma(G)$ is abelian.*

In the case when $\sigma = \{\{2\}, \{3\}, \ldots\}$, we get from Theorem 1 the following results.

Corollary 2 ([17])**.** *If every Schmidt subgroup of G is subnormal in G, then G is metanilpotent.*

Corollary 3 ([12])**.** *If every Schmidt subgroup of G is subnormal in G, then $G/F(G)$ is abelian.*

Corollary 4. *The set of all K-$\mathfrak{N}_\sigma$-subnormal subgroups of G forms a sublattice of the lattice of all subgroups of G.*

Proof. It follows directly from [13, Proposition 2.3], [1, Theorem 6.3.9 and Lemma 6.3.11]. □

Example 1. Let $\mathfrak{N}_\sigma$ be the class of all σ-nilpotent groups. It is not difficult to show that $G \in \mathfrak{N}_\sigma$ if and only if $G = G_1 \times \cdots \times G_t$ for some σ-primary groups $G_1, \cdots, G_t$. Hence $\mathfrak{N}_\sigma$ is a hereditary formation [13]. Moreover, the class $\mathfrak{N}_\sigma$ is a hereditary K-lattice saturated formation [13, Lemma 2.5].

Remark 1. According to Skiba in view of [13, Proposition 2.3], a subgroup A of a σ-group G is σ-subnormal in G if and only if it is K-$\mathfrak{N}_\sigma$-subnormal in G.

Remark 2. G is nilpotent if and only if it is σ^0-nilpotent where $\sigma^0 = \{\{2\}, \{3\}, \ldots\}$.

1. Preliminaries

Recall that G is said to be *π-decomposable* if G is σ-nilpotent, where $\sigma = \{\pi, \pi'\}$, that is, $G = O_\pi(G) \times O_{\pi'}(G)$.

Lemma 1 ([3]). *If G is an $\mathfrak{F}$-critical group, where $\mathfrak{F}$ is the class of all π-decomposable groups, then G is Schmidt group.*

Lemma 2 ([13, Lemma 2.6]). *Let A, K and N be subgroups of G. Suppose that A is σ-subnormal in G and N is normal in G.*

(i) $A \cap K$ is σ-subnormal in K.

(ii) AN/N is σ-subnormal in G/N.

(iii) If A is a σ_i-group, then $A \leq O_{\sigma_i}(G)$.

Lemma 3 ([8, III, 5.2] or [5, Ch. 1, Proposition 1.9]). *If G is Schmidt group, then $G = P \rtimes Q$, where $P = G^{\mathfrak{N}}$ is a Sylow p-subgroup of G and $Q = \langle x \rangle$ is a cyclic Sylow q-subgroup of G. Moreover, $\langle x^q \rangle \leq \Phi(G)$ and P is of exponent p or exponent 4 if P is a non-abelian 2-group.*

Lemma 4 ([6, Lemma 2.1.3]). *Let N be a normal subgroup of G and $\mathfrak{F}$ be a formation. Then $(G/N)^{\mathfrak{F}} = G^{\mathfrak{F}}N/N$.*

Lemma 5 ([13, Lemma 2.5]). *The class of all σ-nilpotent groups $\mathfrak{N}_\sigma$ is closed under taking direct products, homomorphic images and subgroups. Moreover, if E is a normal subgroup of G and $E/E \cap \Phi(G)$ is σ-nilpotent, then E is σ-nilpotent.*

Lemma 6 ([16, Theorem B]). *If G is σ-soluble, then G is a σ-full group.*

Lemma 7 ([14, Proposition 1.6]). *Suppose that G is σ-soluble and let $\mathfrak{F}$ be the class of all σ_i'-closed groups for some i. If G is an $\mathfrak{F}$-critical group, then G is σ_i-closed Schmidt group.*

2. Proof of Theorem

Proof. Suppose that this is false and let G be a counterexample of minimal order. Then $D = G^{\mathfrak{N}_\sigma} \neq 1$ and, $|\sigma(G)| > 1$.

(1) *G is σ-soluble.* Hence G is σ-group and G possesses an abelian minimal normal subgroup L.

Since G is not σ-nilpotent, it has an $\mathfrak{N}_\sigma$-critical subgroup K. Then, for some i, K is an $\mathfrak{N}_{\sigma_0}$-critical group, where $\sigma_0 = \{\sigma_i, \sigma_i'\}$. It follows that G is Schmidt group by [3]. Therefore G is soluble by [6, Theorem 1.10.7]. Hence G is σ-soluble, and G is σ-group by [13]. So (1) holds.

(2) *If H is a proper subgroup of G, then $H^{\mathfrak{N}_\sigma} \leq F_\sigma(H)$. Furthermore, if each Schmidt subgroup of G is K-$\mathfrak{N}_\sigma$-subnormal in G, then $H' \leq F_\sigma(H)$.*

This follows from Claim (1), Remark 2, Lemma 2(i) and the choice of G.

(3) *If N is a minimal normal subgroup of G, then $(G/N)^{\mathfrak{N}_\sigma} \leq F_\sigma(G/N)$. Furthermore, if each Schmidt subgroup of G is K-$\mathfrak{N}_\sigma$-subnormal in G, then $(G/N)' \leq F_\sigma(G/N)$.*

If $G/N \in \mathfrak{N}_\sigma$, it is evident. Now let $G/N \notin \mathfrak{N}_\sigma$. We show that in this case the hypothesis holds for G/N. Assume that H/N is any Schmidt subgroup (respectively, any $\mathfrak{N}_\sigma$-critical subgroup) of G/N. Let E be a minimal supplement to N in H. Then $E/E \cap N \cong EN/N = H/N$ is a Schmidt subgroup (respectively, an $\mathfrak{N}_\sigma$-critical subgroup) and $E \cap N \leq \Phi(E)$. Let $\Phi = \Phi(E)$ and A be a Schmidt subgroup (respectively, an $\mathfrak{N}_\sigma$-critical subgroup) of E. According to Claim (1) and Lemmas 1 and 3 we have

$$(E/E \cap N)/\Phi(E/E \cap N) = (E/E \cap N)/(\Phi/E \cap N) \cong E/\Phi = P \rtimes Q,$$

where P is a Sylow p-subgroup and Q is a Sylow q-subgroup of E/Φ with $|Q| = q$, for some primes $p \neq q$. It follows, again by Lemma 3, that $A = A_p \rtimes A_q$, where $A = (A_q)^A$. Then $A_q \nleq \Phi$, since Φ is nilpotent. It follows that $\Phi A_q/\Phi$ is a Sylow q-subgroup of H/Φ and thus

$$(\Phi A_q/\Phi)^{E/\Phi} = (A_q)^E \Phi/\Phi = E/\Phi.$$

Hence $(A_q)^E = E$, thus $H = EN = (A_q)^E N$. It follows that, if H/N is an $\mathfrak{N}_\sigma$-critical group, then A is an $\mathfrak{N}_\sigma$-critical group and, so A is K-$\mathfrak{N}_\sigma$-subnormal subgroups of G by hypothesis. Hence the subgroup A^E is K-$\mathfrak{N}_\sigma$-subnormal subgroups of G since $\mathfrak{N}_\sigma$ is K-lattice by hypothesis, so $H/N = A^E N/N$ is K-$\mathfrak{N}_\sigma$-subnormal in G/N by Claim (1), Remark 2 and Lemma 2(ii). Therefore the hypothesis holds for G/N, so the choice of G implies that we have (3).

(4) *$L \nleq \Phi(G)$ and for some $p \in \sigma_j$ we have $L = C_G(L) = O_p(G) \leq O_{\sigma_j}(G)$. Moreover, $|L| > p$ and for some maximal subgroup M of G, we have $G = L \rtimes M$.*

It is notable that for some $p \in \sigma_j$ we have $L \leq O_p(G) \leq O_{\sigma_j}(G)$ by Claim (1). Claim (3) and Lemma 4 imply that the group

$$(G/L)^{\mathfrak{N}_\sigma} = G^{\mathfrak{N}_\sigma} L/L = DL/L \cong D/D \cap L$$

(respectively, the derived subgroup $(G/L)' = G'L/L \cong G'/G' \cap L$ of G/L) is σ-nilpotent. Assume that G has a minimal normal subgroup

$R \neq L$. Then $D/D \cap R$ (respectively, $G'/G' \cap R$) is σ-nilpotent. But then $D \cong D/(D \cap L) \cap (D \cap R)$ (respectively, $G' \cong G'/(G' \cap L) \cap (G' \cap R)$) is σ-nilpotent. It follows that $G/F_\sigma(G)$ is σ-nilpotent (respectively, $G/F_\sigma(G)$ is abelian), contrary to the choice of G. Hence L is the unique minimal normal subgroup of G. Furthermore, if $L \leq \Phi(G)$ then D (respectively G') is σ-nilpotent by Lemma 5. Hence $L \nleq \Phi(G)$ and $L = C_G(L) = O_p(G) \leq O_{\sigma_j}(G)$ by [4, A, 15.6(2)]. If $|L| = p$, then $G/C_G(L) = G/L$ is cyclic, so G' is cyclic, it follows that $G/F_\sigma(G)$ is abelian, a contradiction. Hence we have (4).

(5) $L \leq O_{\sigma_i}(G) = F_\sigma(G)$ [This directly follows from Claims (1), (3) and (4)].

(6) $D \leq O_{\sigma_i}(G)$.

Suppose that $\sigma_j \in \sigma \backslash \{\sigma_i\}$. Let G be not σ_j'-closed. Then G possesses an $\mathfrak{M}$-critical subgroup A, where $\mathfrak{M}$ is the class of all σ_j'-closed groups. In view of Claim (1) and Lemma 7, A is a σ_j-closed Schmidt group. Let $P = O_{\sigma_j}(A)$. By hypothesis, A is K-$\mathfrak{N}_\sigma$-subnormal in G, so A is σ-subnormal in G by Claim (1) and Remark 2. But then P is also σ-subnormal in G, therefore $1 < P < O_{\sigma_j}(G)$ by Lemma 2(iii), contrary to Claim (5). Thus G is σ_j'-closed for every $\sigma_j \in \sigma \backslash \{\sigma_i\}$. Therefore we have (6).

(7) *$L = F_\sigma(G) = O_{\sigma_i}(G)$ is a Hall σ_i-group of G.*

According to Claims (1), (4) and Lemma 6 we have $F_\sigma(G) = O_{\sigma_i}(G) \leq H$, where H is a Hall σ_i-group of G. Since $D = G^{\mathfrak{N}_\sigma} \leq O_{\sigma_i}(G)$ by Claim (6), $H/O_{\sigma_i}(G) \trianglelefteq G/O_{\sigma_i}(G)$. Hence $H \leq O_{\sigma_i}(G)$, it follows that $H = O_{\sigma_i}(G)$. Thus $F_\sigma(G) = H$. Now assume that $L < H$. By the Schur-Zassenhaus theorem G has a σ_i-complement, say W. Then $U = LW < G$, so Claim (2) implies that $U/F_\sigma(U)$ is abelian. Clearly, $F_\sigma(U) = L \times (F_\sigma(U) \cap W)$, where $F_\sigma(U) \cap W$ is a σ_i'-group. But Claim (4) implies that $C_G(L) \leq L$. Hence $F_\sigma(U) = L$ and so $W \cong U/L$ is abelian. Therefore $G/F_\sigma(U) = G/O_{\sigma_i}(G) = G/H$ is abelian, contrary to the choice of G. So we have (7).

(8) *A Hall σ_i'-subgroup M of G is a $\mathfrak{U}$-critical group of order q^n for some prime q, where $\mathfrak{U}$ is the class of all abelian groups.*

It is clear that M is a Hall σ_i'-subgroup of G by Claims (4) and (7). Indeed, let S be any maximal subgroup of M. Then $LS/F_\sigma(LS)$ is abelian by Claim (2). In view of Claim (7), $L = (LS)'$ and hence $S \cong LS/L$ is abelian. Thus the choice of G and Claim (7) imply that M is a $\mathfrak{U}$-critical group. Therefore, M is either a Schmidt group or a minimal non-abelian group of prime power order q^n. In the former case, by Lemma 3, $M = Q \rtimes V$, where $Q = M^{\mathfrak{N}} \in Syl_q(M)$, $V \in Syl_r(M)$ and $q \neq r$, and M is

K-$\mathfrak{N}_\sigma$-subnormal subgroups of G. According to Claim (1) and Remark 2 we have M is σ-subnormal in G, so Q is σ-subnormal in G too. By using Claim (7) and Lemma 2(iii), we get that $Q \leq O_{\sigma_j}(G)$ for some $j \neq i$. But this is impossible by Claim (5). This contradiction shows that we have (8).

Final contradiction. According to Claim (8), $Z(M) \neq 1$. Let W be a subgroup of order q in $Z(M)$ and $H = LW$. Then $W \nleq Z(H)$ by Claim (4), thus H is not nilpotent and hence it contains a Schmidt subgroup $A = A_p \rtimes W$. Note that $L = L_1 \times \cdots \times L_m$, where R_k is a minimal normal subgroup of H for all $k = 1, \ldots, m$ by the Mashcke's theorem. By the hypothesis, A is a K-$\mathfrak{N}_\sigma$-subnormal subgroup of G. Thus A is σ-subnormal in G by Claim (1), Remark 2 and Lemma 2(i). Suppose that $A < H$. Then there is a proper subgroup V of H such that $A \leq V$ and either H/V_H is a σ_i-group or V is normal in H. Since $W \leq V_H < H$, for some k we have $L_k \nleq V_H$. Hence $L_k \leq C_H(V_H)$, therefore $L_k \leq N_G(W) = M$. But then $L_k^G = L_k^{LM} = L_k^M \leq M$, which implies that $L \leq M$, a contradiction. Therefore $A = H$, thus $L = A_p$ and W acts irreducibly on L. Clearly, $W \leq \Phi(M)$ and therefore every proper subgroup of M acts irreducibly on L, it follows that every maximal subgroup of M is cyclic. Hence $q = 2$ and so $|L| = p$, contrary to Claim (4). The theorem is proved. □

3. Conflicts of Interest

The author declare no conflicts of interest.

4. Data Availability

No data were used to support this study.

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