© Algebra and Discrete Mathematics
Volume **37** (2024). Number 1, pp. 22–34
DOI:10.12958/adm2231 RESEARCH ARTICLE

Leavitt inverse semigroups of polynomial growth

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Communicated by E. I. Zelmanov

Abstract. We relate growth functions of graph inverse semigroups, Leavitt inverse semigroups and Leavitt path algebras and discuss structure of Leavitt inverse semigroups of polynomially bounded growth.

Introduction

Let F be a field and let $n \geq 1$ be a natural number. In 1962 W. Leavitt [13] constructed an algebra $L_F(1:n)$ with remarkable properties. A C^*-analog of this algebra is known as Cuntz algebra [8].

In 2004, G. Abrams, A. Aranda-Pino [1] and P. Ara, M.A. Moreno, E. Pardo [5] generalized Leavitt's construction and defined the so-called Leavitt path algebras corresponding to an arbitrary quiver.

In [6], C.J. Ash and T.E. Hall introduced graph inverse semigroups $I(\Gamma)$ corresponding to an arbitrary quiver Γ. Finally, in [14], J. Meakin, D. Milan and Z. Wang introduced Leavitt inverse semigroups $LI(\Gamma)$ as multiplicative subsemigroups of Leavitt path algebras $L_F(\Gamma)$ and, simultaneously, homeomorphic images of graph inverse semigroups $I(\Gamma)$. Leavitt path algebras of polynomially bounded growth were described by A. Alahmadi, H. Alsulami, S.K. Jain and E. Zelmanov in [3, 4].

In this paper, we relate growth functions of graph inverse semigroups $I(\Gamma)$, Leavitt inverse semigroups $LI(\Gamma)$ and Leavitt path algebras $L_F(\Gamma)$.

2020 Mathematics Subject Classification: *16S88, 20M18, 16P90.*

Key words and phrases: *(directed) graph, growth function, inverse semigroup, Leavitt inverse semigroup, Leavitt path algebra.*

Then we discuss structure of Leavitt inverse semigroups of polynomially bounded growth.

Theorem 1. *Let Γ be a finite quiver. Let $g_{I(\Gamma)}(n)$, $g_{LI(\Gamma)}(n)$, $g_{L(\Gamma)}(n)$ be growth functions of the inverse semigroups $I(\Gamma)$, $LI(\Gamma)$ and the algebra $L_F(\Gamma)$ respectively, corresponding to natural generators. Then there exists a constant K such that*

$$g_{LI(\Gamma)}(n) \leq g_{I(\Gamma)}(n) \leq K \cdot n \cdot g_{LI(\Gamma)}(n),$$

$$g_{L(\Gamma)}(n) \leq g_{LI(\Gamma)}(n) \leq K \cdot n \cdot g_{L(\Gamma)}(n).$$

If the graph Γ does not contain no-exit cycles then

$$g_{I(\Gamma)}(n) \leq K \cdot g_{LI(\Gamma)}(n), \quad g_{LI(\Gamma)}(n) \leq K \cdot g_{L(\Gamma)}(n).$$

Theorem 2. *Let Γ be a finite quiver such that the Leavitt inverse semigroup $LI(\Gamma)$ has polynomially bounded growth. Then $LI(\Gamma)$ has a finite ascending chain of ideals*

$$I_0 < I_1 < \cdots < I_s = LI(\Gamma)$$

such that

(1) *the ideal I_0 embedds into a block diagonal extended Brandt semigroup $\widetilde{B}(\mathcal{P}, 1)$, 1 is the identity group;*

(2) *each factor I_i/I_{i-1}, $1 \leq i \leq s$, embedds in a block diagonal extended Brandt semigroup $\widetilde{B}(\mathcal{E}, T)$, where T is the infinite cyclic group.*

Theorem 2 is an analog of Theorem 1 in [3, 4]. It generalizes Theorem 4.5 in [14].

1. Definitions

Let Γ be a finite quiver, i.e. a directed graph with the set of vertices V and the set of edges E. Consider the range mapping $r : E \to V$ and the source mapping $s : E \to V$. We think of an edge e as a directed edge from the source $s(e)$ to the range $r(e)$. For a vertex $v \in V$, the number $|s^{-1}(v)|$ is called the *index* of v.

A path p is a finite sequence $p = e_1 \cdots e_n$ of edges $e_i \in E$ such that $r(e_i) = s(e_{i+1})$ for $i = 1, \ldots, n-1$. We define $s(p) = s(e_1)$, $r(p) = r(e_n)$ and refer to p as a path from $s(p)$ to $r(p)$.

If $s(p) = r(p)$, then the path is closed. If $p = e_1 \cdots e_n$ is a closed path and the vertices $s(e_1), s(e_2), \ldots, s(e_n)$ are distinct, then the path p is called a *cycle*.

If $v \in V$, $s^{-1}(v) = \varnothing$, then the vertex v is called a *sink*.

Graph inverse semigroups. Let $\Gamma = (V, E)$ be a quiver. Define the graph inverse semigroup $I(\Gamma)$ as the semigroup with zero 0 presented by generators $\{v, v \in V\}$, $\{e, e^*, e \in E\}$ and the set of relations

(1) $s(e)\, e = e\, r(e) = e$, $r(e)\, e^* = e^* s(e) = e^*$ for all $e \in E$;

(2) $u\, v = 0$ for all $u, v \in V$, $u \neq v$; $v^2 = v$;

(3) $e^* f = 0$ if $e \neq f$, $e, f \in E$;

(4) $e^* e = r(e)$ if $e \in E$.

Recall that a semigroup S is called an *inverse semigroup* if for an arbitrary element $a \in S$ there exists a unique element $a^{-1} \in S$ such that

$$aa^{-1}a = a, \quad a^{-1}aa^{-1} = a^{-1}$$

(see [9, 12]).

It is easy to see that $I(\Gamma)$ is an inverse semigroup. The mapping $v \mapsto v$, $e \mapsto e^*$, $e^* \mapsto e$ extends to an involution $* : I(\Gamma) \to I(\Gamma)$. If $p = e_1 \cdots e_n$ is a path, then $p^* = e_n^* \cdots e_1^*$.

An arbitrary element $a \in I(\Gamma)$ can be uniquely represented as $a = pq^*$, where p, q are paths (possibly of zero length) and $r(p) = r(q)$.

Leavitt path algebras. Let F be a field and let Γ be a finite quiver. The Leavitt path algebra $L_F(\Gamma)$ is presented by generators $\{v,\, v \in V\}$, $\{e, e^*\, e \in E\}$ and defining relations (1)-(4) plus the additional Cuntz-Krieger relations

(5) $v = \sum_{e \in s^{-1}(v)} e\, e^*, \quad v \in V, \quad v$ is not a sink.

Let $F_0 I(\Gamma)$ be the contracted semigroup algebra, that is, the factor algebra of the semigroup algebra $FI(\Gamma)$ modulo the ideal $F\,0$, where 0 is the zero of the semigroup $I(\Gamma)$. Let

$$\operatorname{id}\big(v - \sum_{e \in s^{-1}(v)} e\, e^*, \quad v \in V\big)$$

be the ideal of $F_0 I(\Gamma)$ generated by the Cuntz-Krieger elements. It is clear from the definitions that

$$L_F(\Gamma) \cong F_0 I(\Gamma) \Big/ \operatorname{id}\big(v - \sum_{e \in s^{-1}(v)} e\, e^*, \quad v \in V\big).$$

For more information about Leavitt path algebras, see [2].

Following [14], consider the multiplicative subsemigroup $LI(\Gamma)$ of $L_F(\Gamma)$ generated by elements $v \in V$; $e, e^* \in E \cup E^*$. In [14], it is shown that $LI(\Gamma)$ is an inverse semigroup, which is presented by generators $v \in V$; $e, e^* \in E \cup E^*$, and defining relations (1) – (4) plus the additional relations

(5′) $v = ee^*$, $s(e) = v$, where v runs over all vertices of index 1.

The involution $*$ on $I(\Gamma)$ gives rise to the involution $*$ on $LI(\Gamma)$.

Let us describe the so-called normal basis in $L_F(\Gamma)$ and normal canonical form in $LI(\Gamma)$. For an arbitrary vertex $v \in V$, that is not a sink, choose an edge in $s^{-1}(v)$ and call it *special*.

- [3] Elements v, pq^*, where p, q are paths in Γ (one of them may have zero length), $r(p) = r(q)$, $p = e_1 \cdots e_n$, $q = f_1 \cdots f_m$, and either $e_n \neq f_m$ or $e_n = f_m$, the index of $s(e_n)$ is ≥ 2 and the edge e_n is not special, form a basis in $L_F(\Gamma)$.

•• [14] An arbitrary element of $LI(\Gamma)$ can be uniquely represented as v or pq^*, where p, q are paths in Γ (one of them may have zero length), $r(p) = r(q)$, $p = e_1 \cdots e_n$, $q = f_1 \cdots f_m$, and either $e_n \neq f_m$ or $e_n = f_m$, the index of $s(e_n)$ is ≥ 2.

2. Growth in Leavitt semigroups and algebras

Let S be a semigroup generated by a finite set X. Consider the function $g_X(n)$ defined to be the number of distinct elements in

$$\bigcup_{k=1}^{n} X^k.$$

The weakly increasing function $g_X(n)$ is called the *growth function* of S with respect to the generating set X.

Since the function $g_X(n)$ depends on a choice of X, we introduce the following definition.

Given two weakly increasing functions $f, g : \mathbb{N} \to [1, \infty)$, we say that f is *asymptotically greater than or equal to* g (written $f \succeq g$ or $g \preceq f$) if there exists a positive integer C such that

$$g(n) \leq f(Cn)$$

for all n. If $g \preceq f$ and $f \preceq g$ then we say that the functions f and g are *asymptotically equivalent* (denoted $f \sim g$). If X, Y are finite generating subsets of S, then $g_X(n) \sim g_Y(n)$. The growth of S is the equivalence class of growth functions $g_X(n)$.

For algebras, one can produce analogous functions as follows. If A is a finitely generated algebra over a field F and V is a finite-dimensional subspace that generates A as an F-algebra, then

$$g_V(n) = \dim_F V^n,$$

where V^n is the subspace of A spanned by all k-fold products in V with $1 \le k \le n$. As above, for any two finite-dimensional generating subspaces V and W of the algebra A we have $g_V(n) \sim g_W(n)$.

For more information on growth of groups, semigroups and algebras, see [7, 10, 11].

Let $g_{I(\Gamma)}(n)$, $g_{LI(\Gamma)}(n)$, $g_{L(\Gamma)}(n)$ be the growth functions of the semigroups $I(\Gamma)$, $LI(\Gamma)$, and the algebra $L_F(\Gamma)$ with respect to "natural" generators $v \in V$, e, e^*, $e \in E$.

Definition 1. *A path $p = e_1 \cdots e_n$, $e_i \in E$, is called a* no-exit path *if every vertex $s(e_i)$, $1 \le i \le n$, has index* 1.

Lemma 1. *Every no-exit path p can be represented as $p = e_1 \cdots e_n C^k$, where the vertices $s(e_1), \ldots, s(e_n), r(e_n)$ are distinct, C is a no-exit cycle.*

Proof. Let $p = e_1 \cdots e_m$. If all vertices $s(e_1), \ldots,\ s(e_m), r(e_m)$ are distinct, then we are done. Suppose that $s(e_i) = s(e_j)$, $i < j$, and $j - i$ is minimal with this property. Then $C = e_i e_{i+1} \cdots e_{j-1}$ is a no-exit cycle.

Since $r(e_{j-1}) = s(e_i)$ has index 1, it follows that $e_j = e_i$, $e_{j+1} = e_{i+1}$ and so on. We have

$$p = e_1 \cdots e_{i-1} C^k C',$$

where C' is a beginning of the cycle C, hence $C = C'C''$. Now,

$$p = e_1 \cdots e_{i-1} C' (C''C')^k,$$

where $C''C'$ is a no-exit cycle, all vertices on $e_1 \cdots e_{i-1} C'$ are distinct.

If all vertices $s(e_1), \ldots, s(e_m)$ are distinct but $s(e_j) = r(e_m)$, then $e_j \cdots e_m$ is a no-exit cycle. This completes the proof og the lemma. □

Proof of Theorem 1. The value $g_{I(\Gamma)}(n)$ is the number of elements of length $\le n$ in the normal form pq^*, $r(p) = r(q)$, $\text{length}(p) + \text{length}(q) \le n$. The value $g_{LI(\Gamma)}(n)$ is the number of products of length $\le n$ in the normal form in $LI(n)$.

Let $p = p'p_1$, $q = q'p_1$, where p_1 is a no-exit path. Then pq^* is in the normal form in $I(\Gamma)$ but in $LI(\Gamma)$ $pq^* = p'q'^*$.

Let $\mathcal{P}$ be the set of no-exit paths $e_1 \cdots e_k$ such that the vertices $s(e_1), \ldots, s(e_k), r(e_k)$ are distinct. Since the graph Γ is finite, it follows that $|\mathcal{P}| < \infty$. Let $C_1, \ldots, C_r$ be all distinct no-exit cycles in Γ. Clearly, distinct no-exit cycles do not intersect.

By Lemma 1, any arbitrary no-exit path can be represented as pC_i^t, $p \in \mathcal{P}$, $1 \leq i \leq r$. Hence, the number of no-exit paths of length $\leq n$ is $\leq |\mathcal{P}| \cdot r \cdot n$.

Let pq^* be an element in the normal form in $I(\Gamma)$, length(p) + length$(q) \leq n$; and let $p = p'p_1$, $q = q'p_1$, where p_1 is a no-exit path and p_1 is maximal with this property. Then $p'q'^*$ is in a normal form in $LI(\Gamma)$. Clearly p', q'^* and p_1 uniquely determine the element pq^*. This implies that

$$g_{I(\Gamma)}(n) \leq g_{LI(\Gamma)}(n) \cdot |\mathcal{P}| \cdot r \cdot n, \quad K = |\mathcal{P}| \cdot r.$$

Suppose that for an arbitrary vertex of index ≥ 2, we selected a special edge in $s^{-1}(v)$. If the index of the vertex v is 1, then the only edge in $s^{-1}(v)$ is special.

A path is called *special* if all edges on this path are special. Distinct special cycles do not intersect.

Repeating the proof of Lemma 1, we get

Lemma 2. *Every special path p can be represented as $p = e_1 \cdots e_n C^k$, where the vertices $s(e_1), \ldots, s(e_n), r(e_n)$ are distinct, C is a special cycle.*

An element in the normal basis of the Leavitt inverse semigroup $LI(\Gamma)$ that is not in the normal form in the Leavitt path algebra $L_F(\Gamma)$ looks as

$$p'p_1(q'p_1)^*,$$

where p_1 is a special path. Let p_1 be a maximal special path with this property. The element $p'q'^*$ is in the normal form in $LI(\Gamma)$.

Let S be the set of special paths $e_1 \cdots e_n$ having the vertices $s(e_1)$, $\ldots$, $s(e_n)$, $r(e_n)$ distinct. Let $C_1, \ldots, C_d$ be all distinct special cycles.

By Lemma 2, an arbitrary special path can be represented as pC_i^k, $p \in S$, $1 \leq i \leq d$. Hence, the number of special paths of length $\leq n$ is $\leq |S| \cdot d \cdot n$.

Arguing as above, we get

$$g_{LI(\Gamma)}(n) \leq g_{L(\Gamma)}(n) \cdot |S| \cdot d \cdot n,$$

this completes the proof of the theorem. □

Corollary 1. *The inverse semigroups $I(\Gamma)$, $LI(\Gamma)$, and the algebra $L_F(\Gamma)$ simultaneously have or have not polynomially bounded growths.*

A. Alahmadi, H. Alsulami, S.K. Jain and E. Zelmanov [3] proved that a Leavitt path algebra $L_F(\Gamma)$ has polynomially bounded growth if and only if any two distinct cycles of Γ do not have common vertices. By Theorem 1, this description works also for graph inverse semigroups and Leavitt inverse semigroups.

If distinct cycles have a common vertex, then they generate a free semigroup (see [3]). Hence, a Leavitt path algebra or a Leavitt inverse semigroup can not have an intermediate growth.

3. Structure of Leavitt inverse semigroups with polynomially bounded growth

Given two vertices $v_1, v_2 \in V$, we say that v_2 is a *descendant* of v_1 if there exists a path p such that $s(p) = v_1$, $r(p) = v_2$.

A nonempty subset $W \subseteq V$ is called *hereditary* if all descendants of elements of W again lie in W.

For a hereditary subset W, we may consider a subgraph $\Gamma(W) = (W, E_W)$, $E_W = \{e \in E | s(e) \in W\}$.

For a hereditary subset $W \subseteq V$, we consider the subset

$$\widetilde{W} = \{v \in V \mid \text{ all descendants of } v \text{ lie in } W\}.$$

Clearly, $W \subseteq \widetilde{W}$. The subset $\widetilde{W}$ is called a *hereditary saturated subset* if $\widetilde{W} = W$.

G. Abrams and A. Aranda Pino [1] established a $1-1$ correspondence between ideals of the Leavitt path algebra and hereditary saturated subsets of V. In particular, they showed that if W is a hereditary saturated subset of V and I is the ideal of $L_F(\Gamma)$ generated by W,

$$I = \text{Span } (pq^*;\ p, q \text{ are paths},\ r(p) = r(q) \in W),$$

then $I \cap V = W$.

Let us recall some semigroup constructions. Let $\mathcal{P}$ be a set (of indices), and let G be a group. Let $\mathbb{Z}G$ be the group ring of the group G. Consider the set $M_{\mathcal{P}\times\mathcal{P}}(\mathbb{Z}G)$ of $P \times \mathcal{P}$ matrices over $\mathbb{Z}G$.

For $i, j \in \mathcal{P}$, $a \in ZG$, consider the matrix $e_{ij}(a)$ having the element a at the intersection of the i-th row and j-th column and zeros everywhere else. Let O denote the zero matrix.

The set

$$B(\mathcal{P},G)=\{O,\ e_{ij}(g);\ i,j\in\mathcal{P},\ g\in G\}$$

is an inverse semigroup with zero. It is called a *Brandt semigroup*.

Consider also the set

$$\widetilde{B}(\mathcal{P},G)=\{O,\ e_{i_1j_1}(g)+\cdots+e_{i_kj_k}(g)\};\ i_1,\ldots,i_k,j_1,\ldots,j_k\in\mathcal{P},\ \ g\in G;$$

indices $i_1,\ldots,i_k$ are distinct and indices $j_1,\ldots,j_k$ are distinct.

The subset $\widetilde{B}(\mathcal{P},G)$ consists of $\mathcal{P}\times\mathcal{P}$ matrices: (i) having finitely many nonzero entries, (ii) each row is either a zero row or contains a fixed element $g\in G$ as the only nonzero entry, (iii) each column is either a zero column or contains a fixed element $g\in G$ as the only nonzero entry.

It is easy to see that $\widetilde{B}(\mathcal{P},G)$ is an inverse semigroup. We call it the *extended Brandt semigroup*.

If $\mathcal{P}=\dot{\cup}_k\mathcal{P}_k$ is a disjoint union of subsets, then $B(\dot{\cup}\mathcal{P}_i,G)$ and $\widetilde{B}(\dot{\cup}\mathcal{P}_i,G)$ denote the block diagonal versions of the semigroups $B(\mathcal{P},G)$, $\widetilde{B}(\mathcal{P},G)$: all entries at positions (i,j), where i,j do not belong to the same subset $\mathcal{P}_k$, are equal to zero.

Let S_i, $i\in I$, be a disjoint family of semigroups with zero. Let 0_i be the zero of the semigroup S_i. The semigroup

$$S=\bigcup_i\,(S_i\backslash\{0_i\})\ \dot{\bigcup}\ \{0\}$$

is defined as follows: if $a,b\in S_i\backslash\{0_i\}$ and $ab\neq 0_i$, then a, b are multiplied in S as in S_i. If $ab=0_i$, then we define $ab=0$. If $a\in S_i$, $b\in S_j$, $i\neq j$, then $ab=0$. The semigroup S is called the 0-*disjoint union* of the semigroups S_i.

Let A be an associative F-algebra. Let $\mathcal{P}$ be a countable set. The algebra of $\mathcal{P}\times\mathcal{P}$ matrices over A with finitely many nonzero entries is denoted as $M_\infty(A)$.

Now, let us go back to Leavitt inverse semigroup $LI(\Gamma)$. Let Γ be a finite quiver; and let

$$V_0=\{v\in V\ |\ \text{ no path starting at } v \text{ ends on a cycle}\}.$$

Clearly, all sinks lie in V_0.

It is easy to see that the subset V_0 is hereditary and saturated.

Let I_0 be the ideal of the semigroup $LI(\Gamma)$ generated by V_0,

$$I_0=\{pq^*\ |\ p,q \text{ are paths, } r(p)=r(q)\in V_0\}.$$

In [4], it is shown that the ideal $\mathrm{Span}_F(I_0)$ of the Leavitt path algebra spanned dy I_0 is isomorphic to a finite sum of finite-dimensional matrix algebras $M_n(F)$ and infinite-dimensional matrix algebras $M_\infty(F)$.

The subgraph $\Gamma(V_0)$ is a tree. In [14], it is shown that the Leavitt inverse semigroup of a tree is a finite 0-disjoint union of Brandt semigroups $B(X, 1)$, $|X| < \infty$.

The structure of the ideal I_0 is more complicated.

Let $v_1, \ldots, v_r$ be all distinct sinks of the graph Γ. Let $\mathcal{P}_i$ be the set of paths on Γ that end at v_i.

We will embed the semigroup I_0 in the block diagonal extended Brandt semigroup $\widetilde{B}(\dot{\cup}\mathcal{P}_i, 1)$.

For a vertex $v \in V_0$, let $\mathcal{P}_i(v)$ be the set of paths p such that $s(p) = v$, $r(p) = v_i$. Clearly, $|\mathcal{P}_i(v)| < \infty$. Let

$$\mathcal{P}(v) = \bigcup_{i=1}^{r} \mathcal{P}_i(v).$$

We will define the mapping

$$\varphi : I_0 \to \widetilde{B}\left(\dot{\bigcup_{i=1}^{r}} \mathcal{P}_i, 1\right).$$

Let

$$\varphi(v) = \sum_{p \in \mathcal{P}(v)} e_{p,p}(1) \in \widetilde{B}\left(\dot{\bigcup_{i=1}^{r}} \mathcal{P}_i, 1\right).$$

For an element $pq^* \in I_0$, we have

$$pq^* = \sum_{i=1}^{r} p \left(\sum_{u \in \mathcal{P}_i(r(p))} u\ u^* \right) q^*$$

Define

$$\varphi(pq^*) = \sum_{i=1} \sum_{u \in \mathcal{P}_i(r(p))} e_{pu,qu}(1) \in \widetilde{B}\left(\dot{\bigcup_{i=1}^{r}} \mathcal{P}_i, 1\right).$$

It is straightforward that φ is an embedding of semigroups.

If $v_i, 1 \leq i \leq r$, is one of the sinks, then $\{pq^* \mid p, q \in \mathcal{P}_i\}$ is an ideal in I_0, and the image of this ideal is the Brandt semigroup $B(\mathcal{P}_i, 1)$. Hence, the image $\varphi(I_0)$ contains the block diagonal Brandt semigroup $B(\dot{\cup}\mathcal{P}_i, 1)$ that is isomorphic to the 0-disjoint union of the semigroups $B(\mathcal{P}_i, 1)$.

The semigroup $B(\dot{\cup}\mathcal{P}_i, 1)$ is an ideal in the extended Brandt semigroup $\widetilde{B}(\dot{\cup}\mathcal{P}_i, 1)$. Let us summarize the above.

Lemma 3. *There is an embedding of the ideal I_0 of $LI(\Gamma)$ into the block diagonal extended Brandt semigroup $\widetilde{B}(\dot{\cup}\mathcal{P}_i, 1)$, $\varphi : I_0 \to \widetilde{B}(\dot{\cup}\mathcal{P}_i, 1)$. The image $\varphi(I_0)$ lies between $B(\dot{\cup}\mathcal{P}_i, 1)$ and $\widetilde{B}(\dot{\cup}\mathcal{P}_i, 1)$.*

Let S be a semigroup, and let I be an ideal of S. Consider the semigroup with zero $S/I = (S \setminus I) \cup \{0\}$; $a \cdot b$ is the product of elements a, b in S if $ab \notin I$. If $ab \in I$, then we let $a \cdot b = 0$.

Consider the graph Γ/V_0 with the set of vertices $V \setminus V_0$ and the set of edges $\{e \in E \mid r(e) \in V \setminus V_0\}$. Then

$$LI(\Gamma)/I_0 \cong LI(\Gamma/V_0).$$

Passing to Γ/V_0, we assume that $V_0 = \phi$.

Following [4], we consider the set

$$V_1 = \{v \in V \mid \text{if } p \text{ is a path such that } s(p) = v \text{ and } r(p) \text{ lies on a cycle } C, \\ \text{then } C \text{ is a no-exit cycle}\}.$$

The set V_1 is hereditary and saturated. Let I_1 be the ideal generated by the set V_1 in $LI(\Gamma)$,

$$I_1 = \{pq^* \mid p, q \text{ are paths, } r(p) = r(q) \in V_1\}.$$

Let $C_1, \ldots, C_l$ be all no-exit cycles of Γ. In [14], it is shown that the subsemigroup

$$LI(C_i) = \{pq^* \mid \text{ both paths } p, q \text{ lie on the cycle } C_i\}$$

is isomorphic to the Brandt semigroup $B(X, T)$, where $|X|$ is the number of vertices on C_i; T is the infinite cyclic group. On each no-exit cycle C_i let us fix a vertex v_i.

Let $\mathcal{E}_i$ be the set of paths $p = e_1 \cdots e_n$ such that $s(e_1), \ldots, s(e_n)$ are not equal to v_i, but $r(p) = v_i$. The set $\mathcal{E}_i$ is infinite if and only if there exists a cycle C different from C_i and a path p such that $s(p)$ lies on C, $r(p)$ lies on C_i.

Let $\mathcal{E} = \dot{\cup}_i \mathcal{E}_i$. We will embed the semigroup I_1 into the block diagonal extended Brandt semigroup $\widetilde{B}\,(\dot{\cup}_i \mathcal{E}_i, T)$, where $T = \langle t \rangle$ is the infinite cyclic group generated by the element t.

For a vertex $v \in V_1$, let $\mathcal{E}_i(v)$ be the set of paths p such that $p \in \mathcal{E}_i$, $s(p) = v$.

Let $\mathcal{E}(v) = \dot{\cup}_i \mathcal{E}_i(v)$. Define

$$\psi(v) = \sum_{p\in\mathcal{E}(v)} e_{p,p}(1) \in \widetilde{B}\big(\bigcup_{i=1}^{r} \mathcal{E}_i, T\big).$$

Consider an element $pq^* \in I_1$, $r(p)$ does not lie on any of the cycles $C_1, \ldots, C_l$. We have

$$pq^* = \sum_{i=1}^{l} p \left(\sum_{u\in\mathcal{E}_i(r(p))} uu^* \right) q^*$$

in the Leavitt path algebra $L_F(\Gamma)$.

Define

$$\psi(pq^*) = \sum_i \sum_{u\in\mathcal{E}_i(r(p))} e_{p,u,qu}(1) \in \widetilde{B}\left(\bigcup_{i=1}^{r} \mathcal{E}_i, T\right).$$

Now, suppose that $r(p)$ lies on a cycle C_i. Then $p = p_1 u_1$, $q = q_1 u_2$, where $p_1, q_1 \in \mathcal{E}_i$, u_1, u_2 are paths on the cycle C_i,

$$p\ q^* = p_1\ u_1\ u_2^*\ q_1^*.$$

Either $u_1 u_2^* = p_2$ or $u_1 u_2^* = p_2^*$, where p_2 is a path on the cycle C. Let m be the length of the path p_2. Define

$$\psi(pq^*) = e_{p_1,q_1}(t^m), \quad \text{if } u_1 u_2^* = p_2 \text{ is a path},$$

$$\psi(pq^*) = e_{p_1,q_1}(t^{-m}), \quad \text{if } u_1 u_2^* = p_2^*.$$

We have proved

Lemma 4. *There is an embedding of the ideal I_1 of $LI(\Gamma)$ into the block diagonal extended Brandt semigroup $\widetilde{B}\left(\cup_i \mathcal{E}_i, T\right)$,*

$$\psi : I_1 \to \widetilde{B}\left(\bigcup_i \mathcal{E}_i, T\right).$$

The image $\psi(I_1)$ lies between $B\left(\cup_i \mathcal{E}_i, T\right)$ and $\widetilde{B}\left(\cup_i \mathcal{E}_i, T\right)$.

Let Γ be a finite quiver such that the Leavitt inverse semigroup $LI(\Gamma)$ has polynomially bounded growth. Following [4], we construct an ascending chain of hereditary saturated subsets. The subset $W_0 = V_0$ was defined above. Suppose that subsets $W_0 \subset W_1 \subset \cdots \subset W_i$ have been defined. Consider the graph Γ/W_i. In this graph, let $V_1(\Gamma/W_i)$ be the subset of $V \setminus W_i$ defined above. Let

$$W_{i+1} = V_1(\Gamma/W_i) \bigcup W_i.$$

In [4], it was shown that in finitely many steps we reach the whole set V, $W_0 \subseteq W_1 \subseteq \ldots \subseteq W_s = V$. Let I_i be the ideal of the semigroup $LI(\Gamma)$ generated by W_i.

Now, Theorem 2 immediately follows from Lemmas 3, 4.

References

[1] Abrams, G., Aranda Pino, G.: The Leavitt path algebra of a graph. J. Algebra. **293**, 319–334 (2005). https://doi.org/10.1016/j.jalgebra.2005.07.028

[2] Abrams, G., Ara, P., Siles Molina, M.: Leavitt path algebras. Springer Lecture Notes in Mathematics. **2191** (2017).

[3] Alahmadi, A., Alsulami, H., Jain, S.K., Zelmanov, E.: Leavitt path algebras of finite Gelfand-Kirillov dimension. J. Algebra Appl. **11**(6), 1250225 (2012). https://doi.org/10.1142/S0219498812502258

[4] Alahmadi, A., Alsulami, H., Jain, S.K., Zelmanov, E.: Structure of Leavitt path algebras of polynomial growth. PNAS. **110**(38), 15222–15224 (2013). https://doi.org/10.1073/pnas.1311216110

[5] Ara, P., Moreno, M.A., Pardo, E.: Nonstable K-theory for graph algebras. Algebra Represent. Theory. **10**(2), 157–178 (2007). https://doi.org/10.1007/s10468-006-9044-z

[6] Ash, C.J., Hall, T.E.: Inverse semigroups on graphs. Semigroup Forum. **11**, 140–145 (1975). https://doi.org/10.1007/BF02195262

[7] Bell, J.P., Zelmanov, E.: On the growth of algebras, semigroups and hereditary languages. Invent. Math. **224**, 683–697 (2021). https://doi.org/10.1007/s00222-020-01017-x

[8] Cuntz, J.: Simple C^*-algebras generated by isometries. Comm. Math. Phys. **57**(2), 173–185 (1977).

[9] Ganyushkin, O., Mazorchuk, V.: Classical Finite Transformation Semirgoups. An Introduction. Algebra and Applications. **9**. Springer-Verlag, London (2009).

[10] Grigorchuk, R.I.: Degrees of growth of finitely generated groups and the theory of invariant means. Izv. Akad. Nauk SSSR. **48**(5), 939–985 (1984).

[11] Krause, G.R., Lenagan, T.H.: Growth of algebras and Gelfand-Kirillov dimension. Grad. Studies in Math. **22**. AMS, Providence, RI (2000).

[12] Lawson, M.V.: Inverse semigroups: the Theory of Partial Symmetries. River Edge, NJ (1998).

[13] Leavitt, W.G.: The module type of a ring. Trans. Amer. Math. Soc. **103**, 113–130 (1962). https://doi.org/10.1090/S0002-9947-1962-0132764-X

[14] Meakin, J., Milan, D., Wang, Z.: On a class of inverse semigroups related to Leavitt path algebras. Advances in Math. **384** (2021). https://doi.org/10.1016/j.aim.2021.107729

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Received by the editors: 28.02.2024.