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RESEARCH ARTICLE

# On 3-matrix factorization of polynomials

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Abstract. Let $R = K[x_1, x_2, \cdots, x_r]$ and $S = K[y_1, y_2, \cdots, y_s]$, where $K$ is a field. In this paper, we propose a method showing how to obtain 3-matrix factors for a given polynomial using either the Doolittle or the Crout decomposition techniques that we apply to matrices whose entries are not real numbers but polynomials. We also explicitly define the category of 3-matrix factorizations of a polynomial $f$ whose objects are 3-matrix factorizations of $f$, that is triplets $(P, Q, T)$ of $m \times m$ matrices such that $PQT = fI_m$. Moreover, we construct a bifunctorial operation $\overline{\otimes}_3$ which is such that if $X$ (respectively $Y$) is a $3-$matrix factorization of $f \in R$ (respectively $g \in S$), then $X\overline{\otimes}_3 Y$ is a $3-$matrix factorization of $fg \in K[x_1, x_2, \cdots, x_r, y_1, y_2, \cdots, y_s]$. We call $\overline{\otimes}_3$ the multiplicative tensor product of $3-$matrix factorizations. Finally, we give some properties of the operation $\overline{\otimes}_3$.

In the sequel, $K$ is a field and except otherwise stated, our polynomials will be taken from $R = K[x_1, x_2, \cdots, x_r]$, where $K = \mathbb{R}$, the set of real numbers. Sometimes instead of indexing the indeterminates when they are at most three, we will write $x, y, z$.

## Introduction

Eisenbud was the first to introduce the notion of matrix factorization in his seminal paper [4] in 1980. He proved that matrix factorizations of $f \in K[[x]]$ describe all maximal Cohen-Macaulay modules (MCM modu-

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les) without free summands (see [1] for notes on MCM modules). This notion generalizes the classical polynomial factorization in the sense that classical polynomial factors can now be seen as $1 \times 1$ matrix factors. The polynomial $g = x^3 + y^2$ is irreducible over $\mathbb{R}[x, y]$ but can be factorized as follows:

$$\begin{bmatrix} x & -y \\ y & x^2 \end{bmatrix} \begin{bmatrix} x^2 & y \\ -y & x \end{bmatrix} = (x^3 + y^2) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = gI_2.$$

$\left( \begin{bmatrix} x & -y \\ y & x^2 \end{bmatrix}, \begin{bmatrix} x^2 & y \\ -y & x \end{bmatrix} \right)$ is said to be a $2 \times 2$ matrix factorization of $g$. Matrix factorizations and some of their properties were studied in several papers including [2–4, 8, 10]. In these papers, a matrix factorization of a polynomial $f$ is a pair of $m \times m$ matrices $(P, Q)$ such that $fI_m = PQ$. In this paper, we will refer to this type of matrix factorization as 2-matrix factorization because we have two matrix factors. The category of 2-matrix factorizations was defined in [5]. Its objects are 2-matrix factorizations. We will define the notion of 3-matrix factorization of a polynomial $f$ (which has to do with fatorizing polynomials using three matrices) and we will show that a triplet $(P, Q, T)$ of $m \times m$ matrices whose product equals $fI_m$, is an object in the category of 3-matrix factorizations of $f$ which will be explicitly constructed in the paper (cf. section 3). In the literature, for $n \geq 2$, some authors already mention the category of $n$-matrix factorizations (e.g. [6]), but it seems difficult to find any explicit construction, except in the case $n = 2$ as seen in [5]. One obvious reason for studying matrix factorizations and their properties is that originally, Eisenbud [4] proved that matrix factorizations of $f \in K[[x]]$ describe all maximal Cohen-Macaulay modules (MCM modules) without free summands. Another interesting reason for studying matrix factorizations is that irreducible polynomials can be factorized using matrices. Furthermore, Buchweitz et al. [9] found that matrix factorizations of polynomials (over the reals) of the form $f_n = x_1^2 + \cdots + x_n^2$, for $n = 1, 2, 4$ and 8 are related to the existence of composition algebras over $\mathbb{R}$ of dimension $1, 2, 4$ and 8 namely the complex numbers, the quaternions and the octonians. More on the importance of matrix factorizations with references can be found in the introduction of [8].

Eisenbud (p. 15 of [4]) in 1980 originally defined a matrix factorization of an element $f$ in a ring $R$ (with unity) to be an ordered pair of maps of free $R-$modules $\phi : F \to G$ and $\psi : G \to F$ s.t., $\phi\psi = f \cdot 1_G$ and $\psi\phi = f \cdot 1_F$.

In 1998, Yoshino [5] defined a matrix factorization of $f$ to be a pair of matrices $(P, Q)$ such that $fI = PQ$. Diveris and Crisler in 2016 used this definition (cf. Definition 1 of [3]) of Yoshino. In this paper, we follow suit and we refer to this type of matrix factorization of a polynomial $f$ as a 2-matrix factorization of $f$ as already mentioned above. Next, we extend this definition to $n$-matrix factorizations of a polynomial, for $n = 3$ and $n = 4$. In their paper published in 2016, Carqueville and Murfet defined a matrix factorization using linear factorizations and $\mathbb{Z}_2$-graded modules (cf. p. 8 of [10]). Detailed explanations on how the different definitions mentioned above are interrelated can be found in [11].

Properties of 2-matrix factorizations were used in [3] to give the minimal 2-matrix factorization for a polynomial which is the sum of squares of 8 monomials. They were also used in chapter 6 of [7] to give necessary conditions for the existence of a Morita Context in the bicategory of Landau-Ginzburg models. Moreover, one of the properties of 2-matrix factorizations was used to conclude that a polynomial admits more than one pair of 2-matrix factors (cf. Proposition 2.2 of [7]).

It is good to recall that there is an algorithm referred to as the standard method which produces 2-matrix factors of any given polynomial. Details about this algorithm can be found in [3, 13]. Though in the literature, $n$-matrix factorizations of polynomials are mentioned (e.g. [6]), no technique is given to explicitly find $n$-matrix factors of polynomials, for $n \geq 3$. In this paper, we will use the Doolittle or the Crout matrix decomposition methods on matrices whose entries are polynomials and not just real numbers to obtain an $n$-matrix factorization of a given polynomial $f$ for $n = 3$ and $n = 4$ from any of its 2-matrix factorizations.

Moreover, we propose a bifunctorial operation $\overline{\otimes}_3$ which is such that if $X$ (respectively $Y$) is a 3-matrix factorization of $f \in R$ (respectively $g \in S$), then $X \overline{\otimes}_3 Y$ is a 3-matrix factorization of $fg \in K[x, y]$, where $x = x_1, x_2, \cdots, x_r$ and $y = y_1, y_2, \cdots, y_s$. We call $\overline{\otimes}_3$ the multiplicative tensor product of 3-matrix factorizations. The operation $\overline{\otimes}_3$ is the counterpart of the multiplicative tensor product of 2-matrix factorizations that was defined in [13] and used in conjunction with the Yoshino tensor product of matrix factorizations to reduce the size of matrix factors on the class of summand reducible polynomials. Finally, we give some properties of the operation $\overline{\otimes}_3$.

This paper is organized as follows: In the next section, we give some preliminaries. In section 3, after recalling the definition of 2-matrix factorizations, we define what a 3-matrix factorization of a polynomial is and give some examples. The category of 3-matrix factorizations is explicitly defined in section 4. Finally, in section 5, we define the multiplicative tensor product of 3-matrix factorizations denoted $\overline{\otimes}_3$ and we prove that it is a bifunctorial operation after giving some examples. Moreover, we give some of its properties.

## 1. Preliminaries

In this section, we give some preliminaries.

### 1.1. Doolittle and Crout matrix decomposition methods

Here, we recall two matrix decomposition methods that are well known in the literature, namely Doolittle and Crout matrix decomposition methods.

First, we recall the following definition.

**Definition 1** ($LU$-Factorization, Doolittle and Crout decompositions)**.** A nonsingular matrix $M$ has an $LU$-factorization if it can be expressed as the product of a lower-triangular matrix $L$ and an upper triangular matrix $U$ as follows $M = LU$. If $L$ has $1's$ on its diagonal, then it is called a Doolittle factorization. If $U$ has $1's$ on its diagonal, then it is called a Crout factorization.

When $M = LU$, we say that $M$ admits an $LU$-decomposition.

If $M = LU$ has a Doolittle factorization then we have the following picture:

$$\begin{bmatrix} m_{1,1} & m_{1,2} & m_{1,3} & m_{1,4} \cdots & m_{1,n} \\ m_{2,1} & m_{2,2} & m_{2,3} & m_{2,4} \cdots & m_{2,n} \\ m_{3,1} & m_{3,2} & m_{3,3} & m_{3,4} \cdots & m_{3,n} \\ m_{4,1} & m_{4,2} & m_{4,3} & m_{4,4} \cdots & m_{4,n} \\ \vdots & & & & \\ m_{n,1} & m_{n,2} & m_{n,3} & m_{n,4} \cdots & m_{n,n} \end{bmatrix} = LU,$$

where

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \cdots & 0 \\ l_{2,1} & 1 & 0 & 0 \cdots & 0 \\ l_{3,1} & l_{3,2} & 1 & 0 \cdots & 0 \\ l_{4,1} & l_{4,2} & l_{4,3} & 1 \cdots & 0 \\ \vdots & & & & \\ l_{n,1} & l_{n,2} & l_{n,3} & l_{n,4} \cdots & 1 \end{bmatrix}; U = \begin{bmatrix} u_{1,1} & u_{1,2} & u_{1,3} & u_{1,4} \cdots & u_{1,n} \\ 0 & u_{2,2} & u_{2,3} & u_{2,4} \cdots & u_{2,n} \\ 0 & 0 & u_{3,3} & u_{3,4} \cdots & u_{3,n} \\ 0 & 0 & 0 & u_{4,4} \cdots & u_{4,n} \\ \vdots & & & & \\ 0 & 0 & 0 & 0 \cdots & u_{n,n} \end{bmatrix}.$$

## 2. $n$-matrix factorization of polynomials for $n$=2; 3

### 2.1. 2-matrix factorization of polynomials

**Definition and some examples**

Let $K[[x_1, x_2, \cdots, x_r]]$ be the power series ring in the indeterminates $x_1, x_2, \cdots, x_r$. In the sequel, we will sometimes write $K[[x]]$ instead of $K[[x_1, x_2, \cdots, x_r]]$ for ease of notation. Likewise, we will write $K[x]$ instead of $K[x_1, x_2, \cdots, x_r]$.

The notion of matrix factorization is defined in [5] for nonzero non-invertible $f \in K[[x_1, x_2, \cdots, x_r]]$. We define it as in [3] slightly generalizing the one given in [5] by including elements like $1 \in K$ for convenience. Yoshino [5] requires an element $f \in K[[x]]$ to be nonzero non-invertible because if $f = 0$ then $K[[x]]/(f) = K[[x]]$ and if $f$ is a unit, then $K[[x]]/(f)$ =K[[x]]/K[[x]]= $\{1\}$. But in this work, we will not bother about such restrictions because we will not deal with the homological methods used in [5].

**Definition 2** ([3,5,7])**.** An $m \times m$ **matrix factorization** of a polynomial $f \in R$ is a pair of $m \times m$ matrices $(P, Q)$ such that $PQ = fI_m$, where $I_m$ is the $m \times m$ identity matrix and the coefficients of $P$ and of $Q$ are taken from $R$.

**Example 1.** Let $l = xy + (x^2 + yz)z$. We use the standard method (cf. [3, 8, 13]) to find a matrix factorization of $l$ and quickly find:

$$Q = \left( \begin{bmatrix} x & -(x^2+yz) \\ z & y \end{bmatrix}, \begin{bmatrix} y & x^2+yz \\ -z & x \end{bmatrix} \right).$$

Let $h = xy + x^2z + yz^2$. Observe that $l = h$. We can use the standard method to find a matrix factorization of $h$ with monomial entries. As we can see below, the matrix factors are nicer[1] than the ones obtained above. This approach was studied in [13].

First a matrix factorization of $xy + x^2z$ is

$$\left( \begin{bmatrix} x & -x^2 \\ z & y \end{bmatrix}, \begin{bmatrix} y & x^2 \\ -z & x \end{bmatrix} \right).$$

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[1] “nicer” in the sense that we have matrices with monomial entries

So, a matrix factorization of $h = xy + x^2z + yz^2$ is then:

$$P = \left( \begin{bmatrix} x & -x^2 & -y & 0 \\ z & y & 0 & -y \\ z^2 & 0 & y & x^2 \\ 0 & z^2 & -z & x \end{bmatrix}, \begin{bmatrix} y & x^2 & y & 0 \\ -z & x & 0 & y \\ -z^2 & 0 & x & -x^2 \\ 0 & -z^2 & z & y \end{bmatrix} \right).$$

In this paper, we will not bother much about the type of matrix factors we will obtain, that is; whether they have monomial entries or not.

We will simply be discussing how to factorize a polynomial using two or more matrices. We will refer to the type of factorizations of Definition 2 as 2-matrix factorizations because we have two matrix factors. This is the type that one easily finds in the literature (e.g. [3, 5]). We will generalize Definition 2 below (see Definition 3).

In subsection 2.2, we will show how to obtain a 3-matrix factorization of a polynomial $f$ from any of its 2-matrix factorizations. To that end, we will need to decompose one of the two matrix factors into two matrices using the Doolittle or the Crout matrix decomposition methods that we will apply to matrices whose entries are not real numbers but polynomials. We recalled these methods in subsection 1.1.

## 2.2. 3-matrix factorizations and examples

As mentioned at the introduction, $n$-matrix factorizations (for $n \geq 3$) are mentioned in the literature but no technique is given to explicitly find 3-matrix factors of a polynomial $f$. In this paper, we show how to find it.

**Definition 3.** An $m \times m$ **3-matrix factorization** of a polynomial $f \in R$ is a triplet of $m \times m$ matrices $(A_1, A_2, A_3)$ such that $A_1A_2A_3 = fI_m$, where $I_m$ is the $m \times m$ identity matrix and the coefficients of each matrix $A_i$, $i \in \{1, 2, 3\}$, is taken from the field of fraction of $R$.

In the following example, we find 3-matrix factors of a polynomial in $R$ which have entries in the field of fractions of $R$. That is why in the foregoing definition, we talk of the field of fraction or $R$ instead of just $R$ itself as we did in Definition 2.

**Example 2.** Let $f = x^2 + y^2$. A 2-matrix factorization of $f$ is

$$\left( \begin{bmatrix} x & -y \\ y & x \end{bmatrix}, \begin{bmatrix} x & y \\ -y & x \end{bmatrix} \right) = (x^2 + y^2) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = fI_2.$$

Thus, we have $fI_2 = AB$, where $A = \left(\begin{bmatrix} x & -y \\ y & x \end{bmatrix}\right)$ and $B = \left(\begin{bmatrix} x & y \\ -y & x \end{bmatrix}\right)$.

Let us decompose $A$ as the product of a lower (L) and upper (U) triangular matrices, $A = LU$. We will use the Doolittle's approach (i.e., the main diagonal of $L$ shall be 1's).

We have $\begin{bmatrix} x & -y \\ y & x \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ l_{21} & 1 \end{bmatrix} \begin{bmatrix} U_{11} & U_{12} \\ 0 & U_{22} \end{bmatrix} = \begin{bmatrix} U_{11} & U_{12} \\ l_{21}U_{11} & l_{21}U_{12} + U_{22} \end{bmatrix}$.
Hence,

$$U_{11} = x, \quad U_{12} = -y.$$
$$l_{21}U_{11} = y \Rightarrow l_{21} = \frac{y}{x}.$$
$$l_{21}U_{12} + U_{22} = x \Rightarrow U_{22} = x + \frac{y^2}{x}.$$

Hence, we obtain a 3-matrix factorization of $f$:

$$\begin{bmatrix} 1 & 0 \\ \frac{y}{x} & 1 \end{bmatrix} \begin{bmatrix} x & -y \\ 0 & x + \frac{y^2}{x} \end{bmatrix} \begin{bmatrix} x & -y \\ y & x \end{bmatrix} = (x^2 + y^2) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = fI_2.$$

**Remark 1.** Note that we could decompose $B$ instead of $A$ in order to obtain a 3-matrix factorization of $f$. Also note that the Crout decomposition technique could be used in place of the Doolittle decomposition method.

This remark is also valid for the following example.

**Example 3.** Let $g = xyz + zx^2$.
A 2-matrix factorization of $g$ is

$$\left(\begin{bmatrix} xy & -z \\ x^2 & z \end{bmatrix}, \begin{bmatrix} z & z \\ -x^2 & xy \end{bmatrix}\right) = (xyz + zx^2) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = gI_2.$$

Thus, we have $gI_2 = PQ$, where $P = \left(\begin{bmatrix} xy & -z \\ x^2 & z \end{bmatrix}\right)$, $Q = \left(\begin{bmatrix} z & z \\ -x^2 & xy \end{bmatrix}\right)$.

Let us decompose $P$ as the product of a lower (L) and upper (U) triangular matrices, $P = LU$. We will again use the Doolittle's approach.

We have $\begin{bmatrix} xy & -z \\ x^2 & z \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ l_{21} & 1 \end{bmatrix} \begin{bmatrix} U_{11} & U_{12} \\ 0 & U_{22} \end{bmatrix} = \begin{bmatrix} U_{11} & U_{12} \\ l_{21}U_{11} & l_{21}U_{12} + U_{22} \end{bmatrix}$.

Hence,

$$U_{11} = xy, \quad U_{12} = -z.$$

$$l_{21}U_{11} = x^2 \Rightarrow l_{21} = \frac{x}{y}.$$

$$l_{21}U_{12} + U_{22} = z \Rightarrow U_{22} = z + \frac{zx}{y}.$$

Hence, we obtain a 3-matrix factorization of $g$:

$$\begin{bmatrix} 1 & 0 \\ \frac{x}{y} & 1 \end{bmatrix} \begin{bmatrix} xy & -z \\ 0 & z + \frac{zx}{y} \end{bmatrix} \begin{bmatrix} z & z \\ -x^2 & xy \end{bmatrix} = (xyz + zx^2) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = gI_2.$$

## 3. The category of 3-matrix factorizations of $f \in R$

In this section, we explicitly construct the category of 3-matrix factorizations of $f \in R$. In the literature, some authors already mention the category of $n$-matrix factorizations (e.g. [6]), but we did not find any explicit presentation of it except in the case $n = 2$ as seen in [5].

In the sequel, $K(x_1, \cdots, x_r)$ denotes the fraction field of $K[x_1, \cdots, x_r]$.

The category of 3-matrix factorizations of a polynomial $f \in R = K[x] := K[x_1, \cdots, x_r]$ which we denote by $MF(R, f)_3$ or $MF_R(f)_3$, or $MF(f)_3$ (when there is no risk of confusion) is defined as follows:

- The objects are the 3-matrix factorizations of $f$.
- Given two 3-matrix factorizations of $f$; $(\phi_1, \psi_1, \theta_1)$ and $(\phi_2, \psi_2, \theta_2)$ respectively of sizes $n_1$ and $n_2$, a morphism from $(\phi_1, \psi_1, \theta_1)$ to $(\phi_2, \psi_2, \theta_2)$ is a triplet of matrices $(\alpha, \beta, \delta)$ each of size $n_2 \times n_1$ which makes the following diagram commute:

$$\begin{tikzcd}
K(x)^{n_1} \arrow[r, "\theta_1"] \arrow[d, "\alpha"'] & K(x)^{n_1} \arrow[r, "\psi_1"] \arrow[d, "\delta"'] & K(x)^{n_1} \arrow[r, "\phi_1"] \arrow[d, "\beta"] & K(x)^{n_1} \arrow[d, "\alpha"] \\
K(x)^{n_2} \arrow[r, "\theta_2"] & K(x)^{n_2} \arrow[r, "\psi_2"] & K(x)^{n_2} \arrow[r, "\phi_2"] & K(x)^{n_2}
\end{tikzcd}$$

That is,

$$\begin{cases} \alpha\phi_1 = \phi_2\beta, \\ \psi_2\delta = \beta\psi_1, \\ \delta\theta_1 = \theta_2\alpha. \end{cases}$$

- Given three 3-matrix factorizations of $f$: $(\phi_1, \psi_1, \theta_1)$, $(\phi_2, \psi_2, \theta_2)$ and $(\phi_3, \psi_3, \theta_3)$ respectively of sizes $n_1, n_2$ and $n_3$, the composition

$$(\alpha_2, \beta_2, \delta_2) \circ (\alpha_1, \beta_1, \delta_1) : (\phi_1, \psi_1, \theta_1) \xrightarrow{(\alpha_1, \beta_1, \delta_1)} (\phi_2, \psi_2, \theta_2) \xrightarrow{(\alpha_2, \beta_2, \delta_2)} (\phi_3, \psi_3, \theta_3)$$

is the triplet of matrices $(\alpha_2\alpha_1, \beta_2\beta_1, \delta_2\delta_1)$ such that the following diagram commutes:

$$\begin{tikzcd}
K(x)^{n_1} \arrow[r, "\theta_1"] \arrow[d, "\alpha_1"'] & K(x)^{n_1} \arrow[r, "\psi_1"] \arrow[d, "\delta_1"'] & K(x)^{n_1} \arrow[r, "\phi_1"] \arrow[d, "\beta_1"] & K(x)^{n_1} \arrow[d, "\alpha_1"] \\
K(x)^{n_2} \arrow[r, "\theta_2"] \arrow[d, "\alpha_2"'] & K(x)^{n_2} \arrow[r, "\psi_2"] \arrow[d, "\delta_2"'] & K(x)^{n_2} \arrow[r, "\phi_2"] \arrow[d, "\beta_2"] & K(x)^{n_2} \arrow[d, "\alpha_2"] \\
K(x)^{n_3} \arrow[r, "\theta_3"] & K(x)^{n_3} \arrow[r, "\psi_3"] & K(x)^{n_3} \arrow[r, "\phi_3"] & K(x)^{n_3}
\end{tikzcd}$$

That is,

$$\begin{cases} (\alpha_2\alpha_1)\phi_1 = \phi_3(\beta_2\beta_1), \\ \psi_3(\delta_2\delta_1) = (\beta_2\beta_1)\psi_1, \\ (\delta_2\delta_1)\theta_1 = \theta_3(\alpha_2\alpha_1). \end{cases}$$

- It is easy to see that *associativity of composition of maps of* 3*-matrix factorizations of* $f$ is a consequence of the fact that matrix multiplication is associative.
- For any $n \times n$ 3-matrix factorization $(\phi, \psi, \theta)$ of $f$, there is a map $1_{(\phi,\psi,\theta)} : (\phi, \psi, \theta) \to (\phi, \psi, \theta)$ which is actually the triplet of identity $n \times n$ matrices $(I_n, I_n, I_n)$.
- It is also clear that composing any map of 3-matrix factorizations of $f$ with $1_{(\phi,\psi,\theta)}$ from the left or the right (whenever the composition is possible) leaves the given map unchanged. This ends the definition of the category of 3-matrix factorizations of $f \in R = K[x]$.

## 4. The multiplicative tensor product of 3-matrix factorizations

In this section, we define the multiplicative tensor product of 3-matrix factorizations denoted $\overline{\otimes}_3$ and we prove that it is a bifunctorial operation. $\overline{\otimes}_3$ is the counterpart of the multiplicative tensor product $\overline{\otimes}$ which was constructed in [13] and was used in conjunction with Yoshino tensor product $\widehat{\otimes}$ to reduce the size of matrix factors of summand reducible polynomials. The functoriality of $\overline{\otimes}_3$ will be proved in subsection 4.2. Finally, we give some properties of this operation in subsection 4.3.

## 4.1. Definition and examples

**Definition 4.** Let $X = (\phi, \psi, \theta)$ be a matrix factorization of $f \in K[x]$ of size $n$ and let $X' = (\phi', \psi', \theta')$ be a matrix factorization of $g \in K[y]$ of size $m$. Thus, $\phi, \psi, \phi', \psi', \theta$, and $\theta'$ can be considered as matrices over $L = K(x, y)$. The **multiplicative tensor product of 3-matrix factorizations** $X\overline{\otimes}_3 X'$ is given by

$$X\overline{\otimes}_3 X' = (\phi, \psi, \theta)\overline{\otimes}_3(\phi', \psi', \theta') = ([\phi \otimes \phi'], [\psi \otimes \psi'], [\theta \otimes \theta'])$$

where each component is an endomorphism on $L^n \otimes_L L^m$.

**Example 4.** Consider $g = xyz + zx^2$ and $f = x^2 + y^2$ the polynomials of examples 2 and 3.

$X = \left( \begin{bmatrix} 1 & 0 \\ \frac{y}{x} & 1 \end{bmatrix}, \begin{bmatrix} x & -y \\ 0 & x + \frac{y^2}{x} \end{bmatrix}, \begin{bmatrix} x & y \\ -y & x \end{bmatrix} \right)$ is a 3-matrix factorization of $f$.

$X' = \left( \begin{bmatrix} 1 & 0 \\ \frac{x}{y} & 1 \end{bmatrix}, \begin{bmatrix} xy & -z \\ 0 & z + \frac{zx}{y} \end{bmatrix}, \begin{bmatrix} z & z \\ -x^2 & xy \end{bmatrix} \right)$ is a 3-matrix factorization of $g$.

$$\begin{aligned}
&X\overline{\otimes}_3 X' \\
&= \left( \begin{bmatrix} 1 & 0 \\ \frac{y}{x} & 1 \end{bmatrix}, \begin{bmatrix} x & -y \\ 0 & x + \frac{y^2}{x} \end{bmatrix}, \begin{bmatrix} x & y \\ -y & x \end{bmatrix} \right) \overline{\otimes}_3 \left( \begin{bmatrix} 1 & 0 \\ \frac{x}{y} & 1 \end{bmatrix}, \begin{bmatrix} xy & -z \\ 0 & z + \frac{zx}{y} \end{bmatrix}, \begin{bmatrix} z & z \\ -x^2 & xy \end{bmatrix} \right) \\
&= \left( \begin{bmatrix} 1 & 0 \\ \frac{y}{x} & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ \frac{x}{y} & 1 \end{bmatrix}, \begin{bmatrix} x & -y \\ 0 & x + \frac{y^2}{x} \end{bmatrix} \otimes \begin{bmatrix} xy & -z \\ 0 & z + \frac{zx}{y} \end{bmatrix}, \begin{bmatrix} x & y \\ -y & x \end{bmatrix} \otimes \begin{bmatrix} z & z \\ -x^2 & xy \end{bmatrix} \right) \\
&= \left( \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{x}{y} & 1 & 0 & 0 \\ \frac{y}{x} & 0 & 1 & 0 \\ 1 & \frac{y}{x} & \frac{x}{y} & 1 \end{bmatrix}, \begin{bmatrix} x^2y & -xz & -xy^2 & yz \\ 0 & xz + \frac{zx^2}{y} & 0 & -zy - zx \\ 0 & 0 & x^2y + y^3 & -zx - \frac{zy^2}{x} \\ 0 & 0 & 0 & xz + \frac{zx^2}{y} + \frac{y^2z}{x} + zy \end{bmatrix}, \right. \\
&\quad \left. \begin{bmatrix} xz & xz & yz & yz \\ -x^3 & x^2y & -x^2y & xy^2 \\ -yz & -yz & xz & xz \\ yx^2 & -xy^2 & -x^3 & x^2y \end{bmatrix} \right) \text{ is a 3-matrix factorization of } fg.
\end{aligned}$$

In fact,

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{x}{y} & 1 & 0 & 0 \\ \frac{y}{x} & 0 & 1 & 0 \\ 1 & \frac{y}{x} & \frac{x}{y} & 1 \end{bmatrix} \begin{bmatrix} x^2y & -xz & -xy^2 & yz \\ 0 & xz+\frac{zx^2}{y} & 0 & -zy-zx \\ 0 & 0 & x^2y+y^3 & -zx-\frac{zy^2}{x} \\ 0 & 0 & 0 & xz+\frac{zx^2}{y}+\frac{y^2z}{x}+zy \end{bmatrix}$$

$$\begin{bmatrix} xz & xz & yz & yz \\ -x^3 & x^2y & -x^2y & xy^2 \\ -yz & -yz & xz & xz \\ yx^2 & -xy^2 & -x^3 & x^2y \end{bmatrix}$$

$$= \begin{bmatrix} x^2y & -xz & -xy^2 & yz \\ x^3 & \frac{-x^2z}{y}+xz+\frac{zx^2}{y} & -x^2y & -zy \\ xy^2 & -yz & x^2y & \frac{y^2z}{x}-zx-\frac{zy^2}{x} \\ x^2y & yz & x^3 & xz \end{bmatrix} \begin{bmatrix} xz & 1xz & yz & yz \\ -x^3 & x^2y & -x^2y & xy^2 \\ -yz & -yz & xz & xz \\ yx^2 & -xy^2 & -x^3 & x^2y \end{bmatrix}$$

$= fgI_4.$

## 4.2. Funtoriality of $\overline{\otimes}_3$

This subsection is entirely devoted to the discussion of the bifunctoriality of $\overline{\otimes}_3$.

**Setting the stage:** Let $X_f = (\phi, \psi, \theta)$, $X'_f = (\phi', \psi', \theta')$, and $X''_f = (\phi'', \psi'', \theta'')$ be objects of $MF(K[x], f)_3$ respectively of sizes $n$, $n'$, and $n''$. Let $X_g = (\sigma, \rho, \zeta)$, $X'_g = (\sigma', \rho', \zeta')$, and $X''_g = (\sigma'', \rho'', \zeta'')$ be objects of $MF(K[y], g)_3$ respectively of sizes $m$, $m'$, and $m''$.

**Definition 5.** For morphisms $\Phi_f = (\alpha_f, \beta_f, \delta_f) : X_f = (\phi, \psi, \theta) \to X'_f = (\phi', \psi', \theta')$ and $\Phi_g = (\alpha_g, \beta_g, \delta_g) : X_g = (\sigma, \rho, \zeta) \to X'_g = (\sigma', \rho', \zeta')$ respectively in $MF(K[x], f)_3$ and $MF(K[y], g)_3$, we define $\Phi_f \overline{\otimes}_3 \Phi_g : X_f \overline{\otimes}_3 X_g = (\phi, \psi, \theta) \overline{\otimes}_3 (\sigma, \rho, \zeta) \to X'_f \overline{\otimes}_3 X'_g = (\phi', \psi', \theta') \overline{\otimes}_3 (\sigma', \rho', \zeta')$ by

$$([\alpha_f \otimes \alpha_g], [\beta_f \otimes \beta_g], [\delta_f \otimes \delta_g]).$$

**Lemma 1.** *$\Phi_f \overline{\otimes}_3 \Phi_g$: $X_f \overline{\otimes}_3 X_g = (\phi, \psi, \theta) \overline{\otimes}_3 (\sigma, \rho, \zeta) \to X'_f \overline{\otimes}_3 X'_g = (\phi', \psi', \theta') \overline{\otimes}_3 (\sigma', \rho', \zeta')$ is a morphism in $MF(K[x, y], fg)_3$.*

*Proof.* We need to show that the following diagram commutes:

$$\begin{tikzcd}
K(x,y)^{nm} \arrow[r, "{[\theta\otimes\zeta]}"] \arrow[d, "{[\alpha_f\otimes\alpha_g]}"'] & K(x,y)^{nm} \arrow[r, "{[\psi\otimes\rho]}"] \arrow[d, "{[\delta_f\otimes\delta_g]}"'] & K(x,y)^{nm} \arrow[r, "{[\phi\otimes\sigma]}"] \arrow[d, "{[\beta_f\otimes\beta_g]}"] & K(x,y)^{nm} \arrow[d, "{[\alpha_f\otimes\alpha_g]}"] \\
K(x,y)^{n'm'} \arrow[r, "{[\theta'\otimes\zeta']}"'] & K(x,y)^{n'm'} \arrow[r, "{[\psi'\otimes\rho']}"'] & K(x,y)^{n'm'} \arrow[r, "{[\phi'\otimes\sigma']}"'] & K(x,y)^{n'm'}
\end{tikzcd}$$

viz. all the three squares in the foregoing diagram commute.

• The commutativity of each of these squares from the right to the left is expressed by the following equalities:

$$\begin{aligned}[]
[\alpha_f \otimes \alpha_g][\phi \otimes \sigma] &= [\phi' \otimes \sigma'][\beta_f \otimes \beta_g], \\
[\beta_f \otimes \beta_g][\psi \otimes \rho] &= [\psi' \otimes \rho'][\delta_f \otimes \delta_g], \\
[\delta_f \otimes \delta_g][\theta \otimes \zeta] &= [\theta' \otimes \zeta'][\alpha_f \otimes \alpha_g]
\end{aligned}$$

i.e., all we need to show is the set of equalities:

$$\begin{cases}
\alpha_f\phi \otimes \alpha_g\sigma = \phi'\beta_f \otimes \sigma'\beta_g \cdots (1), \\
\beta_f\psi \otimes \beta_g\rho = \psi'\delta_f \otimes \rho'\delta_g \cdots (2), \\
\delta_f\theta \otimes \delta_g\zeta = \theta'\alpha_f \otimes \zeta'\alpha_g \cdots (3).
\end{cases}$$

Now by hypothesis, $\Phi_f = (\alpha_f, \beta_f, \delta_f) : X_f = (\phi, \psi, \theta) \to X'_f = (\phi', \psi', \theta')$ and $\Phi_g = (\alpha_g, \beta_g, \delta_g) : X_g = (\sigma, \rho, \zeta) \to X'_g = (\sigma', \rho', \zeta')$ are morphisms, meaning that the following diagrams commute

$$\begin{tikzcd}
K(x)^n \arrow[r, "\theta"] \arrow[d, "\alpha_f"'] & K(x)^n \arrow[r, "\psi"] \arrow[d, "\delta_f"'] & K(x)^n \arrow[r, "\phi"] \arrow[d, "\beta_f"] & K(x)^n \arrow[d, "\alpha_f"] \\
K(x)^{n'} \arrow[r, "\theta'"] & K(x)^{n'} \arrow[r, "\psi'"] & K(x)^{n'} \arrow[r, "\phi'"] & K(x)^{n'}
\end{tikzcd}$$

and

$$\begin{tikzcd}
K(y)^m \arrow[r, "\zeta"] \arrow[d, "\alpha_g"'] & K(y)^m \arrow[r, "\rho"] \arrow[d, "\delta_g"'] & K(y)^m \arrow[r, "\sigma"] \arrow[d, "\beta_g"] & K(y)^m \arrow[d, "\alpha_g"] \\
K(y)^{m'} \arrow[r, "\zeta'"] & K(y)^{m'} \arrow[r, "\rho'"] & K(y)^{m'} \arrow[r, "\sigma'"] & K(y)^{m'}
\end{tikzcd}$$

That is,

$$\begin{cases}
\alpha_f\phi = \phi'\beta_f \cdots (i), \\
\psi'\delta_f = \beta_f\psi \cdots (ii), \\
\theta'\alpha_f = \delta_f\theta \cdots (iii)
\end{cases}$$

and

$$\begin{cases}
\alpha_g\sigma = \sigma'\beta_g \cdots (i'), \\
\rho'\delta_g = \beta_g\rho \cdots (ii'), \\
\zeta'\alpha_g = \delta_g\zeta \cdots (iii').
\end{cases}$$

Now considering $(i)$ and $(i')$, we immediately see that equality (1) holds. Similarly, $(ii)$ and $(ii')$ yield (2). Finally, $(iii)$ and $(iii')$ yield (3).

So, $\Phi_f \overline{\otimes}_3 \Phi_g$ is a morphism in $MF(K[x,y],fg)$. □

We can now state the following result.

**Theorem 1.** *1. Let $X$ be a matrix factorization of $f \in K[x]$ of size $n$ and let $Y$ be a matrix factorization of $g \in K[y]$ of size $m$. Then, there is a tensor product $\overline{\otimes}_3$ of 3-matrix factorizations which produces a 3-matrix factorization $X\overline{\otimes}_3 Y$ of the product $fg \in K[x_1,...,x_r, y_1,...,y_s]$ which is of size $nm$. $\overline{\otimes}_3$ is called the multiplicative tensor product of 3-matrix factorizations.*

*2. The multiplicative tensor product of 3-matrix factorizations*

$$(-)\overline{\otimes}_3(-) : MF(K[x],f) \times MF(K[y],g) \to MF(K[x,y],fg)$$

*is a bifunctor.*

*Proof.* 1. This is exactly what we proved above in subsection 4.1.

2. We show that $\overline{\otimes}_3$ is a bifunctor.
In order to ease our computations, let's write $F = (-)\overline{\otimes}_3(-)$. We show that $F$ is a bifunctor. We have:

$$(-)\overline{\otimes}_3(-) : \qquad MF(f) \times MF(g) \quad \longrightarrow \quad MF(fg)$$

$$\begin{tikzcd}
(X_f \arrow[d, "\Phi_f"'] & , & X_g) \arrow[d, "\Phi_g"'] \arrow[r] & X_f\overline{\otimes}_3X_g \arrow[d, "\Phi_f\overline{\otimes}_3\Phi_g:=(\alpha,\beta,\delta)"] \\
(X'_f \arrow[d, "\Phi'_f"'] & , & X'_g) \arrow[d, "\Phi'_g"'] \arrow[r] & X'_f\overline{\otimes}_3X'_g \arrow[d, "\Phi'_f\overline{\otimes}_3\Phi'_g:=(\alpha',\beta',\delta')"] \\
(X''_f & , & X''_g) \arrow[r] & X''_f\overline{\otimes}_3X''_g
\end{tikzcd}$$

We showed in Lemma 1 that $\Phi_f\overline{\otimes}_3\Phi_g := (\alpha,\beta,\delta)$ is a morphism in $MF(K[x,y],fg)$, where

$$(\alpha,\beta,\delta) = ([\alpha_f \otimes \alpha_g],[\beta_f \otimes \beta_g],[\delta_f \otimes \delta_g]).$$

Similarly, if $\Phi'_f := (\alpha'_f,\beta'_f,\delta'_f)$ and $\Phi'_g := (\alpha'_g,\beta'_g,\delta'_g)$ then $\Phi'_f\overline{\otimes}_3\Phi'_g = (\alpha',\beta',\delta')$, where

$$(\alpha',\beta',\delta') = ([\alpha'_f \otimes \alpha'_g],[\beta'_f \otimes \beta'_g],[\delta'_f \otimes \delta'_g]).$$

It now remains to show the composition and the identity axioms.

*Identity Axiom*:
We show that $F(id_{(X_f,X_g)}) = id_{F(X_f,X_g)}$.
Now, $F(id_{(X_f,X_g)}) = F(id_{X_f}, id_{X_g}) := id_{X_f}\overline{\otimes}_3 id_{X_g} : X_f\overline{\otimes}_3 X_g \to X_f\overline{\otimes}_3 X_g$.
And by Definition 5, $id_{X_f}\overline{\otimes}_3 id_{X_g}$ is the triplet of matrices
$([I_n \otimes I_m], [I_n \otimes I_m], [I_n \otimes I_m])$ †

Next, we compute $id_{F(X_f,X_g)} = id_{X_f\overline{\otimes}_3 X_g} : X_f\overline{\otimes}_3 X_g \to X_f\overline{\otimes}_3 X_g$.
By definition of a morphism in the category $MF(fg)$, we know that

$id_{X_f\overline{\otimes}_3 X_g} := ([I_{nm}], [I_{nm}], [I_{nm}])$ ††

Since $I_n \otimes I_m = I_{nm}$, we see that † and †† are the same, therefore $F(id_{(X_f,X_g)}) = id_{F(X_f,X_g)}$ as desired.

*Composition Axiom*:
Consider the situation:

$$\begin{tikzcd} X_f \arrow[r, "\Phi_f"] & X'_f \arrow[r, "\Phi'_f"] & X''_f \end{tikzcd}$$

$$\begin{tikzcd} X_g \arrow[r, "\Phi_g"] & X'_g \arrow[r, "\Phi'_g"] & X''_g \end{tikzcd}$$

$$\begin{tikzcd} X_f\overline{\otimes}_3 X_g \arrow[r, "F(\Phi_f,\Phi_g)"] & X'_f\overline{\otimes}_3 X'_g \arrow[r, "F(\Phi'_f,\Phi'_g)"] & X''_f\overline{\otimes}_3 X''_g \end{tikzcd} .$$

We need to show $F(\Phi'_f \circ \Phi_f, \Phi'_g \circ \Phi_g) = F(\Phi'_f, \Phi'_g) \circ F(\Phi_f, \Phi_g)$.
Now, $\Phi'_f \circ \Phi_f = (\alpha'_f\alpha_f, \beta'_f\beta_f)$ and $\Phi'_g \circ \Phi_g = (\alpha'_g\alpha_g, \beta'_g\beta_g)$.

Thanks to Definition 5, we obtain:

$(\Phi'_f \circ \Phi_f)\overline{\otimes}_3(\Phi'_g \circ \Phi_g)$
$= ([\alpha'_f\alpha_f \otimes \alpha'_g\alpha_g], [\beta'_f\beta_f \otimes \beta'_g\beta_g], [\delta'_f\delta_f \otimes \delta'_g\delta_g]).$ ‡′
Next,

$(\Phi'_f\overline{\otimes}_3\Phi'_g) \circ (\Phi_f\overline{\otimes}_3\Phi_g)$
$= ([\alpha'_f \otimes \alpha'_g], [\beta'_f \otimes \beta'_g], [\delta'_f \otimes \delta'_g]) \circ ([\alpha_f \otimes \alpha_g], [\beta_f \otimes \beta_g], [\delta_f \otimes \delta_g])$
$= ([\alpha'_f\alpha_f \otimes \alpha'_g\alpha_g], [\beta'_f\beta_f \otimes \beta'_g\beta_g], [\delta'_f\delta_f \otimes \delta'_g\delta_g]).$ ‡‡′
From ‡′ and ‡‡′, we see that $F(\Phi'_f \circ \Phi_f, \Phi'_g \circ \Phi_g) = F(\Phi'_f, \Phi'_g) \circ F(\Phi_f, \Phi_g)$. Thus, $(-)\overline{\otimes}_3(-)$ is a bifunctor.

□

### 4.3. Properties of $\overline{\otimes}_3$: associativity, commutativity and distributivity

In this subsection, we give some properties of the multiplicative tensor product of 3-matrix factorizations. We prove that $\overline{\otimes}_3$ is associative, commutative and distributive.

We denote by $X_1 = (\phi_1, \psi_1, \theta_1)$ (resp. $X_2 = (\phi_2, \psi_2, \theta_2)$) an $(n_1 \times n_1)$ (resp. $(n_2 \times n_2)$) 3-matrix factorization of $f \in K[x]$. We also let $X' = (\phi', \psi', \theta')$ (resp. $X'' = (\phi'', \psi'', \theta'')$) denotes a $(p \times p)$ (resp. $(m \times m)$) 3-matrix factorization of $g \in K[y]$ (resp. of $h \in K[z] := K[z_1, \cdots, z_l]$). $X = (\phi, \psi, \theta)$ will also be an $r \times r$ 3-matrix factorization of $f \in K[x]$.

**Proposition 1.** *(Associativity)*
*There is an identity:*

$$(X\overline{\otimes}_3 X')\overline{\otimes}_3 X'' = X\overline{\otimes}_3(X'\overline{\otimes}_3 X'')$$

*in* $MF(fgh)$.

*Proof.* The desired identity follows from the fact that the standard tensor product for matrices is associative. □

To prove the commutativity of $\overline{\otimes}_3$, recall (cf. section 3.1 [12]) that given two matrices $C$ and $D$, the tensor products $C \otimes D$ and $D \otimes C$ are **permutation equivalent**. That is, there exist permutation matrices $P$ and $Q$ (so called commutation matrices) such that: $C \otimes D = P(D \otimes C)Q$. If $C$ and $D$ are square matrices, then $C \otimes D$ and $D \otimes C$ are even **permutation similar**, meaning we can take $P = Q^T$.

To be more precise [12], if $C$ is a $p \times q$ matrix and $D$ is an $r \times s$ matrix, then

$$D \otimes C = S_{p,r}(C \otimes D)S^T_{q,s},$$

where

$$S_{m,n} = \sum_{i=1}^{m}(e_i^T \otimes I_n \otimes e_i) = \sum_{j=1}^{n}(e_j \otimes I_m \otimes e_j^T).$$

$I_n$ is the $n \times n$ identity matrix and $e_i$ is the $i^{th}$ unit vector. $S_{m,n}$ is the **perfect shuffle** permutation matrix.

The commutativity of $\overline{\otimes}_3$ is up to isomorphism. This isomorphism comes from the permutation similarity of the matrices involved.

**Proposition 2.** *(Commutativity)*
*For 3-matrix factorizations* $X \in MF(f)$ *and* $X' \in MF(g)$, *there is a natural isomorphism*

$$X\overline{\otimes}_3X' \cong X'\overline{\otimes}_3X \; in \, MF(fg).$$

*Proof.* We know that $X\overline{\otimes}_3X' = ([\phi\otimes\phi'],[\psi\otimes\psi'],[\theta\otimes\theta']) \, and \, X'\overline{\otimes}_3X = ([\phi'\otimes\phi],[\psi'\otimes\psi],[\theta'\otimes\theta])$. The desired isomorphism then follows from the fact that: $\phi\otimes\phi'$ (respectively $\psi\otimes\psi'$) and $\phi'\otimes\phi$ (respectively $\psi'\otimes\psi$) are permutation similar, and also that $\theta\otimes\theta'$ and $\theta'\otimes\theta$ are permutation equivalent. □

**Proposition 3.** *(Distributivity)*
*If $X_1$ and $X_2$ are 3-matrix factorizations (of $f \in K[x]$) of the same size, then there are identities*

1. $(X_1\oplus X_2)\overline{\otimes}_3X' = (X_1\overline{\otimes}_3X')\oplus(X_2\overline{\otimes}_3X')$.
2. $X'\overline{\otimes}_3(X_1\oplus X_2) = (X'\overline{\otimes}_3X_1)\oplus(X'\overline{\otimes}_3X_2)$.

*Proof.*

1. $(X_1\overline{\otimes}_3X')\oplus(X_2\overline{\otimes}_3X')$

$$\begin{aligned}&= ([\phi_1\otimes\phi'],[\psi_1\otimes\psi'],[\theta_1\otimes\theta'])\oplus([\phi_2\otimes\phi'],[\psi_2\otimes\psi'],[\theta_2\otimes\theta'])\\ &= \left(\begin{bmatrix}\phi_1\otimes\phi' & 0\\ 0 & \phi_2\otimes\phi'\end{bmatrix},\begin{bmatrix}\psi_1\otimes\psi' & 0\\ 0 & \psi_2\otimes\psi'\end{bmatrix},\begin{bmatrix}\theta_1\otimes\theta' & 0\\ 0 & \theta_2\otimes\theta'\end{bmatrix}\right)\cdots(\sharp).\end{aligned}$$

Next,

$(X_1\oplus X_2)\overline{\otimes}_3X'$

$$\begin{aligned}&= ((\phi_1,\psi_1,\theta_1)\oplus(\phi_2,\psi_2,\theta_2))\overline{\otimes}_3(\phi',\psi',\theta')\\ &= \left(\begin{bmatrix}\phi_1 & 0\\ 0 & \phi_2\end{bmatrix},\begin{bmatrix}\psi_1 & 0\\ 0 & \psi_2\end{bmatrix},\begin{bmatrix}\theta_1 & 0\\ 0 & \theta_2\end{bmatrix}\right)\overline{\otimes}(\phi',\psi',\theta')\\ &= \left(\begin{bmatrix}\phi_1 & 0\\ 0 & \phi_2\end{bmatrix}\otimes\phi',\begin{bmatrix}\psi_1 & 0\\ 0 & \psi_2\end{bmatrix}\otimes\psi',\begin{bmatrix}\theta_1 & 0\\ 0 & \theta_2\end{bmatrix}\otimes\theta'\right)\\ &= \left(\begin{bmatrix}\phi_1\otimes\phi' & 0\\ 0 & \phi_2\otimes\phi'\end{bmatrix},\begin{bmatrix}\psi_1\otimes\psi' & 0\\ 0 & \psi_2\otimes\psi'\end{bmatrix},\begin{bmatrix}\theta_1\otimes\theta' & 0\\ 0 & \theta_2\otimes\theta'\end{bmatrix}\right)\cdots(\sharp').\end{aligned}$$

The desired identity now follows from $(\sharp)$ and $(\sharp')$.

2. The proof of this equality is similar to the foregoing proof. □

## References

[1] Leuschke, G.J., Wiegand, R.: Cohen-Macaulay representations. Amer. Math. Soc. **181**, (2012). https://doi.org/10.1090/surv/181

[2] Camacho, A.R.: Matrix factorizations and the Landau-Ginzburg/conformal field theory correspondence. Preprint at https://doi.org/10.48550/arXiv.1507.06494 (2015)

[3] Crisler, D., Diveris, K.: Matrix factorizations of sums of squares polynomials. PME J. **14**(5), 301–306 (2016). https://www.jstor.org/stable/48568151

[4] Eisenbud, D.: Homological algebra on a complete intersection, with an application to group representations. Trans. Amer. Math. Soc. **260**, 35–64 (1980). https://doi.org/10.1090/S0002-9947-1980-0570778-7

[5] Yoshino, Yu.: Tensor products of matrix factorizations. Nagoya Math. J. **152**, 39–56 (1998). https://doi.org/10.1017/S0027763000006796

[6] Leuschke, G.J., Tribone, T: Branched covers and matrix factorizations. Bull. London Math. Soc. **55**(6), 2907–2927 (2023). https://doi.org/10.1112/blms.12901

[7] Fomatati, Y.B.: Multiplicative tensor product of matrix factorizations and some applications. PhD thesis, University of Ottawa (2019). https://ruor.uottawa.ca/bitstreams/db9591eb-e230-43aa-ae82-9f2897ef19a0/download

[8] Fomatati, Y.B.: On tensor products of matrix factorizations. J. Algebra **609**, 180–216 (2022). https://doi.org/10.1016/j.jalgebra.2022.05.034

[9] Buchweitz, R.-O., Eisenbud, D., Herzog, J: Cohen-Macaulay modules on quadrics. Singularities, Representation of Algebras, and Vector Bundles. Lecture Notes in Mathematics, vol. **1273**, 58–116. Springer, Berlin, Heidelberg (1987). https://doi.org/10.1007/BFb0078838

[10] Carqueville, N., Murfet, D.: Adjunctions and defects in Landau–Ginzburg models. Advances Math. **289**, 480–566 (2016). https://doi.org/10.1016/j.aim.2015.03.033

[11] Fomatati, Y.B.: A note on the bicategory of Landau-Ginzburg models ($\mathscr{LG}_K$). Surveys Math. Appl. **19**, 1–40 (2024). https://www.utgjiu.ro/math/sma/v19/a19_01.html

[12] Henderson, H.V., Searle, S.R.: The vec-permutation matrix, the vec operator and Kronecker products: a review. Lin. Multilin. Algebra **9**(4), 271–288 (1981). https://doi.org/10.1080/03081088108817379

[13] Fomatati, Y.B.: Refined multiplicative tensor product of matrix factorizations. J. Pure Appl. Algebra **228**(5), 107556 (2024). https://doi.org/10.1016/j.jpaa.2023.107556

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