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RESEARCH ARTICLE

On the structure of the algebras of derivations of some non-nilpotent Leibniz algebras

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Abstract. Let L be an algebra over a field F with the binary operations $+$ and $[,]$. Then L is called a left Leibniz algebra if it satisfies the left Leibniz identity $[[a, b], c] = [a, [b, c]] - [b, [a, c]]$ for all $a, b, c \in L$. We study the algebras of derivations of non-nilpotent Leibniz algebras of low dimensions.

Let V be a vector space over a field F. Denote by $End_F(V)$ the set of all linear transformations of V. Then $End_F(V)$ is an associative algebra by the operations $+$ and $\circ$. As usual, $End_F(V)$ is a Lie algebra by the operations $+$ and $[,]$ where

$$[f, g] = f \circ g - g \circ f$$

for all $f, g \in End_F(V)$.

Now, let L be an algebra over a field F with the operations $+$ and $[,]$. A linear transformation f of an algebra L is called a *derivation* of L if

$$f([a, b]) = [f(a), b] + [a, f(b)]$$

for all $a, b \in L$.

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For many types of non-associative algebras, the derivations play a very important role in the study of their structure. Such is, in particular, especially true for Lie and Leibniz algebras.

Let L be an algebra over a field F with the binary operations $+$ and $[,]$. Then L is called a *left Leibniz algebra* if it satisfies the left Leibniz identity,

$$[[a,b],c] = [a,[b,c]] - [b,[a,c]]$$

for all $a, b, c \in L$. We will also use another form of this identity:

$$[a,[b,c]] = [[a,b],c] + [b,[a,c]].$$

Leibniz algebras first appeared in the paper of A. Blokh [2], but the term "Leibniz algebra" appears in the book of J.-L. Loday [11] and his article [12]. In [13], J.-L. Loday and T. Pirashvili conducted an in-depth study on Leibniz algebras' properties. The theory of Leibniz algebras has developed very intensely in many different directions. Some of the results of this theory were presented in the book [1]. Note that Lie algebras present a partial case of Leibniz algebras. Conversely, if L is a Leibniz algebra in which $[a,a] = 0$ for every element $a \in L$, then it is a Lie algebra (see, for example, [3]). Thus, Lie algebras can be characterized as anticommutative Leibniz algebras.

Let $Der(L)$ be the subset of all derivations of a Leibniz algebra L. It is possible to prove that $Der(L)$ is a subalgebra of the Lie algebra $End_F(L)$. $Der(L)$ is called the *algebra of derivations* of the Leibniz algebra L.

The influence on the structure of a Leibniz algebra of its algebra of derivations can be observed in the following result: if A is an ideal of a Leibniz algebra, then the factor-algebra of L by the annihilator of A is isomorphic to some subalgebra of $Der(A)$ [4, Proposition 3.2]. Therefore, finding out the structure of algebras of derivations of Leibniz algebras is one of the most important steps in the process of studying the structure of Leibniz algebras. It is natural to start studying the algebra of derivations of a Leibniz algebra whose structure has already been studied quite fully. The structure of the algebra of derivations of finite dimensional one-generator Leibniz algebras was described in the papers [7, 15], and the one belonging to infinite-dimensional one-generator Leibniz algebras was delineated in the paper [10].

The question about the algebras of derivations of Leibniz algebras of small dimensions naturally arises. In contrast to Lie algebra, the situation with Leibniz algebras of dimension 3 is very diverse. The Leibniz algebras of dimension 3 have been described, and their most detailed description can be found in [5]. The papers [6, 8, 9] described the algebras of derivations of nilpotent Leibniz algebras of dimension 3. In this paper, we study the algebras of derivations of some non-nilpotent Leibniz algebras of dimension 3. First, let us recall some definitions.

Every Leibniz algebra L has a specific ideal. Denote by $Leib(L)$ the subspace generated by the elements $[a, a]$, $a \in L$. It is possible to prove that $Leib(L)$ is an ideal of L. The ideal $Leib(L)$ is called the *Leibniz kernel* of algebra L. By the definition, factor-algebra $L/Leib(L)$ is a Lie algebra, and conversely, if K is an ideal of L, such that L/K is a Lie algebra, then K includes the Leibniz kernel.

Let L be a Leibniz algebra. Define the *lower central series* of L,

$$L = \gamma_1(L) \geq \gamma_2(L) \geq \ldots \gamma_\alpha(L) \geq \gamma_{\alpha+1}(L) \geq \ldots \gamma_\delta(L),$$

by the following rule: $\gamma_1(L) = L$, $\gamma_2(L) = [L, L]$, recursively, $\gamma_{\alpha+1}(L) = [L, \gamma_\alpha(L)]$ for every ordinal α, and

$$\gamma_\lambda(L) = \bigcup_{\mu<\lambda} \gamma_\mu(L)$$

for every limit ordinal λ. The last term $\gamma_\delta(L) = \gamma_\infty(L)$ is called the *lower hypocenter* of L. We have: $\gamma_\delta(L) = [L, \gamma_\delta(L)]$.

As usual, we say that a Leibniz algebra L is called *nilpotent* if there exists a positive integer k, such that $\gamma_k(L) = \langle 0 \rangle$. More precisely, L is said to be *nilpotent of nilpotency class c* if $\gamma_{c+1}(L) = \langle 0 \rangle$ but $\gamma_c(L) \neq \langle 0 \rangle$.

The *left* (respectively *right*) *center* $\zeta^{\text{left}}(L)$ (respectively $\zeta^{\text{right}}(L)$) of a Leibniz algebra L is defined by the rule below:

$$\zeta^{\text{left}}(L) = \{x \in L |\ [x, y] = 0 \text{ for each element } y \in L\}$$

(respectively

$$\zeta^{\text{right}}(L) = \{x \in L |\ [y, x] = 0 \text{ for each element } y \in L\}).$$

It is not hard to prove that the left center of L is an ideal, but such is not true for the right center. Moreover, $Leib(L) \leq \zeta^{\text{left}}(L)$, so that $L/\zeta^{\text{left}}(L)$ is a Lie algebra. The right center is a subalgebra of L; in general, the left and right centers are different; they may even have different dimensions (see [4]).

The *center* of L is defined by the rule below:

$$\zeta(L) = \{x \in L |\ [x, y] = 0 = [y, x] \text{ for each element } y \in L\}.$$

The center is an ideal of L. Note that if K is an ideal of L, then the center of K is an ideal of L [14].

Let L be a non-nilpotent Leibniz algebra of dimension 3. As usual, we will suppose that L is not a Lie algebra, so that $Leib(L)$ is non-zero. Thus, we obtain the following two cases:

$$dim_F(Leib(L)) = 2 \text{ or } dim_F(Leib(L)) = 1.$$

Consider the first case. Let $K = Leib(L)$. Since L is not a Lie algebra, there is an element a_1 such that $[a_1, a_1] = a_3 \neq 0$. Using the information about the structure of Leibniz algebras of dimension 2, we obtain that either $[a_1, a_3] = [a_3, a_1] = 0$, or we can choose an element a_1 such that $[a_3, a_1] = 0$, $[a_1, a_3] = a_3$. First, suppose that $[a_1, a_3] = [a_3, a_1] = 0$. Since subalgebra K is abelian of dimension 2, $K = Fa_3 \oplus Fb$ for some element b. Since $[a_3, x] = [x, a_3] = 0$ for every element $x \in K$, the fact that $L = \langle K, a_1\rangle$ implies that $a_3 \in \zeta(L)$. Since L is not nilpotent, factor-algebra L/Fa_3 is not abelian. Then L/Fa_3 has a coset $c + Fa_3$, such that $\langle c, Fa_3\rangle = Leib(L)$, $[a_1 + Fa_3, c + Fa_3] = c + Fa_3$. Put $c = a_2$, then $[a_1, a_2] = a_2 + \lambda a_3$ for some scalar $\lambda \in F$. Furthermore, $[a_3, a_2] = [a_2, a_3] = [a_2, a_1] = 0$, and we come to the following type of Leibniz algebra:

$$Lei_6(3, F) = Fa_1 \oplus Fa_2 \oplus Fa_3 \text{ where}$$
$$[a_1, a_1] = a_3, [a_1, a_2] = a_2 + \lambda a_3, \lambda \in F,$$
$$[a_1, a_3] = [a_2, a_1] = [a_2, a_2] = [a_2, a_3] = [a_3, a_1] = [a_3, a_2] = [a_3, a_3] = 0.$$

In other words, $Lei_6(3, F) = L$ is the sum of the Leibniz kernel $Leib(L) = A_2 = Fa_2 \oplus Fa_3$ and the nilpotent one-generator Leibniz algebras $A_1 = Fa_1 \oplus Fa_3$ of dimension 2, $[A_1, A_2] = A_2$, $Leib(L) = [L, L] = \zeta^{\text{left}}(L)$, $\zeta^{\text{right}}(L) = \zeta(L) = Fa_3$.

Now, suppose that a subalgebra $\langle a_1\rangle$ is not nilpotent. Using the information about the structure of Leibniz algebras of dimension 2, we can choose an element a_1, such that $[a_1, a_3] = a_3$. Since subalgebra K is abelian of dimension 2, $K = Fa_3 \oplus Fb$ for some element b. Moreover, $[a_3, x] = [x, a_3] = 0$ for every element $x \in K$. It follows that a subalgebra $\langle a_3\rangle$ is an ideal of L. Since $\langle a_3\rangle \neq Leib(L)$, the factor-algebra L/Fa_3 is not a Lie algebra. Then L/Fa_3 has a coset $c+Fa_3$, such that $\langle c, Fa_3\rangle = Leib(L)$, $[a_1+Fa_3, c+Fa_3] = c+Fa_3$. Put $c = a_2$, then $[a_1, a_2] = a_2+\lambda a_3$ for some scalar $\lambda \in F$. Furthermore, $[a_3, a_2] = [a_2, a_3] = [a_2, a_1] = 0$, and we come to the following type of Leibniz algebra:

$$Lei_7(3,F) = Fa_1 \oplus Fa_2 \oplus Fa_3 \text{ where}$$
$$[a_1,a_1] = [a_1,a_3] = a_3, [a_1,a_2] = a_2 + \lambda a_3, \lambda \in F,$$
$$[a_2,a_1] = [a_2,a_2] = [a_2,a_3] = [a_3,a_1] = [a_3,a_2] = [a_3,a_3] = 0.$$

In other words, $Lei_7(3,F) = L$ is the sum of the Leibniz kernel $Leib(L) = Fa_2 \oplus Fa_3 = A_2$ and the non-nilpotent one-generator Leibniz algebras $A_1 = Fa_1 \oplus Fa_3$ of dimension 2, $[A_1,A_2] = A_2$, $Leib(L) = [L,L] = \zeta^{\text{left}}(L)$, $\zeta^{\text{right}}(L) = \zeta(L) = \langle 0 \rangle$.

We begin with some general properties of the algebra of derivations of Leibniz algebras. Here, we show some basic elementary properties of derivations that have been proven in the paper [10].

Lemma 1. *Let L be a Leibniz algebra over a field F and f be a derivation of L. Then $f(\zeta^{\text{left}}(L)) \leq \zeta^{\text{left}}(L)$, $f(\zeta^{\text{right}}(L)) \leq \zeta^{\text{right}}(L)$, and $f(\zeta(L)) \leq \zeta(L)$.*

Corollary 1. *Let L be a Leibniz algebra over a field F and f be a derivation of L. Then $f(\zeta_\alpha(L)) \leq \zeta_\alpha(L)$ for every ordinal α.*

Lemma 2. *Let L be a Leibniz algebra over a field F and f be a derivation of L. Then $f([L,L]) \leq [L,L]$.*

Proof. We have

$$f([a,b]) = [f(a),b] + [a,f(b)] \in [L,L]$$

for each elements $a,b \in L$. If x is an arbitrary element of $[L,L]$, then

$$x = [a_1,b_1] + \ldots + [a_n,b_n]$$

for some elements $a_1,b_1,\ldots,a_n,b_n \in L$. Then

$$f(x) = f([a_1,b_1]) + \ldots + f([a_n,b_n]) \in [L,L].$$

□

Lemma 3. *Let L be a Leibniz algebra over a field F and D be an algebra of derivation of L. If A is a D-invariant subspace of L, then $Ann_D(L/A)$ is an ideal of D.*

Proof. If $f, g \in Ann_D(L/A)$, then $f(a), g(a) \in A$ for every element $a \in L$. Then

$$(f - g)(a) = f(a) - g(a) \in A.$$

Let $f \in Ann_D(L/A)$, $h \in D$. Then

$$\begin{aligned}[f, h](a) &= (f \circ h - h \circ f)(a)\\ &= (f \circ h)(a) - (h \circ f)(a)\\ &= f(h(a)) - h(f(a)).\end{aligned}$$

Since $f(a) \in A$, and A is a D-invariant subspace of L, $h(f(a)) \in A$. The choice of f implies that $f(h(a)) \in A$, so that $[f, h](a) \in A$. Since it is valid for each element $a \in L$, we obtain that $[f, h] \in Ann_D(L/A)$. Hence, $Ann_D(L/A)$ is an ideal of D. □

Lemma 4. *Let L be a Leibniz algebra over a field F of dimension 2, $L = Fa_1 \oplus Fa_2$,*

$$[a_1, a_2] = a_2,\ [a_2, a_1] = [a_2, a_2] = [a_1, a_1] = 0.$$

If D is an algebra of derivations of L, then D is isomorphic to a Lie subalgebra of $M_2(F)$ consisting of the matrices having the following form:

$$\begin{pmatrix} 0 & 0 \\ 0 & \beta \end{pmatrix},$$

$\beta \in F$. In particular, D is abelian and isomorphic to the additive group of field F.

Proof. Let $f \in D$. By Lemma 2, $f(Fa_2) \leq Fa_2$, so that

$$\begin{aligned}f(a_1) &= \alpha_1 a_1 + \alpha_2 a_2,\\ f(a_2) &= \beta a_2.\end{aligned}$$

Then

$$\begin{aligned}f(a_2) = f([a_1, a_2]) &= [f(a_1), a_2] + [a_1, f(a_2)]\\ &= [\alpha_1 a_1 + \alpha_2 a_2, a_2] + [a_1, \beta a_2]\\ &= \alpha_1 [a_1, a_2] + \beta [a_1, a_2] = \alpha_1 a_2 + \beta a_2\\ &= (\alpha_1 + \beta) a_2,\end{aligned}$$

and we obtain:

$$(\alpha_1 + \beta) a_2 = \beta a_2 \text{ and } \alpha_1 + \beta = \beta.$$

It follows that $\alpha_1 = 0$. Furthermore,

$$\begin{aligned} 0 = f([a_1, a_1]) &= [f(a_1), a_1] + [a_1, f(a_1)] \\ &= [\alpha_1 a_1 + \alpha_2 a_2, a_1] + [a_1, \alpha_1 a_1 + \alpha_2 a_2] \\ &= \alpha_2 [a_1, a_2] = \alpha_2 a_2. \end{aligned}$$

It follows that $\alpha_2 = 0$.

Denote by Ξ the canonical monomorphism of $End_{[,]}(L)$ in $M_2(F)$. Then $\Xi(f)$ is the following matrix:

$$\begin{pmatrix} 0 & 0 \\ 0 & \beta \end{pmatrix},$$

$\beta \in F$. Thus, we can see that D is abelian and isomorphic to the additive group of field F. □

Denote by Ξ the canonical monomorphism of $End(L)$ in $M_3(F)$ (i.e., the mapping), assigning to each endomorphism its matrix with respect to the basic $\{a_1, a_2, a_3\}$.

Theorem 1. *Let D be an algebra of derivations of the Leibniz algebra $Lei_6(3, F)$. Then D is isomorphic to a Lie subalgebra of $M_3(F)$ consisting of the matrices of the following form:*

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & \beta & 0 \\ \alpha & \lambda\beta & 0 \end{pmatrix},$$

$\alpha, \beta \in F$. Furthermore, D is abelian and isomorphic to the direct sum of two copies of the additive group of field F.

Proof. Let $L = Lei_6(3, F)$ and $f \in Der(L)$. By Lemma 2, $f([L, L]) \leq [L, L]$, and by Lemma 1, $f(\zeta(L)) \leq \zeta(L) = Fa_3$. So that

$$\begin{aligned} f(a_1) &= \alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3, \\ f(a_2) &= \beta_2 a_2 + \beta_3 a_3, \\ f(a_3) &= \gamma a_3, \end{aligned}$$

$\alpha_1, \alpha_2, \alpha_3, \beta_2, \beta_3, \gamma \in F$. Then

$$\begin{aligned}
f(a_3) &= f([a_1, a_1]) = [f(a_1), a_1] + [a_1, f(a_1)] \\
&= [\alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3, a_1] + [a_1, \alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3] \\
&= \alpha_1 [a_1, a_1] + \alpha_1 [a_1, a_1] + \alpha_2 [a_1, a_2] \\
&= \alpha_1 a_3 + \alpha_1 a_3 + \alpha_2 (a_2 + \lambda a_3) = \alpha_2 a_2 + (2\alpha_1 + \lambda \alpha_2) a_3, \\
f([a_1, a_2]) &= f(a_2 + \lambda a_3) = f(a_2) + \lambda f(a_3) = \beta_2 a_2 + \beta_3 a_3 + \lambda \gamma_3 \\
&= \beta_2 a_2 + (\beta_3 + \lambda \gamma) a_3, \\
f([a_1, a_2]) &= [f(a_1), a_2] + [a_1, f(a_2)] \\
&= [\alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3, a_2] + [a_1, \beta_2 a_2 + \beta_3 a_3] \\
&= \alpha_1 [a_1, a_2] + \beta_2 [a_1, a_2] = (\alpha_1 + \beta_2)[a_1, a_2] \\
&= (\alpha_1 + \beta_2)(a_2 + \lambda a_3) = (\alpha_1 + \beta_2) a_2 + \lambda (\alpha_1 + \beta_2) a_3.
\end{aligned}$$

Thus, we obtain

$$\begin{aligned}
\alpha_2 a_2 + (2\alpha_1 + \lambda \alpha_2) a_3 &= \gamma a_3, \\
\beta_2 a_2 + (\beta_3 + \lambda \gamma) a_3 &= (\alpha_1 + \beta_2) a_2 + \lambda (\alpha_1 + \beta_2) a_3
\end{aligned}$$

or

$$\alpha_2 = 0,\ 2\alpha_1 + \lambda \alpha_2 = \gamma,\ \beta_2 = \alpha_1 + \beta_2,\ \beta_3 + \lambda \gamma = \lambda \alpha_1 + \lambda \beta_2.$$

It follows that

$$\alpha_2 = 0,\ \alpha_1 = 0,\ \gamma = 2\alpha_1 + \lambda \alpha_2 = 0,\ \beta_3 = \lambda \beta_2.$$

Hence, $\Xi(f)$ is the following matrix:

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & \beta_2 & 0 \\ \alpha_3 & \lambda \beta_2 & 0 \end{pmatrix},$$

$\alpha_3, \beta_2 \in F$.

Conversely, let x, y be arbitrary elements of L,

$$\begin{aligned}
x &= \xi_1 a_1 + \xi_2 a_2 + \xi_3 a_3, \\
y &= \eta_1 a_1 + \eta_2 a_2 + \eta_3 a_3
\end{aligned}$$

where $\xi_1, \xi_2, \xi_3, \eta_1, \eta_2, \eta_3$ are arbitrary scalars. Then

$$\begin{aligned}
[x, y] &= [\xi_1 a_1 + \xi_2 a_2 + \xi_3 a_3, \eta_1 a_1 + \eta_2 a_2 + \eta_3 a_3] \\
&= \xi_1\eta_1[a_1, a_1] + \xi_1\eta_2[a_1, a_2] = \xi_1\eta_1 a_3 + \xi_1\eta_2(a_2 + \lambda a_3) \\
&= \xi_1\eta_2 a_2 + (\xi_1\eta_1 + \lambda\xi_1\eta_2)a_3, \\
f(x) &= f(\xi_1 a_1 + \xi_2 a_2 + \xi_3 a_3) = \xi_1\alpha_3 a_3 + \xi_2(\beta_2 a_2 + \lambda\beta_2 a_3) \\
&= \xi_2\beta_2 a_2 + (\xi_1\alpha_3 + \lambda\xi_2\beta_2)a_3, \\
f(y) &= \eta_2\beta_2 a_2 + (\eta_1\alpha_3 + \lambda\eta_2\beta_2)a_3, \\
f([x, y]) &= f(\xi_1\eta_2 a_2 + (\xi_1\eta_1 + \lambda\xi_1\eta_2)a_3) \\
&= \xi_1\eta_2 f(a_2) + (\xi_1\eta_1 + \lambda\xi_1\eta_2)f(a_3) \\
&= \xi_1\eta_2(\beta_2 a_2 + \lambda\beta_2 a_3) = \xi_1\eta_2\beta_2 a_2 + \lambda\xi_1\eta_2\beta_2 a_3.
\end{aligned}$$

Therefore,

$$\begin{aligned}
[f(x), y] + [x, f(y)] &= [\xi_2\beta_2 a_2 + (\xi_1\alpha_3 + \lambda\xi_2\beta_2)a_3, \eta_1 a_1 + \eta_2 a_2 + \eta_3 a_3] \\
&\quad + [\xi_1 a_1 + \xi_2 a_2 + \xi_3 a_3, \eta_2\beta_2 a_2 + (\eta_1\alpha_3 + \lambda\eta_2\beta_2)a_3] \\
&= \xi_1\eta_2\beta_2[a_1, a_2] = \xi_1\eta_2\beta_2(a_2 + \lambda a_3) \\
&= \xi_1\eta_2\beta_2 a_2 + \lambda\xi_1\eta_2\beta_2 a_3,
\end{aligned}$$

so that $f([x, y]) = [f(x), y] + [x, f(y)]$.

Furthermore, the equality

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & \beta & 0 \\ \alpha & \lambda\beta & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & \gamma & 0 \\ \mu & \lambda\gamma & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \beta\gamma & 0 \\ 0 & \lambda\beta\gamma & 0 \end{pmatrix}$$

shows that D is abelian and isomorphic to the direct sum of two copies of the additive group of field F. □

Theorem 2. *Let D be an algebra of derivations of the Leibniz algebra $Lei_7(3, F)$.*

If $\lambda = 0$, then D is isomorphic to a Lie subalgebra of $M_3(F)$ consisting of the matrices having the following form:

$$\begin{pmatrix} 0 & 0 & 0 \\ \alpha_2 & \beta_2 & \alpha_2 \\ \alpha_3 & \beta_3 & \alpha_3 \end{pmatrix},$$

$\alpha_2, \alpha_3, \beta_2, \beta_3 \in F$. In this case, D is isomorphic to Lie algebra $M_2(F)$.

If $\lambda \neq 0$ and $char(F) = 2$, then D is isomorphic to a Lie subalgebra of $M_3(F)$ consisting of the matrices of the following form:

$$\begin{pmatrix} \alpha_1 & 0 & 0 \\ \lambda^{-1}\alpha_1 & \alpha_3 & \lambda^{-1}\alpha_1 \\ \alpha_3 & \beta_3 & \alpha_1 + \alpha_3 \end{pmatrix},$$

$\alpha_1, \alpha_3, \beta_3 \in F$. In this case, D is isomorphic to Lie algebra $M_2(F)$.

If $\lambda \neq 0$ and $char(F) \neq 2$, then D is isomorphic to a Lie subalgebra of $M_3(F)$ consisting of the matrices of the following form:

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & \alpha_3 & 0 \\ \alpha_3 & \beta_3 & \alpha_3 \end{pmatrix},$$

$\alpha_3, \beta_3 \in F$. Furthermore, $D = A_1 \oplus B_1 \oplus E_1$ where $A_1 \oplus B_1$ is an abelian ideal of D and $B_1 \oplus E_1$ is an abelian subalgebra of D, $dim_F(A_1) = dim_F(B_1) = dim_F(E_1) = 1$.

Proof. Let $L = Lei_7(3, F)$, $f \in Der(L)$. By Lemma 2, $f([L, L]) \leq [L, L]$, so that $f(a_1) = \alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3$, $f(a_2) = \beta_2 a_2 + \beta_3 a_3$, $f(a_3) = \gamma_2 a_2 + \gamma_3 a_3$, $\alpha_1, \alpha_2, \alpha_3, \beta_2, \beta_3, \gamma_2, \gamma_3 \in F$. Then

$$\begin{aligned} f(a_3) &= f([a_1, a_1]) = [f(a_1), a_1] + [a_1, f(a_1)] \\ &= [\alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3, a_1] + [a_1, \alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3] \\ &= \alpha_1 [a_1, a_1] + \alpha_1 [a_1, a_1] + \alpha_2 [a_1, a_2] + \alpha_3 [a_1, a_3] \\ &= \alpha_1 a_3 + \alpha_1 a_3 + \alpha_2 (a_2 + \lambda a_3) + \alpha_3 a_3 \\ &= \alpha_2 a_2 + (2\alpha_1 + \lambda \alpha_2 + \alpha_3) a_3, \\ f(a_3) &= f([a_1, a_3]) = [f(a_1), a_3] + [a_1, f(a_3)] \\ &= [\alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3, a_3] + [a_1, \gamma_2 a_2 + \gamma_3 a_3] \\ &= \alpha_1 [a_1, a_3] + \gamma_2 [a_1, a_2] + \gamma_3 [a_1, a_3] \\ &= \alpha_1 a_3 + \gamma_2 (a_2 + \lambda a_3) + \gamma_3 a_3 \\ &= \gamma_2 a_2 + (\alpha_1 + \lambda \gamma_2 + \gamma_3) a_3. \end{aligned}$$

Furthermore,

$$\begin{aligned} f([a_1,a_2]) &= f(a_2+\lambda a_3) = f(a_2)+\lambda f(a_3) \\ &= \beta_2 a_2+\beta_3 a_3+\lambda(\gamma_2 a_2+\gamma_3 a_3) \\ &= (\beta_2+\lambda\gamma_2)a_2+(\beta_3+\lambda\gamma_3)a_3, \\ f([a_1,a_2]) &= [f(a_1),a_2]+[a_1,f(a_2)] \\ &= [\alpha_1 a_1+\alpha_2 a_2+\alpha_3 a_3, a_2]+[a_1,\beta_2 a_2+\beta_3 a_3] \\ &= \alpha_1[a_1,a_2]+\beta_2[a_1,a_2]+\beta_3[a_1,a_3] \\ &= \alpha_1(a_2+\lambda a_3)+\beta_2(a_2+\lambda a_3)+\beta_3 a_3 \\ &= (\alpha_1+\beta_2)a_2+(\lambda\alpha_1+\lambda\beta_2+\beta_3)a_3. \end{aligned}$$

Thus, we obtain

$$\begin{aligned} \alpha_2 a_2+(2\alpha_1+\lambda\alpha_2+\alpha_3)a_3 &= \gamma_2 a_2+(\alpha_1+\lambda\gamma_2+\gamma_3)a_3, \\ (\beta_2+\lambda\gamma_2)a_2+(\beta_3+\lambda\gamma_3)a_3 &= (\alpha_1+\beta_2)a_2+(\lambda\alpha_1+\lambda\beta_2+\beta_3)a_3, \end{aligned}$$

or

$$\begin{aligned} \alpha_2 &= \gamma_2, \\ 2\alpha_1+\lambda\alpha_2+\alpha_3 &= \alpha_1+\lambda\gamma_2+\gamma_3, \\ \beta_2+\lambda\gamma_2 &= \alpha_1+\beta_2, \\ \beta_3+\lambda\gamma_3 &= \lambda\alpha_1+\lambda\beta_2+\beta_3. \end{aligned}$$

It follows that

$$\alpha_2=\gamma_2,\ \alpha_1+\alpha_3=\gamma_3,\ \lambda\gamma_2=\alpha_1,\ \lambda\gamma_3=\lambda\alpha_1+\lambda\beta_2.$$

If $\lambda=0$, then $\alpha_2=\gamma_2,\ \alpha_1=0,\ \alpha_3=\gamma_3$. In this case, $\Xi(f)$ is the following matrix:

$$\begin{pmatrix} 0 & 0 & 0 \\ \alpha_2 & \beta_2 & \alpha_2 \\ \alpha_3 & \beta_3 & \alpha_3 \end{pmatrix},$$

$\alpha_2,\alpha_3,\beta_2,\beta_3\in F$.

If $\lambda\neq 0$, then $\alpha_2=\gamma_2,\ \alpha_1+\alpha_3=\gamma_3,\ \lambda\gamma_2=\alpha_1,\ \gamma_3=\alpha_1+\beta_2$, or $\alpha_2=\gamma_2,\ \alpha_1+\alpha_3=\gamma_3,\ \lambda\gamma_2=\alpha_1,\ \alpha_3=\beta_2$. In this case, $\Xi(f)$ is the following matrix:

$$\begin{pmatrix} \alpha_1 & 0 & 0 \\ \lambda^{-1}\alpha_1 & \alpha_3 & \lambda^{-1}\alpha_1 \\ \alpha_3 & \beta_3 & \alpha_1+\alpha_3 \end{pmatrix},$$

$\alpha_1, \alpha_3, \beta_3 \in F$.

Conversely, let x, y be arbitrary elements of L,

$$x = \xi_1 a_1 + \xi_2 a_2 + \xi_3 a_3,$$
$$y = \eta_1 a_1 + \eta_2 a_2 + \eta_3 a_3$$

where $\xi_1, \xi_2, \xi_3, \eta_1, \eta_2, \eta_3$ are arbitrary scalars. First, suppose that $\lambda = 0$. Then

$$\begin{aligned}
[x, y] &= [\xi_1 a_1 + \xi_2 a_2 + \xi_3 a_3, \eta_1 a_1 + \eta_2 a_2 + \eta_3 a_3] \\
&= \xi_1\eta_1[a_1, a_1] + \xi_1\eta_2[a_1, a_2] + \xi_1\eta_3[a_1, a_3] \\
&= \xi_1\eta_1 a_3 + \xi_1\eta_2 a_2 + \xi_1\eta_3 a_3 \\
&= \xi_1\eta_2 a_2 + (\xi_1\eta_1 + \xi_1\eta_3)a_3, \\
f(x) &= f(\xi_1 a_1 + \xi_2 a_2 + \xi_3 a_3) \\
&= \xi_1(\alpha_2 a_2 + \alpha_3 a_3) + \xi_2(\beta_2 a_2 + \beta_3 a_3) + \xi_3(\alpha_2 a_2 + \alpha_3 a_3) \\
&= (\xi_1\alpha_2 + \xi_2\beta_2 + \xi_3\alpha_2)a_2 + (\xi_1\alpha_3 + \xi_2\beta_3 + \xi_3\alpha_3)a_3, \\
f(y) &= (\eta_1\alpha_2 + \eta_2\beta_2 + \eta_3\alpha_2)a_2 + (\eta_1\alpha_3 + \eta_2\beta_3 + \eta_3\alpha_3)a_3, \\
f([x, y]) &= f(\xi_1\eta_2 a_2 + (\xi_1\eta_1 + \xi_1\eta_3)a_3) \\
&= \xi_1\eta_2 f(a_2) + (\xi_1\eta_1 + \xi_1\eta_3)f(a_3) \\
&= \xi_1\eta_2(\beta_2 a_2 + \beta_3 a_3) + (\xi_1\eta_1 + \xi_1\eta_3)(\alpha_2 a_2 + \alpha_3 a_3) \\
&= (\xi_1\eta_2\beta_2 + \xi_1\eta_1\alpha_2 + \xi_1\eta_3\alpha_2)a_2 \\
&+ (\xi_1\eta_2\beta_3 + \xi_1\eta_1\alpha_3 + \xi_1\eta_3\alpha_3)a_3.
\end{aligned}$$

Therefore,

$$\begin{aligned}
[f(x), y] + [x, f(y)] &= [(\xi_1\alpha_2 + \xi_2\beta_2 + \xi_3\alpha_2)a_2 + (\xi_1\alpha_3 + \xi_2\beta_3 + \xi_3\alpha_3)a_3, \\
&\quad \eta_1 a_1 + \eta_2 a_2 + \eta_3 a_3] + [\xi_1 a_1 + \xi_2 a_2 + \xi_3 a_3, \\
&\quad (\eta_1\alpha_2 + \eta_2\beta_2 + \eta_3\alpha_2)a_2 + (\eta_1\alpha_3 + \eta_2\beta_3 + \eta_3\alpha_3)a_3] \\
&= \xi_1(\eta_1\alpha_2 + \eta_2\beta_2 + \eta_3\alpha_2)a_2 \\
&+ \xi_1(\eta_1\alpha_3 + \eta_2\beta_3 + \eta_3\alpha_3)a_3,
\end{aligned}$$

so that $f([x, y]) = [f(x), y] + [x, f(y)]$.

Now, suppose that $\lambda \neq 0$. Then

$$\begin{aligned}
[x, y] &= [\xi_1 a_1 + \xi_2 a_2 + \xi_3 a_3, \eta_1 a_1 + \eta_2 a_2 + \eta_3 a_3] \\
&= \xi_1\eta_1[a_1, a_1] + \xi_1\eta_2[a_1, a_2] + \xi_1\eta_3[a_1, a_3] \\
&= \xi_1\eta_1 a_3 + \xi_1\eta_2(a_2 + \lambda a_3) + \xi_1\eta_3 a_3 \\
&= \xi_1\eta_2 a_2 + (\xi_1\eta_1 + \lambda\xi_1\eta_2 + \xi_1\eta_3)a_3.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
f(x) &= f(\xi_1 a_1 + \xi_2 a_2 + \xi_3 a_3) \\
&= \xi_1(\alpha_1 a_1 + \lambda^{-1}\alpha_1 a_2 + \alpha_3 a_3) \\
&+ \xi_2(\alpha_3 a_2 + \beta_3 a_3) + \xi_3(\lambda^{-1}\alpha_1 a_2 + (\alpha_1 + \alpha_3)a_3) \\
&= \xi_1\alpha_1 a_1 + (\xi_1\lambda^{-1}\alpha_1 + \xi_2\alpha_3 + \xi_3\lambda^{-1}\alpha_1)a_2 \\
&+ (\xi_1\alpha_3 + \xi_2\beta_3 + \xi_3\alpha_1 + \xi_3\alpha_3)a_3, \\
f(y) &= \eta_1\alpha_1 a_1 + (\eta_1\lambda^{-1}\alpha_1 + \eta_2\alpha_3 + \eta_3\lambda^{-1}\alpha_1)a_2 \\
&+ (\eta_1\alpha_3 + \eta_2\beta_3 + \eta_3\alpha_1 + \eta_3\alpha_3)a_3, \\
f([x,y]) &= f(\xi_1\eta_2 a_2 + (\xi_1\eta_1 + \lambda\xi_1\eta_2 + \xi_1\eta_3)a_3) \\
&= \xi_1\eta_2 f(a_2) + (\xi_1\eta_1 + \lambda\xi_1\eta_2 + \xi_1\eta_3)f(a_3) \\
&= \xi_1\eta_2(\alpha_3 a_2 + \beta_3 a_3) \\
&+ (\xi_1\eta_1 + \lambda\xi_1\eta_2 + \xi_1\eta_3)(\lambda^{-1}\alpha_1 a_2 + (\alpha_1 + \alpha_3)a_3) \\
&= (\xi_1\eta_2\alpha_3 + (\xi_1\eta_1 + \lambda\xi_1\eta_2 + \xi_1\eta_3)\lambda^{-1}\alpha_1)a_2 \\
&+ (\xi_1\eta_2\beta_3 + (\xi_1\eta_1 + \lambda\xi_1\eta_2 + \xi_1\eta_3)(\alpha_1 + \alpha_3))a_3 \\
&= (\xi_1\eta_2\alpha_3 + \lambda^{-1}\alpha_1\xi_1\eta_1 + \alpha_1\xi_1\eta_2 + \lambda^{-1}\alpha_1\xi_1\eta_3)a_2 \\
&+ (\xi_1\eta_2\beta_3 + \alpha_1\xi_1\eta_1 + \lambda\xi_1\eta_2\alpha_1 + \xi_1\eta_3\alpha_1 + \alpha_3\xi_1\eta_1 \\
&+ \alpha_3\lambda\xi_1\eta_2 + \xi_1\eta_3\alpha_3)a_3,
\end{aligned}$$

Therefore,

$$\begin{aligned}
[f(x),y] + [x,f(y)] &= [\xi_1\alpha_1 a_1 + (\xi_1\lambda^{-1}\alpha_1 + \xi_2\alpha_3 + \xi_3\lambda^{-1}\alpha_1)a_2 \\
&+ (\xi_1\alpha_3 + \xi_2\beta_3 + \xi_3\alpha_1 + \xi_3\alpha_3)a_3, \eta_1 a_1 + \eta_2 a_2 + \eta_3 a_3] \\
&+ [\xi_1 a_1 + \xi_2 a_2 + \xi_3 a_3, \eta_1\alpha_1 a_1 \\
&+ (\eta_1\lambda^{-1}\alpha_1 + \eta_2\alpha_3 + \eta_3\lambda^{-1}\alpha_1)a_2 \\
&+ (\eta_1\alpha_3 + \eta_2\beta_3 + \eta_3\alpha_1 + \eta_3\alpha_3)a_3] \\
&= \xi_1\alpha_1\eta_1[a_1,a_1] + \xi_1\alpha_1\eta_2[a_1,a_2] + \xi_1\alpha_1\eta_3[a_1,a_3] \\
&+ \xi_1\eta_1\alpha_1[a_1,a_1] + \xi_1(\eta_1\lambda^{-1}\alpha_1 + \eta_2\alpha_3 + \eta_3\lambda^{-1}\alpha_1)[a_1,a_2] \\
&+ \xi_1(\eta_1\alpha_3 + \eta_2\beta_3 + \eta_3\alpha_1 + \eta_3\alpha_3)[a_1,a_3] \\
&= \xi_1\alpha_1\eta_1 a_3 + \xi_1\alpha_1\eta_2(a_2 + \lambda a_3) + \xi_1\alpha_1\eta_3 a_3 + \xi_1\eta_1\alpha_1 a_3 \\
&+ \xi_1(\eta_1\lambda^{-1}\alpha_1 + \eta_2\alpha_3 + \eta_3\lambda^{-1}\alpha_1)(a_2 + \lambda a_3) \\
&+ \xi_1(\eta_1\alpha_3 + \eta_2\beta_3 + \eta_3\alpha_1 + \eta_3\alpha_3)a_3 \\
&= (\xi_1\alpha_1\eta_2 + \xi_1\eta_1\lambda^{-1}\alpha_1 + \xi_1\eta_2\alpha_3 + \xi_1\eta_3\lambda^{-1}\alpha_1)a_2 \\
&+ (\xi_1\alpha_1\eta_1 + \xi_1\alpha_1\eta_2\lambda + \xi_1\alpha_1\eta_3 + \xi_1\eta_1\alpha_1 + \xi_1\eta_1\alpha_1 + \lambda\xi_1\eta_2\alpha_3 \\
&+ \xi_1\eta_3\alpha_1 + \xi_1\eta_1\alpha_3 + \xi_1\eta_2\beta_3 + \xi_1\eta_3\alpha_1 + \xi_1\eta_3\alpha_3)a_3.
\end{aligned}$$

Since $f([x,y]) = [f(x),y] + [x,f(y)]$, we obtain

$$(\xi_1\eta_2\beta_3 + \alpha_1\xi_1\eta_1 + \lambda\xi_1\eta_2\alpha_1 + \xi_1\eta_3\alpha_1 + \alpha_3\xi_1\eta_1 + \alpha_3\lambda\xi_1\eta_2 + \xi_1\eta_3\alpha_3)a_3 =$$
$$(\xi_1\alpha_1\eta_1 + \xi_1\alpha_1\eta_2\lambda + \xi_1\alpha_1\eta_3 + \xi_1\eta_1\alpha_1 + \xi_1\eta_1\alpha_1 + \lambda\xi_1\eta_2\alpha_3 + \xi_1\eta_3\alpha_1$$
$$+\xi_1\eta_1\alpha_3 + \xi_1\eta_2\beta_3 + \xi_1\eta_3\alpha_1 + \xi_1\eta_3\alpha_3)a_3$$

or $2\xi_1\alpha_1\eta_3 + 2\xi_1\eta_1\alpha_1 = 0$. Hence, if $char(F) = 2$, then we obtain that the matrix

$$\begin{pmatrix} \alpha_1 & 0 & 0 \\ \lambda^{-1}\alpha_1 & \alpha_3 & \lambda^{-1}\alpha_1 \\ \alpha_3 & \beta_3 & \alpha_1+\alpha_3 \end{pmatrix}$$

defines a derivation of L. If $char(F) \neq 2$, then we obtain $\xi_1\alpha_1\eta_3 + \xi_1\eta_1\alpha_1 = 0$. Since it is true for all ξ_1, η_3, η_1, we obtain that $\alpha_1 = 0$. Thus, if $\lambda \neq 0$ and $char(F) \neq 2$, then $\Xi(f)$ is the following matrix:

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & \alpha_3 & 0 \\ \alpha_3 & \beta_3 & \alpha_3 \end{pmatrix},$$

$\alpha_3, \beta_3 \in F$.

Furthermore, let $\lambda = 0$. Then by what is proven above, $\Xi(D)$ consists of the matrices having the following form:

$$\begin{pmatrix} 0 & 0 & 0 \\ \alpha_2 & \beta_2 & \alpha_2 \\ \alpha_3 & \beta_3 & \alpha_3 \end{pmatrix},$$

$\alpha_2, \alpha_3, \beta_2, \beta_3 \in F$.

Consider the mapping

$$\vartheta : \Xi(D) \to M_2(F)$$

defined by the rule

$$\begin{pmatrix} 0 & 0 & 0 \\ \alpha_2 & \beta_2 & \alpha_2 \\ \alpha_3 & \beta_3 & \alpha_3 \end{pmatrix} \to \begin{pmatrix} \beta_2 & \alpha_2 \\ \beta_3 & \alpha_3 \end{pmatrix}.$$

The equalities

$$\begin{pmatrix} 0 & 0 & 0 \\ \alpha_2 & \beta_2 & \alpha_2 \\ \alpha_3 & \beta_3 & \alpha_3 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ \mu_2 & \nu_2 & \mu_2 \\ \mu_3 & \nu_3 & \mu_3 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 0 & 0 \\ \beta_2\mu_2 + \alpha_2\mu_3 & \beta_2\nu_2 + \alpha_2\nu_3 & \beta_2\mu_2 + \alpha_2\mu_3 \\ \beta_3\mu_2 + \alpha_3\mu_3 & \beta_3\nu_2 + \alpha_3\nu_3 & \beta_3\mu_2 + \alpha_3\mu_3 \end{pmatrix}$$

and

$$\begin{pmatrix} \beta_2 & \alpha_2 \\ \beta_3 & \alpha_3 \end{pmatrix} \begin{pmatrix} \nu_2 & \mu_2 \\ \nu_3 & \mu_3 \end{pmatrix} = \begin{pmatrix} \beta_2\nu_2 + \alpha_2\nu_3 & \beta_2\mu_2 + \alpha_2\mu_3 \\ \beta_3\nu_2 + \alpha_3\nu_3 & \beta_3\mu_2 + \alpha_3\mu_3 \end{pmatrix}$$

show that mapping ϑ is a homomorphism. Clearly, ϑ is an epimorphism. It is not hard to see that $Ker(\vartheta)$ is zero, so that ϑ is an isomorphism. Thus, we obtain that the algebra of derivations D of L is isomorphic to Lie algebra $M_2(F)$.

Now, suppose that $\lambda \neq 0$ and $char(F) = 2$. Then, by what is proven above, $\Xi(D)$ consists of the matrices having the following form:

$$\begin{pmatrix} \alpha_1 & 0 & 0 \\ \alpha_2 & \alpha_3 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{pmatrix},$$

where $\alpha_2 = \gamma_2 = \lambda^{-1}\alpha_1$, $\gamma_3 = \alpha_1 + \alpha_3$. Consider again the mapping

$$\vartheta : \Xi(D) \to M_2(F)$$

defined by the rule

$$\begin{pmatrix} \alpha_1 & 0 & 0 \\ \alpha_2 & \alpha_3 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{pmatrix} \to \begin{pmatrix} \alpha_3 & \gamma_2 \\ \beta_3 & \gamma_3 \end{pmatrix}.$$

The equalities

$$\begin{pmatrix} \alpha_1 & 0 & 0 \\ \alpha_2 & \alpha_3 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{pmatrix} \begin{pmatrix} \mu_1 & 0 & 0 \\ \mu_2 & \mu_3 & \sigma_2 \\ \mu_3 & \nu_3 & \sigma_3 \end{pmatrix}$$
$$= \begin{pmatrix} \alpha_1\mu_1 & 0 & 0 \\ \alpha_2\mu_1 + \alpha_3\mu_2 + \gamma_2\mu_3 & \alpha_3\mu_3 + \gamma_2\nu_3 & \alpha_3\sigma_2 + \gamma_2\sigma_3 \\ \alpha_3\mu_1 + \beta_3\mu_2 + \gamma_3\mu_3 & \beta_3\mu_3 + \gamma_3\nu_3 & \beta_3\sigma_2 + \gamma_3\sigma_3 \end{pmatrix}$$

and

$$\begin{pmatrix} \alpha_3 & \gamma_2 \\ \beta_3 & \gamma_3 \end{pmatrix} \begin{pmatrix} \mu_3 & \sigma_2 \\ \nu_3 & \sigma_3 \end{pmatrix} = \begin{pmatrix} \alpha_3\mu_3 + \gamma_2\nu_3 & \alpha_3\sigma_2 + \gamma_2\sigma_3 \\ \beta_3\mu_3 + \gamma_3\nu_3 & \beta_3\sigma_2 + \gamma_3\sigma_3 \end{pmatrix}$$

show that a mapping ϑ is a homomorphism. Clearly, ϑ is an epimorphism. We note that $Ker(\vartheta)$ consists of the matrices such that $0 = \alpha_3 = \gamma_2 = \beta_3 = \gamma_3$. It follows that $\alpha_2 = \alpha_1 = 0$, so that $Ker(\vartheta)$ is zero and ϑ

is an isomorphism. Thus, we obtain again that the algebra of derivations D of L is isomorphic to Lie algebra $M_2(F)$.

At last, assume that $\lambda \neq 0$ and $char(F) \neq 2$. Then, by what is proven above, $\Xi(D)$ consists of the matrices having the following form:

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & \alpha_3 & 0 \\ \alpha_3 & \beta_3 & \alpha_3 \end{pmatrix},$$

$\alpha_3, \beta_3 \in F$.

Let C be a set of matrices having the following form:

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & \gamma & 0 \\ 0 & \beta & \gamma \end{pmatrix},$$

$\gamma, \beta \in F$. The equality

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & \gamma & 0 \\ 0 & \beta & \gamma \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & \mu & 0 \\ 0 & \nu & \mu \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \gamma\mu & 0 \\ 0 & \beta\mu + \gamma\nu & \gamma\mu \end{pmatrix}$$

shows that C is an abelian subalgebra of $\Xi(D)$ of dimension 2.

Let B be a set of matrices having the following form:

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \beta & 0 \end{pmatrix},$$

$\beta \in F$, and let E be a set of matrices having the following form:

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & \gamma & 0 \\ 0 & 0 & \gamma \end{pmatrix},$$

$\gamma \in F$. Clearly, B, E are subalgebras of C, $C = B \oplus E$. The equalities

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & \gamma & 0 \\ 0 & \beta & \gamma \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \sigma & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \gamma\sigma & 0 & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \sigma & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & \gamma & 0 \\ 0 & \beta & \gamma \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

show that the set A consisting of the matrices having the form

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \sigma & 0 & 0 \end{pmatrix},$$

$\sigma \in F$, is an ideal of $\Xi(D)$ of dimension 1. Moreover, $\Xi(D) = A \oplus B \oplus E$ where $A \oplus B$ is an abelian ideal of $\Xi(D)$ and $B \oplus E$ is an abelian subalgebra of $\Xi(D)$. □

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