© Algebra and Discrete Mathematics
Volume **37** (2024). Number 1, pp. 44–84
DOI:10.12958/adm2224

RESEARCH ARTICLE

Spectral multiplicity functions of adjacency operators of graphs and cospectral infinite graphs

Pierre de la Harpe

Communicated by R. I. Grigorchuk

Abstract. The adjacency operator of a graph has a spectrum and a class of scalar-valued spectral measures which have been systematically analyzed; it also has a spectral multiplicity function which has been less studied. The first purpose of this article is to review some examples of infinite graphs for which the spectral multiplicity function of the adjacency operator has been determined. The second purpose of this article is to show explicit examples of infinite connected graphs which are cospectral, i.e., which have unitarily equivalent adjacency operators, and also explicit examples of infinite connected graphs which are uniquely determined by their spectrum.

1. Introduction

Let G be a **graph** with vertex set V and edge set E. Here graphs are without loops and multiple edges (except in Section 6), and E is a set of unordered pairs of vertices. The degree $\deg(u)$ of a vertex $u \in V$ is

The author acknowledges support of the Swiss NSF grant 200020-20040. I am grateful to Jonathan Breuer, Chris Godsil, Rostislav Grigorchuk, and Tatiana Nagnibeda, for useful indications and remarks during the writing of this paper.

2020 Mathematics Subject Classification: *05C50, 47A10.*

Key words and phrases: *spectral graph theory, adjacency operator, spectral measure, spectral multiplicity function, unitarily equivalent operators, cospectral graphs, Jacobi matrix.*

the number of edges incident to u. We assume below that V is nonempty, countable (infinite or finite), and that G is of bounded degree, i.e., that $\max_{u \in V} \deg(u) < \infty$. Let $\ell^2(V)$ denote the complex Hilbert space of functions $\xi : V \to \mathbf{C}$ such that $\sum_{u \in V} \vert \xi(u) \vert^2 < \infty$. It has a canonical orthonormal basis $(\delta_u)_{u \in V}$; the value of $\delta_u \in \ell^2(V)$ is 1 at u and 0 at other vertices. The **adjacency operator** of G is the bounded self-adjoint linear operator A_G on $\ell^2(V)$ defined by

$$(A_G \xi)(u) = \sum_{v \in V, \ \{u,v\} \in E} \xi(v) \quad \text{for all } \ \xi \in \ell^2(V) \ \text{ and } \ u \in V.$$

Adjacency operators appear in the theory of both finite graphs and infinite graphs. From the vast literature, we quote [16], [17], [7], [27], [11] for finite graphs, and [33], [36], [26], [37], [5], [24], [29] for infinite graphs.

As for any self-adjoint operator, the Hahn–Hellinger Multiplicity Theorem implies that A_G is characterized up to unitary equivalence by three invariants (see Section 2):

— the **spectrum** $\Sigma(A_G)$, also called the spectrum of G, which is a nonempty compact subset of $\mathbf{R}$;

— a **scalar-valued spectral measure** μ_G which is a finite Borel measure on $\Sigma(A_G)$, well-defined up to equivalence, sometimes viewed as a measure on $\mathbf{R}$ with closed support $\Sigma(A_G)$;

— the **spectral multiplicity function** $\mathfrak{m}_G$, which is a measurable function from $\Sigma(A_G)$ to $\{1, 2, \dots, \infty\}$, well defined up to equality μ_G-almost everywhere.

We define the **marked spectrum** of G to be the triple

$$(\Sigma(A_G), [\mu_G], \mathfrak{m}_G),$$

where $[\mu_G]$ denotes the class of a spectral valued measure μ_G. Two graphs are **cospectral** if they have the same marked spectrum. A graph G is **determined by its marked spectrum** if any graph of bounded degree with the same marked spectrum is isomorphic to G.

Let $G = (V, E)$ be a graph which is finite, or more generally a graph such that $\ell^2(V)$ has an orthonormal basis of eigenvectors of A_G, for example the Cayley graph of a lamplighter group as in [30] and [6]. The scalar-valued spectral measures of A_G are precisely the measures which charge every eigenvalue of A_G, so that the meaningful part of the marked spectrum of G reduces to the pair $(\Sigma(A_G), \mathfrak{m}_G)$. For such a graph, the spectral multiplicity functions can be defined as in finite graph theory: $\mathfrak{m}_G(x) = \dim\ker(x\mathrm{Id} - A_G)$ for all $x \in \Sigma(A_G)$. For more general graphs, see Definition 2.10.

For finite graphs, spectra and multiplicities of eigenvalues have been studied intensively. For infinite graphs, spectra of adjacency operators have attracted a lot of attention, but in contrast spectral measures a bit less, and spectral multiplicity functions even less (even if there are precise computations of multiplicities for some classes of graphs, for example for sparse trees [9]).

The first purpose of this article is to review a small number of examples of infinite connected graphs $G = (V, E)$ for which the spectral multiplicity function of A_G has been determined. All this is well-known to experts, but we did not find good references in the literature. In Section 2, we review various kinds of multiplication operators, the Hahn–Hellinger Multiplicity Theorem, and the definition of the spectral multiplicity function for a bounded self-adjoint operator. In Propositions 3.1, 3.2, and 3.4, we show:

Proposition 1.1. *The adjacency operator of the infinite ray R has spectrum $[-2, 2]$, scalar-valued spectral measure equivalent to Lebesgue measure, and uniform multiplicity one.*

The adjacency operator of the infinite line L has spectrum $[-2, 2]$, scalar-valued spectral measure equivalent to Lebesgue measure, and uniform multiplicity two.

For $d \geq 2$, the adjacency operator of the lattice L_d has spectrum $[-2d, 2d]$, scalar-valued spectral measure equivalent to Lebesgue measure, and infinite uniform multiplicity.

Section 4 is a study of spherically symmetric rooted trees. For the particular case of regular trees, we need in Section 5 to recall results on operators defined by infinite Jacobi matrices. In Propositions 4.7 and 5.3, we show:

Proposition 1.2. *For $d \geq 2$, the adjacency operator of the infinite regular rooted tree T_d^{root} of branching degree d has spectrum $[-2\sqrt{d}, 2\sqrt{2}]$, scalar-valued spectral measure equivalent to Lebesgue measure, and infinite uniform multiplicity.*

For $d \geq 3$, the adjacency operator of the regular rooted tree T_d of degree d has spectrum $[-2\sqrt{d-1}, 2\sqrt{d-1}]$, scalar-valued spectral measure equivalent to Lebesgue measure, and infinite uniform multiplicity.

Examples of cospectral finite graphs date back to the very first papers in spectral graph theory. They include a pair of graphs with 5 vertices, a pair of connected graphs with 6 vertices, a pair of trees with 8 vertices (already in [16]), and pairs of regular connected graphs with 10 vertices; for these and much more, see [11] and [31]. It is striking that examples of cospectral pairs appear that early in spectral graph theory. In contrast, the study of the spectrum of the Laplacian of geometric objects like bounded open domains in Euclidean spaces goes back to [46], and the question of existence of cospectral plane domains (rather called isospectral plane domains) was open for a long time, indeed from before [32], until the discovery of explicit examples of cospectral plane domains [28].

The second purpose of this article is to show explicit examples of cospectral infinite connected graphs. To our knowledge, such examples do not appear explicitly in the literature. As an immediate consequence of the two previous propositions, we have Corollaries 4.8 and 5.4:

Corollary 1.3. *For any integer $d \geq 2$, the graphs L_d, $T_{d^2}^{\mathrm{root}}$ and T_{d^2+1} are cospectral.*

Note that L_d and T_{d^2+1} are Cayley graphs. Further examples of multiplets of cospectral spherically symmetric rooted trees are shown in Example 4.9. The final Section 6 is a very short account of an uncountable family of cospectral Schreier graphs, from [29].

Our third purpose is to show examples of graphs determined by their spectra. There are well-known finite graphs determined by their spectra: finite paths, cycles, complete graphs K_n, complete bipartite graphs $K_{n,n}$, triangular graphs $T(n)$ with $n \neq 8$; to cite but a few. For some experts "it seems more likely that almost all graphs are determined by their spectrum, than that almost all graphs are not"; see [11, Chapter 14, and in particular Section 14.4]. Some finite graphs are determined by their spectra *among connected graphs*, but not among all finite graphs; this is the case for finite graphs G with $\|A_G\| \leq 2$ [18]. For infinite graphs, we have Propositions 3.8 and 3.9:

Proposition 1.4. *The infinite graph R is determined by its marked spectrum.*

Each of the following three graphs is determined by its marked spectrum among connected graphs of bounded degree: R, the graph D_∞ of Proposition 3.7, and the infinite line L.

2. Spectral measures and the Hahn–Hellinger Multiplicity Theorem

This section is a reminder on various notions of spectral measures and on the theorem of the title, which is due to E. Hellinger in 1907 and H. Hahn in 1912; references to the original papers can be found in [23, Section X.6, p. 928]. All Hilbert spaces which appear here are complex, and separable whenever needed. The scalar product of two vectors ξ, η in a Hilbert space $\mathcal{H}$ is denoted by $\langle \xi \mid \eta \rangle$; it is linear in ξ and antilinear in η. We use the following notation: $\mathbf{N} = \{0, 1, 2, \dots, \}$ and $\overline{\mathbf{N}^*} = \{1, 2, \dots, \infty\}$.

2.A. Spectrum, spectral measures, and dominant vectors

Let $\mathcal{H}$ be a Hilbert space, $\mathcal{L}(\mathcal{H})$ the algebra of bounded linear operators on $\mathcal{H}$, and $X \in \mathcal{L}(\mathcal{H})$. The **spectrum** of X is the set $\Sigma(X)$ of $\lambda \in \mathbf{C}$ such that $\lambda \mathrm{Id} - X$ is not invertible in $\mathcal{L}(\mathcal{H})$. It is a compact subset of $\mathbf{C}$, and a non-empty one unless $\mathcal{H} = \{0\}$. Assume from now on that X is **self-adjoint**, so that $\Sigma(X)$ is a compact subset of $\mathbf{R}$. Denote by $\mathcal{B}_{\Sigma(X)}$ the σ-algebra of Borel subsets of $\Sigma(X)$. By the spectral theorem, there exists a **projection-valued spectral measure** $E_X : \mathcal{B}_{\Sigma(X)} \to \mathrm{Proj}(\mathcal{H})$ such that $X = \int_{\Sigma(X)} x dE_X(x)$. A vector $\xi \in \mathcal{H}$ determines a **local spectral measure at** ξ on $\Sigma(X)$, denoted by μ_ξ, defined by $\mu_\xi(B) = \langle E_X(B)\xi \mid \xi \rangle$ for all $B \in \mathcal{B}_{\Sigma(X)}$; then $\langle X\xi \mid \xi \rangle = \int_{\Sigma(X)} x d\mu_\xi(x)$. A vector ξ is **dominant** for X if μ_η is absolutely continuous with respect to μ_ξ for all $\eta \in \mathcal{H}$. ("Dominant vector" is the terminology of [44, p. 306]; the terminology of [8, p. 446] is "vector of maximal type", and that of [20] is "separating vector" for the W*-algebra generated by X). A **scalar-valued spectral measure for** X is a measure on $\Sigma(X)$ of the form μ_ξ, for ξ dominant. Two scalar-valued spectral measures for X are equivalent, i.e., are absolutely continuous with respect to each other. A vector $\xi \in \mathcal{H}$ is **cyclic** for X if the closed linear span of $\{X^n \xi\}_{n \in \mathbf{N}}$ is the whole of $\mathcal{H}$.

We denote by $\mathcal{B}(\Sigma(X))$ the algebra of bounded Borel-measurable functions on $\Sigma(X)$. For f in this algebra, the operator $f(X)$ is defined by Borel functional calculus.

Proposition 2.1 (**existence and characterizations of dominant vectors for self-adjoint operators**). *Let X be a bounded self-adjoint operator on a separable Hilbert space $\mathcal{H}$. Let $\mathcal{B}(\Sigma(X))$ and E_X be as above.*

(1) There exist dominant vectors for X. More precisely, for any $\eta \in \mathcal{H}$, there exists a dominant vector ξ for X such that η is in the closed linear span of $\{X^n \xi\}_{n\in\mathbf{N}}$.

(2) A vector $\xi \in \mathcal{H}$ is dominant for X if and only if, for any $f \in \mathcal{B}(\Sigma(X))$, the equality $f(X)\xi = 0$ implies $f(X) = 0$.

(3) A vector $\xi \in \mathcal{H}$ is dominant for X if and only if, for any Borel subset B of $\Sigma(X)$, the equality $\mu_\xi(B) = 0$ is equivalent to the equality $E_X(B) = 0$.

(4) Cyclic vectors for X are dominant vectors for X.

(5) If X has at least one cyclic vector, dominant vectors for X are cyclic vectors for X.

Let $(\varepsilon_j)_{j\geq 1}$ be an orthonormal basis of $\mathcal{H}$. For $j \geq 1$, let μ_j denote the local spectral measure at ε_j.

(6) If $\xi \in \mathcal{H}$ is such that the local spectral measure μ_ξ dominates μ_j for all $j \geq 1$, then ξ is a dominant vector.

References for the proof. For (1) and (2), see [44, Lemma 5.4.7 and Problem 3 of § 5.4.]. For (3), see [15, Theorem IX.8.9]. For (4), let $\xi \in \mathcal{H}$ and $f \in \mathcal{B}(\Sigma(X))$ be such that $f(X)\xi = 0$; then $f(X)X^n\xi = X^n f(X)\xi = 0$ for all $n \geq 0$, hence $f(X)\eta = 0$ for all η in the closed convex hull of $\{X^n\xi\}_{n\in\mathbf{N}}$; if ξ is cyclic then $f(X)\eta = 0$ for all $\eta \in \mathcal{H}$, hence $f(X) = 0$, and therefore ξ is dominant. We leave the proofs of (5) and (6) to the reader; alternatively, see [14, Proposition 2.2 and Corollary 2.5]. □

The marked spectrum of a scalar multiple of a bounded self-adjoint operator can easily be written in terms of the marked spectrum of the original operator. For future reference, we make this precise in the following proposition, which is an immediate consequence of the definitions.

Proposition 2.2. *Let $\mathcal{H}$ be a separable Hilbert space, X a bounded self-adjoint operator on $\mathcal{H}$, and $\xi \in \mathcal{H}$. Let $k > 0$ be a positive real number and let $Y = kX$. Denote by μ_ξ^X the local spectral measure of X at ξ and by μ_ξ^Y the local spectral measure of Y at ξ. Let $[m, M]$ be the convex hull of the spectrum of X. Assume that μ_ξ^X is of the form $\rho_\xi^X \lambda$, where ρ_ξ^X is a function in $L^1([m, M], \lambda)$ with values in $\mathbf{R}_+$ and where λ denotes the Lebesgue measure on $[m, M]$. Then:*

(1) $\|Y\| = k\|X\|$;

(2) $\Sigma(Y) = k\Sigma(X)$;

(3) $\mu_\xi^Y = \rho_\xi^Y \lambda$ *where* $\rho_\xi^Y(x) = k^{-1}\rho_\xi^X(x/k)$ *for all* $x \in [km, kM]$.

For our analysis of lattice graphs L_d in Section 3, we will need the following facts on local spectral measures of some operators defined on tensor products. Let $\mathcal{H}_1, \mathcal{H}_2$ be two separable Hilbert spaces. For $j \in \{1, 2\}$, let X_j be a bounded self-adjoint operator on $\mathcal{H}_j$; choose a vector ξ_j in $\mathcal{H}_j$, and let μ_j be the local spectral measure of X_j at ξ_j; we view μ_j as a finite measure on $\mathbf{R}$ with closed support contained in $\Sigma(X_j)$. Let Id_j denote the identity operator on $\mathcal{H}_j$. Let $\mathcal{H}$ be the Hilbert space tensor product $\mathcal{H}_1 \otimes \mathcal{H}_2$ and let $X \in \mathcal{L}(\mathcal{H})$ be the operator $X_1 \otimes \mathrm{Id}_2 + \mathrm{Id}_1 \otimes X_2$. It is well-known that the operator X is bounded, self-adjoint, of norm $\|X\| = \|X_1\| + \|X_2\|$, and of spectrum

$$\Sigma(X) = \{z \in \mathbf{R} : z = x + y \text{ for some } x \in \Sigma(X_1) \text{ and } y \in \Sigma(X_2)\}$$

(see [13] or [43]). Let $\xi = \xi_1 \otimes \xi_2 \in \mathcal{H}$ and let μ be the local spectral measure of X at ξ.

Proposition 2.3. *Let* X_1 *be a self-adjoint operator on* $\mathcal{H}_1$ *and* X_2 *a self-adjoint operator on* $\mathcal{H}_2$*; let* ξ_1*,* ξ_2*,* μ_1*,* μ_2*,* $X = X_1 \otimes \mathrm{Id}_2 + \mathrm{Id}_1 \otimes X_2$*,* $\xi = \xi_1 \otimes \xi_2$*, and* μ *be as above.*

Then μ *is the convolution product* $\mu_1 * \mu_2$*.*

Proof. Recall that the convolution of two finite measure ν_1, ν_2 on $\mathbf{R}$ is the direct image of the measure $\nu_1 \otimes \nu_2$ on $\mathbf{R}^2$ by the map $\mathbf{R}^2 \to \mathbf{R}$, $(x, y) \mapsto x + y$. We have

$$\int_{\mathbf{R}} f(z) d(\nu_1 * \nu_2)(z) = \int_{\mathbf{R}} \int_{\mathbf{R}} f(x + y) d\nu_1(x) d\nu_2(y)$$

for any continuous function $f : \mathbf{R} \to \mathbf{C}$ which tends to zero at infinity; when ν_1 and ν_2 are measures with compact support, this holds more generally for any continuous function $f : \mathbf{R} \to \mathbf{C}$. See [42, Chapter 7, Exercise 5].

For the next computation, observe that the operators $X_1 \otimes \mathrm{Id}_2$ and $\mathrm{Id}_1 \otimes X_2$ commute, and that $(X_1 \otimes \mathrm{Id}_2)^j (\mathrm{Id}_1 \otimes X_2)^k = X_1^j \otimes X_2^k$ for all

$j, k \geq 0$. For all $n \in \mathbf{N}$, we have

$$\begin{aligned}\int_{\Sigma(X)} z^n d\mu(z) &= \langle (X_1 \otimes \mathrm{Id}_2 + \mathrm{Id}_1 \otimes X_2)^n (\xi_1 \otimes \xi_2) \,|\, \xi_1 \otimes \xi_2 \rangle \\ &= \left\langle \sum_{j=0}^{n} \binom{n}{j} (X_1^j \otimes X_2^{n-j})(\xi_1 \otimes \xi_2) \;\middle|\; \xi_1 \otimes \xi_2 \right\rangle \\ &= \sum_{j=0}^{n} \binom{n}{j} \langle X_1^j \xi_1 \,|\, \xi_1 \rangle \langle X_2^{n-j} \xi_2 \,|\, \xi_2 \rangle,\end{aligned}$$

hence

$$\begin{aligned}\int_{\Sigma(X)} z^n d\mu(z) &= \sum_{j=0}^{n} \binom{n}{j} \int_{\Sigma(X_1)} x^j d\mu_1(x) \int_{\Sigma(X_2)} y^{n-j} d\mu_2(y) \\ &= \int_{\Sigma(X_1)} \int_{\Sigma(X_2)} \left(\sum_{j=0}^{n} \binom{n}{j} x^j y^{n-j} \right) d\mu_1(x) d\mu_2(y) \\ &= \int_{\Sigma(X_1)} \int_{\Sigma(X_2)} (x+y)^n d\mu_1(x) d\mu_2(y) \\ &= \int_{\Sigma(X)} z^n d(\mu_1 * \mu_2)(z).\end{aligned}$$

This shows that the moments of μ are the same as the moments of $\mu_1 * \mu_2$. Since μ and $\mu_1 * \mu_2$ are measures with compact support, it follows that $\mu = \mu_1 * \mu_2$. □

Remark 2.4. For each positive integer d, there is a similar fact which holds for operators of the form

$$X = X_1 \otimes \mathrm{Id}_2 \otimes \cdots \otimes \mathrm{Id_d} + \mathrm{Id}_1 \otimes X_2 \otimes \cdots \otimes \mathrm{Id_d} + \cdots + \mathrm{Id}_1 \otimes \mathrm{Id}_2 \otimes \cdots \otimes X_d$$

which have spectral measures of the form $\mu = \mu_1 * \mu_2 * \cdots * \mu_d$.

2.B. Multiplication operators

We recall successively the definition of the Hilbert space $L^2(\Sigma, \mu, \mathfrak{m})$, some facts on functions $\varphi \in L^\infty(\Sigma, \mu)$, and on multiplications operators $M_{\Sigma,\mu,\mathfrak{m},\varphi}$.

Let Σ be a non-empty metrizable compact space. Let $\mathcal{B}_\Sigma$ the σ-algebra of Borel subsets of Σ, and μ a finite positive measure on $(\Sigma, \mathcal{B}_\Sigma)$.

Let $\mathfrak{m} : \Sigma \to \overline{\mathbf{N}^*}$ be a measurable function. Denote by ℓ^2_∞ the Hilbert space of square summable sequences $(z_j)_{j\geq 1}$ of complex numbers and, for each $n \geq 1$, by ℓ^2_n the subspace of sequences such that $z_j = 0$ for all $j \geq n+1$. Let $L^2(\Sigma, \mu, \mathfrak{m})$ be the separable Hilbert space of measurable functions $\xi : \Sigma \to \ell^2_\infty$ such that $\xi(x) \in \ell^2_{\mathfrak{m}(x)}$ for all $x \in \Sigma$ and $\int_\Sigma \|\xi(x)\|^2_{\ell^2_\infty} d\mu(x) < \infty$. In more sophisticated terms, $L^2(\Sigma, \mu, \mathfrak{m})$ is the Hilbert space of square summable vector fields of the μ-measurable field of Hilbert spaces $(\mathcal{H}_x)_{x\in\Sigma}$, where $\mathcal{H}_x = \ell^2_{m(x)}$ for all $x \in \Sigma$. The space $L^2(\Sigma, \mu, \mathfrak{m})$ can also be seen as a Hilbert direct sum

$$\bigoplus_{n\in\overline{\mathbf{N}^*}} L^2(\Sigma_n, \mu_n, \ell^2_n),$$

where $\Sigma_n = \mathfrak{m}^{-1}(n)$, the measure μ_n is defined by $\mu_n(B) = \mu(B \cap \Sigma_n)$ for all Borel sets $B \in \mathcal{B}_\Sigma$, and $L^2(\Sigma_n, \mu_n, \ell^2_n)$ is the Hilbert space of square-summable ℓ^2_n-valued functions on (Σ_n, μ_n). Note that $\Sigma_n = \emptyset$ when $\mathfrak{m}(x) \neq n$ for all $x \in \Sigma$, and more generally that $\mu_n = 0$ and $L^2(\Sigma_n, \mu_n, \ell^2_n) = \{0\}$ when $\mathfrak{m}(x) \neq n$ for μ-almost all $x \in \Sigma$. Note also that μ_n can be seen either as a measure on Σ_n, or as a measure on Σ such that $\mu_n(\Sigma \smallsetminus \Sigma_n) = 0$; in the latter case, the measures μ_n 's are pairwise singular with each other.

Let $\varphi : \Sigma \to \mathbf{R}$ be a measurable complex-valued function on Σ. The **essential supremum** of φ is the infimum $\|\varphi\|_\infty$ of the numbers $c \geq 0$ such that $\mu(\{x \in \Sigma : |\varphi(x)| > c\}) = 0$. We assume from now on that φ is **essentially bounded**, i.e., that $\|\varphi\|_\infty < \infty$. The **essential range** of φ is the set R_φ of complex numbers z such that $\mu(\{x \in \Sigma : |\varphi(x) - z| < \varepsilon\}) > 0$ for all $\varepsilon > 0$; we have $\|\varphi\|_\infty = \sup\{|z| : z \in R_\varphi\}$. In other words, R_φ is the closed support of the measure $\varphi_*(\mu)$ on $\mathbf{C}$, the push forward of μ by φ, and therefore R_φ is a closed subset of $\mathbf{C}$, indeed a compact subset of $\mathbf{C}$ since φ is essentially bounded. Below, $\|\varphi\|_\infty$ and R_φ will be the norm and the spectrum of a multiplication operator.

For $z \in \mathbf{C}$ and $\varepsilon > 0$, let $D_\varepsilon(z)$ denote the closed disc $\{w \in \mathbf{C} : |w - z| \leq \varepsilon\}$. Note that $\mu(\varphi^{-1}(D_\varepsilon(z))) > 0$ for all $\varepsilon > 0$ when $z \in R_\varphi$. For $z \in R_\varphi$, the **essential pre-image** $\varphi^{-1}_\mu(z)$ is defined as the set of those $x \in \Sigma$ for which, for every neighborhood V of x in Σ, we have

$$\liminf_{\varepsilon\to 0} \frac{\mu\left(V \cap \varphi^{-1}(D_\varepsilon(z))\right)}{\mu\left(\varphi^{-1}(D_\varepsilon(z))\right)} > 0.$$

For $z \in \mathbf{C} \smallsetminus R_\varphi$, set $\varphi^{-1}_\mu(z) = \emptyset$. When φ is continuous, $\varphi^{-1}_\mu(z)$ is contained in $\varphi^{-1}(z)$ [2, Theorem 6]; equality need not hold [2, p. 853–854].

Below, the cardinalities of the essential pre-images of φ will be the values of the spectral multiplicity function of a multiplication operator.

Let $\varphi, \varphi' : \Sigma \to \mathbf{C}$ be two measurable functions which are equal μ-almost every where; then the norms $\|\varphi\|_\infty$, $\|\varphi'\|_\infty$ are equal, φ, φ' have the same essential range, and φ, φ' have the same essential pre-images. From now on, we consider such functions as being equal, and write (abusively) "function" for "equivalence class of functions modulo equality μ-almost everywhere". The space $L^\infty(X, \mu)$ of essentially bounded complex-valued functions on (Σ, μ) is a Banach space for the norm $\| \cdot \|_\infty$. It is the dual of $L^1(X, \mu)$, hence it can be considered with both its norm topology and its w*-topology (see for example [22, Theorem 1.45]).

Suppose that Σ is a nonempty compact subset of the real line. Denote by $\mathcal{C}(\Sigma)$ the algebra of continuous functions on Σ, with the sup-norm, and by $\mathcal{P}(\Sigma)$ the subalgebra of functions which are restrictions to Σ of polynomial functions on $\mathbf{R}$. Then $\mathcal{P}(\Sigma)$ is dense in $\mathcal{C}(\Sigma)$, by the Stone–Weierstrass theorem, and the natural image of $\mathcal{C}(\Sigma)$ in $L^\infty(\Sigma, \mu)$ is w*-dense, see [22, Corollary 4.53]. It follows that $\mathcal{P}(\Sigma)$ is w*-dense in $L^\infty(\Sigma, \mu)$.

Definition 2.5. Let $\Sigma, \mu, \mathfrak{m}$ and φ be as above. The **multiplication operator** $M_{\Sigma,\mu,\mathfrak{m},\varphi}$ is the operator defined on the space $L^2(\Sigma, \mu, \mathfrak{m})$ by

$$(M_{\Sigma,\mu,\mathfrak{m},\varphi}\xi)(x) = \varphi(x)\xi(x) \quad \text{for all } \xi \in L^2(\Sigma, \mu, \mathfrak{m}) \text{ and } x \in \Sigma.$$

When $\mathfrak{m}$ is the constant function of value 1, we write $M_{\Sigma,\mu,\varphi}$ instead of $M_{\Sigma,\mu,\mathfrak{m},\varphi}$.

A **straight multiplication operator** $M_{\Sigma,\mu,\mathfrak{m}}$ is an operator of this type in the particular case of a compact subset Σ of the real line and of the function φ given by the inclusion $\Sigma \subset \mathbf{R}$, so that $(M_{\Sigma,\mu,\mathfrak{m}})(x) = x\xi(x)$ for all $\xi \in L^2(\Sigma, \mu, \mathfrak{m})$ and $x \in \Sigma$.

Proposition 2.6. *Let Σ, μ, $\mathfrak{m} : \Sigma \to \overline{\mathbf{N}^*}$, $L^2(\Sigma, \mu, \mathfrak{m})$, $\varphi \in L^\infty(\Sigma, \mu)$ be as above, and $M_{\Sigma,\mu,\mathfrak{m},\varphi}$ the corresponding multiplication operator, as in Definition 2.5. Suppose now that φ is a real-valued function.*

(1) $M_{\Sigma,\mu,\mathfrak{m},\varphi}$ is a bounded self-adjoint operator, $\|M_{\Sigma,\mu,\mathfrak{m},\varphi}\| = \|\varphi\|_\infty$.

(2) The spectrum of $M_{\Sigma,\mu,\mathfrak{m},\varphi}$ is the essential range R_φ of φ, and $\lambda \in \mathbf{R}$ is an eigenvalue of $M_{\Sigma,\mu,\mathfrak{m},\varphi}$ if and only if $\mu(\{x \in \Sigma : \varphi(x) = \lambda\}) > 0$.

(3) The spectral measure $E_{M_{\Sigma,\mu,\mathfrak{m},\varphi}}$ is given by

$$E_{M_{\Sigma,\mu,\mathfrak{m},\varphi}}(B) = M_{\Sigma,\mu,\mathfrak{m},\chi_{\varphi^{-1}(B)}}$$

for any Borel subset B of R_φ, where $\chi_{\varphi^{-1}(B)}$ stands for the characteristic function of the inverse image of B by φ.

(4) The measure μ is a scalar-valued spectral measure for $M_{\Sigma,\mu,\mathfrak{m},\varphi}$.

Suppose that, in particular, $\Sigma \subset \mathbf{R}$ and that φ is given by the inclusion $\Sigma \subset \mathbf{R}$; let $M_{\Sigma,\mu,\mathfrak{m}}$ be the corresponding straight multiplication operator, as in Definition 2.5. Let Σ_μ denote the closed support of μ.

(5) $\|M_{\Sigma,\mu,\mathfrak{m}}\| = \sup\{|x| : x \in \Sigma_\mu\}$.

(6) The spectrum of $M_{\Sigma,\mu,\mathfrak{m}}$ is Σ_μ.

(7) $E_{M_{\Sigma,\mu,\mathfrak{m}}}(B) = M_{\Sigma,\mu,\mathfrak{m},\chi_B}$ *for any Borel subset B of Σ_μ.*

(8) μ is a scalar-valued spectral measure for $M_{\Sigma,\mu,\mathfrak{m}}$.

Suppose moreover that $\mathfrak{m} = \mathbf{1}_\Sigma$ is the constant function of value 1, so that the operator $M = M_{\Sigma,\mu,\mathbf{1}_\Sigma}$ acts on $L^2(\Sigma,\mu)$.

(9) For $\xi \in L^2(\Sigma,\mu)$, the following conditions are equivalent:

(i) ξ is cyclic for M;

(ii) ξ is dominant for M;

(iii) $\mu(\{x \in \Sigma : \xi(x) = 0\}) = 0$.

(10) In particular, the function on Σ of constant value 1 is a cyclic vector for M.

On the proof. Let $\mathfrak{m}_\mu$ denote the restriction of the function $\mathfrak{m}$ to Σ_μ. The spaces $L^2(\Sigma,\mu,\mathfrak{m})$ and $L^2(\Sigma_\mu,\mu,\mathfrak{m}_\mu)$ are canonically isomorphic, and M can be seen as an operator on $L^2(\Sigma_\mu,\mu,\mathfrak{m}_\mu)$. It follows that we can assume without loss of generality that $\Sigma = \Sigma_\mu$, namely that the closed support of μ is the whole of Σ.

The arguments to prove Claims (1) to (4) are standard; see for example Sections 4.20 to 4.28 in [22], or any of [1, 2, 34].

Let $\xi \in L^2(\Sigma,\mu)$. Suppose first that the condition $\mu(\{x \in \Sigma : \xi(x) = 0\}) = 0$ of (9) (iii) is satisfied. Let $\eta \in L^2(\Sigma,\mu)$ be orthogonal to $M^n\xi$ for all $n \in \mathbf{N}$; we are going to show that $\eta = 0$. Note that the product $\xi\overline{\eta}$ is in the weak* dual $L^1(\Sigma,\mu)$ of $L^\infty(\Sigma,\mu)$, because ξ and η are in $L^2(\Sigma,\mu)$. Since $\langle M^n\xi \,|\, \eta\rangle = \int_\Sigma x^n\xi(x)\overline{\eta(x)}\mu(x) = 0$ for all $n \in \mathbf{N}$, we have

$$\int_\Sigma f(x)\xi(x)\overline{\eta(x)}d\mu(x) = 0$$

for all $f \in \mathcal{P}(\Sigma)$, and therefore also for all $f \in L^\infty(\Sigma, \mu)$ because $\mathcal{P}(\Sigma)$ is w*-dense in $L^\infty(\Sigma, \mu)$. This implies that $\xi\overline{\eta} = 0$ in $L^1(\Sigma, \mu)$, hence that $\xi(x)\eta(x) = 0$ for μ-almost all $x \in \Sigma$, hence by hypothesis on ξ that $\eta(x) = 0$ for μ-almost all $x \in \Sigma$, hence that $\eta = 0$. It follows that ξ is cyclic for M.

This shows (10) because the condition of (9) (iii) is clearly satisfied for ξ the constant function of value 1. Moreover, a vector in $L^2(\Sigma, \mu)$ is cyclic for M if and only if it is dominant for M, by Proposition 2.1.

Suppose now on the contrary that $\xi \in L^2(\Sigma, \mu)$ is such that $\mu(\{x \in \Sigma : \xi(x) = 0\}) > 0$. Define a Borel function $\chi : \Sigma \to \mathbf{C}$ by $\chi(x) = 1$ when x is such that $\xi(x) \neq 0$ and $\chi(x) = 0$ otherwise. Then $\chi(M) \neq 0$ and $\chi(M)\xi = 0$. It follows that ξ is not dominant for M.

This concludes the proof of (9). □

An operator X_1 on a Hilbert space $\mathcal{H}_1$ and an operator X_2 on a Hilbert space $\mathcal{H}_2$ are **unitarily equivalent** if there exists a unitary operator (= a surjective isometry) $U : \mathcal{H}_1 \to \mathcal{H}_2$ such that $X_2 = UX_1U^*$.

If $X_1 \in \mathcal{L}(\mathcal{H}_1)$ and $X_2 \in \mathcal{L}(\mathcal{H}_2)$ are two self-adjoint operators which are unitarily equivalent, their spectra coincide, $\Sigma(X_1) = \Sigma(X_2)$, and their scalar-valued spectral measures are the same.

Example 2.7 (**unitarily equivalent pairs of multiplication operators**). Let $[a_1, b_1]$, $[a_2, b_2]$ be two intervals of the real line, with $-\infty < a_1 < b_1 < \infty$ and $-\infty < a_2 < b_2 < \infty$. We consider the Hilbert spaces $L^2([a_1, b_1], \lambda)$ and $L^2([a_2, b_2], \lambda)$, where λ is the Lebesgue measure. Let

$$\varphi_2 : [a_2, b_2] \xrightarrow{\approx} [a_1, b_1]$$

be a function of class $\mathcal{C}^1$, injective, mapping $[a_2, b_2]$ onto $[a_1, b_1]$, and such that $|\varphi_2'(x)| > 0$ for all $x \in \,]a_2, b_2[$. Define an operator $M_1 = M_{[a_1,b_1],\lambda,\mathbf{1}}$ on $L^2([a_1, b_1], \lambda)$ by

$$(M_1\xi_1)(x) = x\xi_1(x) \quad \text{for all } \xi_1 \in L^2([a_1, b_1]) \text{ and } x \in [a_1, b_1]$$

and an operator $M_2 = M_{[a_2,b_2],\lambda,\mathbf{1},\varphi_2}$ on $L^2([a_2, b_2], \lambda)$ by

$$(M_2\xi_2)(x) = \varphi_2(x)\xi_2(x) \quad \text{for all } \xi_2 \in L^2([a_2, b_2]) \text{ and } x \in [a_2, b_2].$$

Then M_1 and M_2 are unitarily equivalent.

Proof. Let $U : L^2([a_1, b_1], \lambda) \to L^2([a_2, b_2], \lambda)$ be the operator defined by

$$(U\xi_1)(x) = \sqrt{|\varphi_2'(x)|}\, \xi_1(\varphi_2(x))$$

for all $\xi_1 \in L^2([a_1, b_1], \lambda)$ and $x \in [a_2, b_2]$. Then U is unitary. Indeed, for $\xi_1 \in L^2([a_1, b_1], \lambda)$ and $\xi_2 \in L^2([a_2, b_2], \lambda)$, we have

$$\begin{aligned}\|U\xi_1\|^2 &= \int_{a_2}^{b_2} |(U\xi_1)(x)|^2 dx = \int_{a_2}^{b_2} |\xi_1(\varphi_2(x))|^2 |\varphi_2'(x)| dx \\ &= \int_{a_1}^{b_1} |\xi_1(y)|^2 dy = \|\xi_1\|^2,\end{aligned}$$

and similarly $\|U^{-1}\xi_2\|^2 = \|\xi_2\|^2$.

For $\xi_1 \in L^2([a_1, b_1], \lambda)$, we have

$$\begin{aligned}(M_2 U\xi_1)(x) &= \varphi_2(x) \left(\sqrt{|\varphi_2'(x)|}\, \xi_1\big(\varphi_2(x)\big) \right) \\ &= \sqrt{|\varphi_2'(x)|} \Big(\varphi_2(x) \xi_1(\varphi_2(x)) \Big) \\ &= \sqrt{|\varphi_2'(x)|}\, (M_1\xi_1)(\varphi_2(x)) = (UM_1\xi_1)(x).\end{aligned}$$

It follows that $M_2 U = UM_1$, and this ends the proof. $\square$

Here are two particular cases; this will be useful in the proof of Proposition 3.2.

Example 2.8. (1) Let $[a_1, b_1] = [a_2, b_2] = [0, 1]$ and $\varphi_2(x) = x^\alpha$ for some $\alpha \in \mathbf{R}, \alpha > 0$. The operator M_1 of multiplication by x and the operator M_α of multiplication by x^α on $L^2([0, 1], \lambda)$ are unitarily equivalent. The unitary operator U on $L^2([0, 1], \lambda)$ is given by $(U\xi)(x) = \sqrt{\alpha x^{\alpha-1}}\xi(x^\alpha)$, and $M_\alpha U = UM_1$.

(2) Let $[a_1, b_1] = [-2, 2]$, $[a_2, b_2] = [0, \pi]$, and $\varphi_2(x) = 2\cos(x)$. The operator M_1 of multiplication by x on $L^2([-2, 2], \lambda)$ and the operator $M_{2\cos}$ of multiplication by $2\cos(x)$ on $L^2([0, \pi], \lambda)$ are unitarily equivalent. Similarly, the operator M_1 on $L^2([-2, 2], \lambda)$ and the operator of multiplication by $2\cos(x)$ on $L^2([\pi, 2\pi], \lambda)$ are unitarily equivalent.

2.C. The Hahn–Hellinger Multiplicity Theorem, and spectral multiplicity functions

The following Theorem 2.9 is the keystone of Hahn–Hellinger theory.

Theorem 2.9. *Any self-adjoint operator X on a separable Hilbert space $\mathcal{H}$ is unitarily equivalent to the straight multiplication operator $M_{\Sigma,\mu,\mathfrak{m}}$ of Definition 2.5 for the spectrum $\Sigma = \Sigma(X)$ of X, a scalar-valued spectral measure μ for X, and a measurable function $\mathfrak{m} : \Sigma \to \{1, 2, \ldots, \infty\}$.*

Moreover, if μ' is a measure on Σ and $\mathfrak{m}' : \Sigma \to \{1, 2, \dots, \infty\}$ a measurable function, then X is unitarily equivalent to the straight multiplication operator $M_{\Sigma,\mu',\mathfrak{m}'}$ if and only if the measures μ, μ' are equivalent, and the functions $\mathfrak{m}, \mathfrak{m}'$ are equal μ-almost everywhere.

For a sample of other formulations of the theorem and for proofs, see [23, Theorem X.5.10], [20, Chap. II, § 6], [4, Section 2.2], [34], [15, Theorem 10.16 and Theorem 10.20], [44, Section 5.4], and [8, Theorem 10.4.6].

Definition 2.10. Let $\mathcal{H}$, X, Σ, μ and $\mathfrak{m}$ be as in the previous theorem. The function $\mathfrak{m}$ is the **spectral multiplicity function** of X. The operator X is of **finite multiplicity** if there exists a finite constant N such that $\mathfrak{m}(x) \le N$ for μ-almost all $x \in \Sigma$. The operator X is **multiplicity-free**, or simple, if $\mathfrak{m}(x) = 1$ for μ-almost all $x \in \Sigma$, equivalently if it is unitarily equivalent to the operator of multiplication by x on the Hilbert space $L^2(\Sigma, \mu)$, where μ is a scalar-valued spectral measure on the spectrum Σ of X. The operator X is of **uniform multiplicity** $n \in \overline{\mathbf{N}^*}$ if $\mathfrak{m}(x) = n$ for μ-almost all $x \in \Sigma(X)$, equivalently if X is unitarily equivalent to a direct sum $X_1 \oplus \cdots \oplus X_n$ of pairwise unitarily equivalent multiplicity-free self-adjoint operators $X_1, \dots, X_n$.

Corollary 2.11 (reformulation of part of Theorem 2.9). *Let X_1, X_2 be two self-adjoint operators on two Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$. Suppose that X_1 and X_2 have same spectrum, equivalent scalar-valued spectral measures, and spectral multiplicity functions which are equal almost everywhere; in other words, suppose that X_1 and X_2 have the same marked spectrum.*

Then X_1 and X_2 are unitarily equivalent.

Proposition 2.12. *For a self-adjoint operator X on a separable Hilbert space $\mathcal{H}$, the following properties are equivalent:*

(i) X is multiplicity-free;

(ii) X has a cyclic vector.

Reference for a proof, and comments. Suppose that X satisfies Condition (i). Let $M_{\Sigma,\mu,\mathfrak{m}} = M_{\Sigma,\mu,\mathbf{1}_\Sigma}$ be as in Theorem 2.9. Then $M_{\Sigma,\mu,\mathbf{1}_\Sigma}$ has a cyclic vector by (10) of Proposition 2.6, hence X has a cyclic vector.

The converse implication (ii) $\Rightarrow$ (i) can be seen as one form of the spectral theorem; we refer to [44, Theorem 5.1.7]. $\square$

The next proposition is a complement to Proposition 2.6 for the spectral multiplicity function, in the simple case of $\mathfrak{m} = \mathbf{1}_\Sigma$. We use it below in the proof of Proposition 3.4. For the proof, we refer to [1, Theorem 5].

Proposition 2.13. *Let Σ be a non-empty metrizable compact space and μ a finite positive measure on Σ; assume that the closed support of μ is the whole of Σ. Let φ be a* continuous *real-valued function on Σ, viewed as $\varphi \in L^\infty(\Sigma, \mu)$. Let $M = M_{\Sigma,\mu,\mathbf{1}_\Sigma,\varphi}$ be the multiplication operator by φ on $L^2(\Sigma, \mu)$, as in Definition 2.5; recall from Proposition 2.6 that the spectrum of M is the essential range R_φ.*

Then the spectral multiplicity function $\mathfrak{m}$ for $M_{\Sigma,\mu,\mathbf{1}_\Sigma,\varphi}$ satisfies

$$\mathfrak{m}(x) = \sharp\left(\varphi_\mu^{-1}(x)\right)$$

for μ-almost all $x \in \Sigma(M) = R_\varphi$.

For infinite connected graphs, spectral multiplicity functions of adjacency operators have not been much studied. It would be interesting (at least for us!) to understand which of these graphs have multiplicity-free adjacency operators.

In contrast, many results have been shown concerning finite graphs and adjacency operators with simple eigenvalues; we quote a few.

All eigenvalues are simple for finite paths, and for all trees with at most 10 vertices (see the tables of [17]).

If G is a finite graph such that all eigenvalues of A_G are simple, any automorphism of G is of order 2; more precisely, the automorphism group of G is an elementary abelian 2-group ([38], see also [11, Corollary 1.6.1]).

Let $G = (V, E)$ be a finite graph with $n = |V|$ vertices and A_G its adjacency matrix. We denote by $\mathbf{1}_V$ the vector in $\ell^2(V)$ defined by $\mathbf{1}_V(v) = 1$ for all $v \in V$. Say G is **controllable** if $\mathbf{1}_V$ is a cyclic vector for A_G. It is conjectured in [25] and proved in [40] that *almost all finite graphs are controllable*, and therefore multiplicity-free. This is made precise as follows. Consider a positive integer n and a probability $p \in]0, 1[$. Let $\mathcal{G}(n, p)$ be the set of all graphs with vertex set $\{1, 2, \dots, n\}$ having $\lfloor p\binom{n}{2}\rfloor$ edges. Let $\mathcal{MFG}(n, p)$ be the subset of $\mathcal{G}(n, p)$ of multiplicity-free graphs. We denote by $\sharp S$ the cardinality of a set S. Then

$$\lim_{n\to\infty} \sharp\mathcal{MFG}(n, p)/\sharp\mathcal{G}(n, p) = 1.$$

3. The infinite ray, the infinite line, and the lattices

Let again $G = (V, E)$ be a graph, $(\delta_v)_{v\in V}$ the standard orthonormal basis of the Hilbert space $\ell^2(V)$, and A_G its adjacency operator on $\ell^2(V)$. A vertex $v \in V$ is **dominant** if the vector δ_v is dominant for A_G, and v is **cyclic** if the vector δ_v is cyclic for A_G. The **vertex spectral measure at** $v \in V$ is the local spectral measure at δ_v on the spectrum $\Sigma(A_G)$ of A_G.

The **infinite ray** is the graph R with vertex set $\mathbf{N} = \{0, 1, 2, 3, \dots\}$ and edge set $E = \{\{j, j+1\} : j \in \mathbf{N}\}$. The adjacency operator A_R of R is defined by

$$(A_R\xi)(u) = \xi(u-1) + \xi(u+1) \quad \text{for all } \xi \in \ell^2(\mathbf{N}) \text{ and } u \in \mathbf{N},$$

where $\xi(-1)$ should be read as 0. With respect to the standard basis $(\delta_n)_{n\in\mathbf{N}}$ of the Hilbert space $\ell^2(\mathbf{N})$ the adjacency operator A_R is the **free Jacobi matrix**:

$$A_R = J = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots \\ 1 & 0 & 1 & 0 & \cdots \\ 0 & 1 & 0 & 1 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

with entries $J_{m,n} = 1$ if $|m-n| = 1$ and $J_{m,n} = 0$ otherwise. The following proposition collects standard results on J.

Proposition 3.1. *Let R be the infinite ray and let $A_R = J$ be its adjacency operator.*

(1) The norm of A_R is 2.

(2) The spectrum of A_R is $[-2, 2]$, and A_R does not have any eigenvalue.

(3) The vertex spectral measure of A_R at 0 is given by

$$d\mu(x) = \frac{1}{2\pi}\sqrt{4-x^2}\,dx$$

for $x \in [-2, 2]$; it is a scalar-valued spectral measure for A_R.

(4) The vertex 0 is cyclic in R and the operator A_R is multiplicity-free.

(It is known that all vertices of R are cyclic [14, Proposition 7.1].)

Proof. The strategy of the proof is to view J as the matrix of an operator of multiplication by x on a Hilbert space of functions on $[-2, 2]$ with respect to an appropriate basis of orthogonal polynomials. For some background on orthogonal polynomials and their relations with Jacobi matrices, see [44, Section 4.1].

Consider the sequence $(P_n)_{n=0}^{\infty}$ of functions defined on the interval $[-2, 2]$ of the real line by

$$P_n(2\cos\theta) = \frac{\sin((n+1)\theta)}{\sin(\theta)}$$

for $\theta \in [0, \pi]$. Note that $P_0(x) = 1$, $P_1(x) = x$, $P_2(x) = x^2 - 1$, for all $x \in [-2, 2]$. Define P_{-1} to be the zero function. From the trigonometric formula

$$2\cos(\theta)\sin(n\theta) = \sin((n-1)\theta) + \sin((n+1)\theta),$$

it follows that

$$xP_{n-1}(x) = P_{n-2}(x) + P_n(x) \quad \text{for all } n \geq 1. \tag{3.1}$$

This implies, by induction on n, that P_n is a polynomial, of the form $P_n(x) = x^n + (\text{lower order terms})$ for all $n \geq 0$.

The P_n 's are Chebychev polynomials, up to a scale change. More precisely, if $U_n(x)$ denotes the Chebychev polynomial of the second kind of degree n, defined by $U_n(\cos\theta) = \sin((n+1)\theta)/\sin(\theta)$, then $P_n(x) = U_n(x/2)$.

Define a probability measure μ on $[-2, 2]$ by

$$d\mu(x) = \frac{1}{2\pi}\sqrt{4 - x^2}\, dx \quad \text{for } x \in [-2, 2].$$

Let $m, n \geq 0$; using the change of variables $x = 2\cos(\theta)$, we compute

$$\begin{aligned}
&\int_{-2}^{2} P_m(x)P_n(x)d\mu(x) = \\
&\quad = \frac{1}{2\pi}\int_{-2}^{2} P_m(x)\sqrt{4-x^2}\; P_n(x)\sqrt{4-x^2}\; \frac{dx}{\sqrt{4-x^2}} \\
&\quad = \frac{1}{2\pi}\int_{0}^{\pi} P_m(2\cos(\theta))2\sin(\theta)\; P_n(2\cos(\theta))2\sin(\theta)\; d\theta \\
&\quad = \frac{2}{\pi}\int_{0}^{\pi} \sin((m+1)\theta)\; \sin((n+1)\theta)\; d\theta
\end{aligned}$$

$$= \frac{1}{\pi} \int_0^\pi \Big[\cos \big[(m+1)\theta - (n+1)\theta\big] - \cos \big[(m+1)\theta) + (n+1)\theta\big] \Big] d\theta$$
$$= \frac{1}{\pi} \int_0^\pi \Big[\cos \big[(m-n)\theta\big] - \cos \big[(m+n+2)\theta\big] \Big] d\theta$$
$$= 0 \ \text{ if } \ m \neq n \ \text{ and } \ 1 \ \text{ if } \ m = n.$$

It follows that $(P_n)_{n \geq 0}$ is an orthonormal basis of $L^2([-2,2],\mu)$. If M_μ denotes the operator of multiplication by x on this space, we have by Equation (3.1) above

$$M_\mu P_n = P_{n-1} + P_{n+1} \quad \text{for all } \ n \geq 0, \tag{3.2}$$

where P_{-1} should be read as 0.

This shows that J is the matrix of M_μ with respect to the basis $(P_n)_{n\geq 0}$. The claims of Proposition 3.1 follow therefore from the corresponding facts of Proposition 2.6. □

The **line** is the graph L with vertex set $\mathbf{Z} = \{\dots, -1, 0, 1, \dots\}$ and edge set $E = \{\{j, j+1\} : j \in \mathbf{Z}\}$. The line can be seen as the Cayley graph of the infinite cyclic group $\mathbf{Z}$ generated by $\{1, -1\}$. The adjacency operator A_L of L is defined by

$$(A_L\xi)(u) = \xi(u-1) + \xi(u+1) \quad \text{for all } \ \xi \in \ell^2(\mathbf{Z}) \ \text{ and } \ u \in \mathbf{Z}.$$

The following proposition can be viewed as an exercise in Fourier series.

Proposition 3.2. *Let L be the infinite line and let A_L be its adjacency operator.*

(1) The norm of A_L is 2.

(2) The spectrum of A_L is $[-2,2]$.

(3) For all $j \in \mathbf{Z}$, the vertex spectral measure μ_j of A_L at j is given by its density with respect to the Lebesgue measure:

$$d\mu_j(x) = \frac{1}{\pi\sqrt{4-x^2}} dx \quad \textit{for } \ x \in [-2,2].$$

The measure μ_j is independent of j, it is a scalar-valued spectral measure for A_L, and the vertex j is dominant.

(4) The operator A_L is of uniform multiplicity 2.

Lemma 3.3. *The adjacency operator A_L of the line L is unitarily equivalent to the operator $M_{[0,2\pi],\lambda,2\cos}$ of multiplication by the function $2\cos$ on the Hilbert space $L^2([0,2\pi],\lambda)$, where λ stands for the Lebesgue measure on the interval.*

Proof. The Fourier transform

$$U : \ell^2(\mathbf{Z}) \to L^2([0,2\pi],\lambda), \qquad (U\xi)(x) = \sum_{n\in\mathbf{Z}} \xi(n)e^{inx}$$

is a surjective isometry with inverse

$$U^{-1} : L^2([0,2\pi],\lambda) \to \ell^2(\mathbf{Z}), \qquad (U^{-1}\eta)(n) = \frac{1}{2\pi}\int_0^{2\pi} \eta(x)e^{-inx}dx.$$

For any $\eta \in L^2([0,2\pi],\lambda)$, we have

$$\begin{aligned}
\left(UA_LU^{-1}\eta\right)(x) &= \sum_{n\in\mathbf{Z}} \left(A_LU^{-1}\eta\right)(n)e^{inx} \\
&= \sum_{n\in\mathbf{Z}} \Big(\left(U^{-1}\eta\right)(n-1)e^{inx} + \left(U^{-1}\eta\right)(n+1)e^{inx}\Big) \\
&= \Big(\sum_{k\in\mathbf{Z}} \left(U^{-1}\eta\right)(k)e^{ikx}\Big)e^{ix} \; + \; \Big(\sum_{k\in\mathbf{Z}} \left(U^{-1}\eta\right)(k)e^{ikx}\Big)e^{-ix} \\
&= \big(U(U^{-1}\eta)\big)(x)e^{ix} + \big(U(U^{-1}\eta)\big)(x)e^{-ix} = 2\cos(x)\eta(x)
\end{aligned}$$

for all $x \in [0,2\pi]$, so that

$$UA_LU^{-1} = M_{[0,2\pi],\lambda,2\cos}$$

as was to be proved. $\square$

Proof of Proposition 3.2. For (1) and (2), use Lemma 3.3: the norm of A_L is the norm of $M_{[0,2\pi],\lambda,2\cos}$, which is $\sup_{-2\le x\le 2}|2\cos(x)| = 2$, and the spectrum of A_L is the spectrum of $M_{[0,2\pi],\lambda,2\cos}$, which is the range of the function $2\cos$, namely which is $[-2,2]$.

(3) Let $j \in \mathbf{Z}$, viewed as a vertex of L. The vertex spectral measure μ_j at j is defined by

$$\int_{[-2,2]} f(x)d\mu_j(x) = \langle f(A_L)\delta_j \,|\, \delta_j\rangle$$

for all continuous function f on the spectrum of A_L. For $n \in \mathbf{N}$, its n^{th} moment is

$$\int_{[-2,2]} x^n d\mu_j(x) = \langle (A_L)^n \delta_j \mid \delta_j \rangle.$$

This number is also the number of paths of length n from j to j in the graph L. When n is odd, this number is clearly 0. When $n = 2m$ is even, each such path has m left steps and m right steps, so that this number is the binomial coefficient $\binom{2m}{m}$.

The moments of the measure $\frac{1}{\pi\sqrt{4-x^2}}dx$ on $[-2, 2]$ are also easy to compute. Moments of odd order vanish, because $\int_{-2}^{2} \frac{f(x)}{\pi\sqrt{4-x^2}}dx = 0$ when f is an odd function, in particular when $f(x) = x^{2m+1}$ for some $m \in \mathbf{N}$. For moments of even order $2m$, using again the change of variables $x = 2\cos\theta$, we have

$$\begin{aligned}
\int_{-2}^{2} \frac{x^{2m}}{\pi\sqrt{4-x^2}}\, dx &= \frac{1}{\pi}\int_0^{\pi} \frac{(2\cos\theta)^{2m}}{2\sin\theta} 2\sin\theta\, d\theta \\
&= \frac{1}{\pi}\int_0^{\pi} \left(e^{i\theta} + e^{-i\theta}\right)^{2m} d\theta \\
&= \frac{1}{\pi}\sum_{k=0}^{2m} \binom{2m}{k} \int_0^{\pi} e^{i2(m-k)\theta}\, d\theta = \binom{2m}{m},
\end{aligned}$$

because all but one term $(k = m)$ vanish in the sum over k.

These computations show that the measures μ_j and $\frac{dx}{\pi\sqrt{4-x^2}}$ on $[-2, 2]$ have the same moments, hence they are equal. In particular μ_j is independent of j. It follows from Proposition 2.1 (6) that this measure is a scalar-valued spectral measure for A_L, and that the vertex j is dominant.

On the one hand, Claim (4) follows from Proposition 2.13. On the other hand, we prefer to show it with a more elementary argument, as follows.

We view the operator $M_{[0,2\pi],\lambda,2\cos}$ of Lemma 3.3 as the direct sum of two operators: the operator $M_{[0,\pi],\lambda,2\cos}$ of multiplication by $2\cos$ on $L^2([0, \pi], \lambda)$ and the operator $M_{[\pi,2\pi],\lambda,2\cos}$ of multiplication by $2\cos$ on $L^2([\pi, 2\pi], \lambda)$. By Example 2.8, each of these two operators is unitarily equivalent to the operator M_1 of multiplication by x on $L^2([-2, 2], \lambda)$. It follows that $M_{[0,2\pi],\lambda,2\cos}$, and therefore also the adjacency operator A_L of the line, are unitarily equivalent to the operator of multiplication by x on the space $L^2([-2, 2], \lambda, \mathbf{C}^2)$, so that A_L is of uniform multiplicity 2. $\square$

Let now d be an integer, $d \geq 1$. Let $\{e_1, \dots, e_d\}$ be the canonical basis of the free abelian group $\mathbf{Z}^d$. The **lattice** L_d is the graph with vertex set $\mathbf{Z}^d$ and edge set

$$E = \{\{u, v\} : u \in \mathbf{Z}^d,\ v = u + e_j \ \text{ for some } \ j \in \{1, \dots, d\}\}.$$

In other words, L_d is the Cayley graph of the group $\mathbf{Z}^d$ with respect to the generating set $\{\pm e_1, \dots, \pm e_d\}$. The adjacency operator A_d of L_d is given by

$$(A_d\xi)(u) = \sum_{j=1}^{d} \xi(u - e_j) + \xi(u + e_j) \quad \text{ for all } \ \xi \in \ell^2(\mathbf{Z}^d) \ \text{ and } \ u \in \mathbf{Z}^d.$$

When $d = 1$, the lattice L_1 is the infinite line L of Proposition 3.2; now we denote by μ_1 the vertex spectral measure of the line, given by $d\mu_1(x) = \frac{1}{\pi\sqrt{4-x^2}}dx$ for all $x \in [-2, 2]$.

Proposition 3.4. *Let $d \geq 2$. Let L_d be the lattice graph of dimension d and let A_d be its adjacency operator.*

(1) The norm of A_d is 2.

(2) The spectrum of A_d is $[-2d, 2d]$.

(3) The vertex spectral measure μ_d of a vertex v in L_d is independent of v; it is the convolution of d copies of the spectral measure μ_1 of Proposition 3.2. It is a scalar-valued spectral measure for A_d and it is equivalent to the Lebesgue measure supported on $[-2d, 2d]$.

(4) The operator A_d had infinite uniform multiplicity.

For the proof of the proposition above, we begin as for Proposition 3.2, with minor modifications. Much of what follows holds for $d \geq 1$, rather than for $d \geq 2$ only. Proposition 2.13 is used for the only slightly delicate point, which is our proof of (4).

Lemma 3.5. *Let λ denote the Lebesgue measure on $[0, 2\pi]^d$. The Fourier transform*

$$U : \ell^2(\mathbf{Z}^d) \to L^2([0, 2\pi]^d, \lambda), \qquad (U\xi)(x) = \sum_{u \in \mathbf{Z}^d} \xi(u) e^{i\langle u | x \rangle}$$

(where $\langle u \,|\, x\rangle = \sum_{j=1}^{d} u_j x_j$*) is a surjective isometry with inverse*

$$\begin{aligned} &U^{-1} : L^2([0,2\pi]^d, \lambda) \to \ell^2(\mathbf{Z}^d), \\ &(U^{-1}\eta)(u) = \frac{1}{(2\pi)^d} \int_{[0,2\pi]^d} \eta(x) e^{-i\langle u|x\rangle} dx. \end{aligned}$$

Let $2\sum\cos$ *be the function* $[0,2\pi]^d \to \mathbf{R}$, $x = (x_j)_{j=1}^d \mapsto 2\sum_{j=1}^{d} \cos(x_j)$.

The operators A_d *and* $M_{[0,2\pi]^d,\lambda,2\sum\cos}$ *are unitarily equivalent; more precisely:*

$$UA_dU^{-1} = M_{[0,2\pi]^d,\lambda,2\sum\cos}.$$

Proof. For any $\eta \in L^2([0,2\pi]^d, \lambda)$, we have

$$\begin{aligned} &\left(UA_dU^{-1}\eta\right)(x) = \sum_{u\in\mathbf{Z}^d} \left(A_dU^{-1}\eta\right)(u) e^{i\langle u|x\rangle} \\ &= \sum_{u\in\mathbf{Z}^d} \sum_{j=1}^{d} \Big(\left(U^{-1}\eta\right)(u-e_j) e^{i\langle u|x\rangle} + \left(U^{-1}\eta\right)(u+e_j) e^{i\langle u|x\rangle} \Big) \\ &= \sum_{j=1}^{d} \Big(\sum_{k\in\mathbf{Z}^d} \left(U^{-1}\eta\right)(k) e^{i\langle k|x\rangle} \Big) e^{ix_j} + \sum_{j=1}^{d} \Big(\sum_{k\in\mathbf{Z}^d} \left(U^{-1}\eta\right)(k) e^{i\langle k|x\rangle} \Big) e^{-ix_j} \\ &= \sum_{j=1}^{d} U(U^{-1}\eta)(x) e^{ix_j} + \sum_{j=1}^{d} U(U^{-1}\eta)(x) e^{-ix_j} = \left(2\sum_{j=1}^{d} \cos(x_j) \right) \eta(x), \end{aligned}$$

so that UA_LU^{-1} is the operator of multiplication by $2\sum\cos$ on the Hilbert space $L^2([0,2\pi]^d, \lambda)$. □

Proof of Proposition 3.4. By Proposition 2.6, $M_{[0,2\pi]^d,\lambda,2\sum\cos}$, the operator of multiplication by the function $2\sum_{j=1}^d \cos(x_j)$ on $L^2([0,2\pi]^d,\lambda)$, has norm $2d$ and spectrum $[-2d, 2d]$. Claims (1) and (2) follow from Lemma 3.5.

Observe that there is a natural isomorphism

$$\ell^2(\mathbf{Z}) \otimes \ell^2(\mathbf{Z}) \otimes \cdots \otimes \ell^2(\mathbf{Z}) \to \ell^2(\mathbf{Z}^d)$$

by which we can identify the operators

$$A_1 \otimes \mathrm{Id} \otimes \cdots \otimes \mathrm{Id} + \cdots + \mathrm{Id} \otimes \cdots \otimes \mathrm{Id} \otimes A_1 \quad \text{and} \quad A_d.$$

By Proposition 2.3, the vertex spectral measure of A_d at a vertex of L_d is the convolution of d copies of the vertex spectral measure of A_1 at a vertex of L_1. It follows from Proposition 2.1 (6) that the vertex spectral measure of A_d at a vertex of L_d is a scalar-valued spectral measure for A_d. This proves the first part of Claim (3).

By Proposition 3.2, the vertex spectral measure at a vertex of the line L_1 is $d\mu_1(x) = f(x)dx$, where $f(x) = \frac{1}{\pi\sqrt{4-x^2}}$ if $-2 < x < 2$ and $f(x) = 0$ otherwise. The vertex spectral measure at a vertex of the lattice L_d, which is the convolution power $\mu_d \doteqdot \mu_1 * \mu_1 * \cdots * \mu_1$ (d factors), is consequently of the form $f_d(x)dx$, where f_d is a continuous function, $f_d(x) > 0$ for all $x \in \,]-2d, 2d[$, and $f_d(x) = 0$ for all x such that $|x| \geq 2d$. In particular, this measure μ_d is equivalent to the Lebesgue measure on the interval $[-2d, 2d]$. This concludes the proof of Claim (3).

By Proposition 2.13, the operator $M_{[0,2\pi]^d,\lambda,2\sum\cos}$ has uniform infinite spectral multiplicity. By Lemma 3.5, Claim (4) follows. $\square$

Remark 3.6. Consider the so-called *discrete Laplacian* $D_d = 2d\,\mathrm{Id} - A_d$ on the lattice L_d, acting on $\ell^2(\mathbf{Z}^d)$. Proposition 3.4 shows that D_d has spectrum $[0, 4d]$ and uniform multiplicity, 2 when $d = 1$ and ∞ when $d \geq 2$. The *continuous* Laplacian $\Delta_d = -\sum_{j=1}^{d} \frac{\partial^2}{\partial x_j^2}$ on the Euclidean space $\mathbf{R}^d$ is an unbounded self-adjoint operator with domain

$$Dom(\Delta_d) = \left\{ \xi \in L^2(\mathbf{R}^d, \lambda) \ : \ \int_{\mathbf{R}^d} \|k\|^2 \, |\widehat{\xi}(k)|^2 d\lambda(k) < \infty \right\},$$

where λ denotes the Lebesgue measure and $\widehat{\xi}$ the Fourier transform of ξ. The spectrum of Δ_d is $[0, \infty[$. The operators D_d and Δ_d share the same multiplicities: it is known that Δ_d has uniform multiplicity, 2 when $d = 1$ and ∞ when $d \geq 2$.

Let $D_\infty = (V', E')$ be the graph obtained from $R = (V, E)$ by adding one vertex $0'$ to the set of vertices V of R and one edge $\{0', 1\}$ to the set of edges E of R. Thus $V' = \{0'\} \cup \mathbf{N}$ and

$$E' = \big\{ \{0', 1\}, \{0, 1\}, \{1, 2\}, \{2, 3\}, \dots \big\} = \{0', 1\} \cup E.$$

Let A_D be the adjacency operator of D_∞.

Proposition 3.7. *The spectrum of A_D is $[-2, 2]$ and 0 is an eigenvalue of A_D. The vertices 0 and $0'$ are cyclic, and A_D is multiplicity-free.*

Proof. The spectrum $\Sigma(X)$ of a bounded self-adjoint operator X is the union of the essential spectrum $\Sigma_{\text{ess}}(X)$ and a discrete set of points in $\mathbf{R} \smallsetminus \Sigma_{\text{ess}}(X)$ which are eigenvalues of finite multiplicity. In particular $\Sigma_{\text{ess}}(J_1) = \Sigma(J_1) = [-2, 2]$ by Proposition 3.1.

Let $R' = (V', E)$ be the graph obtained from $R = (V, E)$ by adding one isolated vertex $\{0'\}$, and let A'_R be its adjacency operator. The marked spectrum of A'_R is the union of that of $A_R = J_1$ and of the simple eigenvalue 0. The operator A_D is a perturbation of A'_R by an operator of finite rank, indeed of rank 2. If K is a compact self-adjoint operator on the same space as X, it is a theorem of Weyl that $\Sigma_{\text{ess}}(X+K) = \Sigma_{\text{ess}}(X)$ [44, Theorem 3.14.1]. In particular

$$\Sigma_{\text{ess}}(A_D) \overset{\text{(by Weyl)}}{=} \Sigma_{\text{ess}}(A'_R) = \Sigma(A'_R) = [-2, 2]. \tag{3.3}$$

Let $n \geq 4$. The finite graph D_n has vertex set $\{0', 0, 1, \dots, n-2\}$ and edge set $\Big\{\{0', 1\}, \{0, 1\}, \{1, 2\}, \dots, \{n-3, n-2\}\Big\}$. The spectrum $\Sigma(D_n)$ of its adjacency operator is well-known [11, Theorem 3.1.3] to be a finite subset of $]-2, 2[$. Let D'_n be the graph with vertex set V' and the same edge set as D_n. Since $0 \in \Sigma(D_n)$, the spectrum of D'_n is the same as that of D_n. For $n \to \infty$, the sequence of the adjacency operators of D'_n converges strongly to A_D. It follows that $\Sigma(A_D)$ is contained in the union $\bigcup_{n \geq 4} \Sigma(D'_n)$, hence in $[-2, 2]$; see [23, Section X.7]. Together with (3.3), this shows that $\Sigma(A_D) = [-2, 2]$.

Let $\xi \in \ell^2(V')$ be defined by $\xi(0) = 1$, $\xi(0') = -1$ and $\xi(j) = 0$ for all $j \geq 1$. Then $A_D \xi = 0$, so that 0 is an eigenvalue of A_D. It is easy to check that 0 and $0'$ are cyclic vertices; if necessary, see [14, Example 7.2]. □

Proposition 3.8. *Let G be an infinite connected graph of bounded degree with adjacency operator A_G such that $\|A_G\| \leq 2$. Then $\|A_G\| = 2$, $\Sigma(A_G) = [-2, 2]$, and G is isomorphic to one of the three following graphs:*

- *the infinite ray R and then A_G is multiplicity-free, without eigenvalue;*
- *the graph D_∞ and then A_G is multiplicity-free, with an eigenvalue;*
- *the infinite line L and then A_G is of uniform multiplicity two, without eigenvalue.*

It follows that these three graphs are determined by their marked spectrum among connected graphs of bounded degree.

Proof. Let $F = (V_F, E_F)$ be a finite subgraph of $G = (V_G, E_G)$, and let $F_{\mathrm{ind}} = (V_F, E_{\mathrm{ind}})$ be the subgraph of G induced by V_F. Then $\|A_F\| \leq \|A_{F_{\mathrm{ind}}}\|$ by Perron–Frobenius theory and $\|A_{F_{\mathrm{ind}}}\| \leq \|A_G\|$ by standard arguments (details in [14, proof of Proposition 3.1]), so that $\|A_F\| \leq 2$.

Computations with finite graphs show we would have $\|A_F\| > 2$ if F was a connected finite graph containing strictly one of $\widetilde{A}_n$ $(n \geq 2)$, $\widetilde{D}_n$ $(n \geq 4)$, $\widetilde{E}_n$ $(n = 6, 7, 8)$, and this is not possible. Here $\widetilde{A}_n$ denotes the cycle with $n+1$ vertices, $\widetilde{D}_n$ the graph obtained from a segment with vertices $v_1, \ldots, v_{n-1}$ and edges $\{v_j, v_{j+1}\}$ $(1 \leq j \leq n-2)$ by adding two vertices v_0, v_n and two edges $\{v_0, v_2\}$, $\{v_{n-2}, v_n\}$, and $\widetilde{E}_6, \widetilde{E}_7, \widetilde{E}_8$ the stars with respectively $7, 8, 9$ vertices described in [11]; see Theorem 3.1.3 in this book. It follows that G is a tree, because it does not contain strictly any $\widetilde{A}_n$ $(n \geq 2)$. Also G does not have vertices of degree ≥ 4, and G has at most one vertex of degree 3, because it does not contain strictly any $\widetilde{D}_n$ $(n \geq 4)$. And finally, if G contains a vertex of degree 3, two of the segments starting from this vertex must be of length 1, because it does not contain strictly any $\widetilde{E}_n$ $(n = 6, 7, 8)$. It follows that G is isomorphic to one of R, D_∞, L, hence $\|A_G\| = 2$ and $\Sigma(A_G) = [-2, 2]$.

Since R, D_∞, and L have different multiplicity functions, each of them is determined by its marked spectrum among connected graphs. □

Proposition 3.9. *The infinite ray R is determined by its marked spectrum.*

Proof. Let G be a graph of bounded degree with the same marked spectrum as that of R. Let $(G_i)_{i \in I}$ be the connected components of G. Denote by A_G the adjacency operator of G and, for each $i \in I$, by A_i that of G_i. There cannot exist $i \in I$ with G_i finite; otherwise $\Sigma(A_i)$ would consist of eigenvalues, and thus $\Sigma(A_G)$ would contain eigenvalues, but this is impossible since $\Sigma(A_R)$ does not; hence each G_i is infinite. By Proposition 3.8, each G_i is isomorphic to one of R, D_∞, or L; but D_∞ is impossible because A_R does not have eigenvalue and L is impossible because A_R has uniform spectral multiplicity 1; hence each G_i is isomorphic to R. The graph G cannot be the union of 2 or more connected components isomorphic to R, again because A_R has uniform spectral multiplicity 1; hence G is isomorphic to R. □

Note that the infinite line L is not characterized by its marked spectrum. Indeed, the adjacency operator A_L and the adjacency operator of a graph with two connected components isomorphic to R are unitarily equivalent, as it follows from Corollary 2.11 and Propositions 3.1 & 3.2.

It is natural to ask whether there are other infinite connected graphs G with $\|A_G\| < \sqrt{2+\sqrt{5}} \sim 2.058$ which are characterized by their marked spectrum among connected graphs; see [12]. The range $\sqrt{2+\sqrt{5}} \leq \|A_G\| \leq \frac{3}{2}\sqrt{2} \sim 2.121$ could also be investigated [47].

4. Spherically symmetric infinite rooted trees

Let $T = (V, E)$ be a spherically symmetric rooted tree, of bounded degree and without leaves, and let A_T be its adjacency operator. The main technical result of this section is Proposition 4.6, showing an orthogonal decomposition of $\ell^2(V)$ in subspaces invariant by A_T on each of which A_T is an infinite Jacobi matrix. This is standard, it has been used for trees as here and in other contexts; see [41], [3, Lemma 1], [45, Theorem 3.2], [9, Theorem 2.4].

Let $T = (V, E)$ be a tree. Choose a root $v_0 \in V$. For $v \in V$, denote by $|v|$ the distance from v to v_0. For an integer $r \geq 0$, let $S_r = \{v \in V : |v| = r\}$ be the sphere in V of radius r around v_0. For $v \in V$, denote by N_v^+ the set of neighboring vertices of V at distance $|v|+1$ from v_0. For $v \in V$ different from v_0, denote by v_- the neighboring vertex of v at distance $|v|-1$ from v_0; note that, for $v \neq v_0$, the set of neighbors of v is $\{v_-\} \cup N_v^+$, and therefore the degree of v is $\deg(v) = 1 + |N_v^+|$. The set of neighbors of v_0 is $N_{v_0}^+ = S_1$.

The infinite rooted tree T is **spherically symmetric** if, for every $r \geq 0$, every vertex in S_r has exactly $d_r \geq 1$ adjacent vertices in S_{r+1}, for some sequence $(d_r)_{r \geq 0}$ of positive integers, the sequence of **branching degrees** of T. From now on, we consider an infinite spherically symmetric rooted tree T of bounded degree, with sequence of branching degrees such that

$$d_r \geq 2 \ \text{ for all } \ r \geq 0 \quad \text{ and } \quad \sup_r d_r < \infty. \tag{4.1}$$

For $r \geq 0$, we identify $\ell^2(S_r)$ with the subspace of $\ell^2(V)$ of functions which vanish on $V \smallsetminus S_r$. We set $\ell^2(S_{-1}) = \{0\}$. Define an operator H on $\ell^2(V)$ by

$$(H\xi)(v) = \xi(v_-) \ \text{ if } \ |v| \geq 1 \ \text{ and } \ (H\xi)(v_0) = 0 \ \text{ for all } \ \xi \in \ell^2(V). \tag{4.2}$$

Proposition 4.1. *Let $T = (V, E)$ be a spherically symmetric infinite rooted tree with root $v_0 \in V$, and with sequence of branching degrees $(d_r)_{r \geq 0}$ such that Condition (4.1) holds. Let A_T and H be as above.*

(1) *The operator H is bounded on $\ell^2(V)$ of norm $\sqrt{\max_{r\geq 0} d_r}$, and is injective.*

(2) *The adjoint H^* of H is given by*

$$(H^*\xi)(v) = \sum_{w\in N_v^+} \xi(w) \quad \textit{for all } \xi \in \ell^2(V) \textit{ and } v \in V, \tag{4.3}$$

and we have

$$A_T = H + H^*.$$

(3) *For all $r \geq 0$:*

- *the restriction $\frac{1}{\sqrt{d_r}} H\big|_{\ell^2(S_r)}$ is an isometry from $\ell^2(S_r)$ into $\ell^2(S_{r+1})$ and $\frac{1}{d_r} H^*H\big|_{\ell^2(S_r)} = \mathrm{Id}_{\ell^2(S_r)}$;*
- *$H^*(\ell^2(S_r)) = \ell^2(S_{r-1})$ and $HH^*(\ell^2(S_r)) \subset \ell^2(S_r)$.*

(4) *Let $r \geq 0$ and $k \geq 0$. If ξ and η in $\ell^2(S_r)$ are orthogonal, then $H^k\xi$ and $H^k\eta$ in $\ell^2(S_{r+k})$ are also orthogonal.*

Proof. (1) Let $\xi \in \ell^2(V)$. We have

$$\begin{aligned}\|H\xi\|^2 &= \sum_{v\in V} |(H\xi)(v)|^2 = \sum_{v\in V, v\neq v_0} |\xi(v_-)|^2 = \sum_{w\in V} d_{|w|}\, |\xi(w)|^2 \\ &\leq \big(\max_{r\geq 0} d_r\big) \sum_{w\in V} |\xi(w)|^2 = \big(\max_{r\geq 0} d_r\big) \|\xi\|^2,\end{aligned}$$

hence $\|H\| \leq \sqrt{\max_{r\geq 0} d_r}$. For the equality, see the end of (3) below.

If $H\xi = 0$, i.e., if $\xi(v_-) = 0$ for all $v \in V \smallsetminus \{v_0\}$, then $\xi = 0$, hence H is injective.

(2) We use temporarily Formula (4.3) as a definition of H^*. Then H^* is bounded; indeed, using the Cauchy–Schwarz inequality, we have for all $\xi \in \ell^2(V)$

$$\begin{aligned}\sum_{v\in V} |(H^*\xi)(v)|^2 &= \sum_{v\in V} \Big| \sum_{w\in N_v^+} \xi(w)\Big|^2 \leq \sum_{v\in V} d_{|v|} \sum_{w\in N_v^+} |\xi(w)|^2 \\ &= \sum_{w\in V, w\neq v_0} d_{|w|-1} |\xi(w)|^2 \leq \big(\max_{r\geq 0} d_r\big) \|\xi\|^2.\end{aligned}$$

And H^* is the adjoint of H because, for $\xi, \eta \in \ell^2(V)$, we have

$$\begin{aligned}\langle H^*\xi \,|\, \eta\rangle &= \sum_{v \in V} (H^*\xi)(v)\overline{\eta(v)} = \sum_{v \in V} \sum_{w \in N_v^+} \xi(w)\overline{\eta(v)} \\ &= \sum_{w \neq v_0} \xi(w)\overline{\eta(w_-)} = \sum_{w \neq v_0} \xi(w)\overline{(H\eta)(w)} = \langle \xi \,|\, H\eta\rangle.\end{aligned}$$

The equality $A_T = H + H^*$ follows from (4.2) and (4.3).

(3) Let $\xi \in \ell^2(S_r)$. It is obvious that $H\xi \in \ell^2(S_{r+1})$. Moreover, the computation of the proof of (1) continues as

$$\|H\xi\|^2 = \sum_{w \in V} d_{|w|} \, |\xi(w)|^2 = d_r \sum_{w \in S_r} |\xi(w)|^2 = d_r \|\xi\|^2,$$

hence $\frac{1}{\sqrt{d_r}} H\big|_{\ell^2(S_r)}$ is an isometry from $\ell^2(S_r)$ into $\ell^2(S_{r+1})$. We have also

$$(H^*H\xi)(v) = \sum_{w \in N_v^+} (H\xi)(w) = d_r \xi(v) \quad \text{for all } \ v \in V,$$

hence

$$\frac{1}{d_r} H^*H\big|_{\ell^2(S_r)} = \mathrm{Id}_{\ell^2(S_r)}. \tag{4.4}$$

It follows that H^* maps $\ell^2(S_{r+1})$ onto $\ell^2(S_r)$, and also that $\|H\| \geq \sqrt{d_r}$.

It follows now that $\|H\| = \sqrt{\max_{r \geq 0} d_r}$.

(4) For ξ and η orthogonal in $\ell^2(S_r)$ we have, using Equality (4.4),

$$\langle H\xi \,|\, H\eta\rangle = \langle H^*H\xi \,|\, \eta\rangle = d_r \langle \xi \,|\, \eta\rangle = 0,$$

so that $H\xi$ and $H\eta$ are orthogonal in $\ell^2(S_{r+1})$. For $k \geq 2$, the same argument repeated k times shows that $H^k\xi$ and $H^k\eta$ are orthogonal. □

Set

$$\mathcal{U}_{0,0} = \ell^2(S_0) \quad \text{and} \quad \mathcal{U}_{0,r} = H^r(\mathcal{U}_{0,0}) \quad \text{for each integer } \ r \geq 0.$$

Note that $\mathcal{U}_{0,r}$ is the one-dimensional subspace of $\ell^2(V)$ of functions on V which vanish outside S_r and which are constant on S_r. Set

$$\mathcal{V}_0 = \bigoplus_{r=0}^{\infty} \mathcal{U}_{0,r},$$

which is the subspace of $\ell^2(V)$ of functions which are constant on each sphere.

We define now subspaces $\mathcal{U}_{n,r}$ and $\mathcal{V}_n$ for $n \geq 1$ and $r \geq n$, by induction on n. Let $n \geq 1$; assume that $\mathcal{U}_{m,q}$ has already been defined when $0 \leq m < n$ and $q \geq m$. Define

$$\begin{aligned}
\mathcal{U}_{n,n} &= \text{ orthogonal complement of } \mathcal{U}_{0,n} \oplus \mathcal{U}_{1,n} \oplus \cdots \oplus \mathcal{U}_{n-1,n} \text{ in } \ell^2(S_n),\\
\mathcal{U}_{n,r} &= H^{r-n}(\mathcal{U}_{n,n}) \text{ in } \ell^2(S_r) \text{ for all } r \geq n,\\
\mathcal{V}_n &= \bigoplus_{r=n}^{\infty} \mathcal{U}_{n,r}.
\end{aligned}$$

Observe that

$$\ell^2(V) = \bigoplus_{r=0}^{\infty} \ell^2(S_r) \quad \text{and} \quad \ell^2(S_r) = \bigoplus_{n=0}^{r} \mathcal{U}_{n,r} \quad \text{for all } r \geq 0. \tag{4.5}$$

Proposition 4.2. *Let the notation be as above. There are orthogonal direct sums decompositions*

$$\ell^2(V) = \bigoplus_{n=0}^{\infty} \mathcal{V}_n = \bigoplus_{n=0}^{\infty} \bigoplus_{r=n}^{\infty} \mathcal{U}_{n,r}.$$

For each $n \geq 0$, the subspace $\mathcal{V}_n$ of $\ell^2(V)$ is invariant by H, H^*, and A_T.

Proof. We continue to follow [3].

We first check that the direct sums are orthogonal. Let n_1, r, s be nonnegative integers such that $r \neq s$ and $0 \leq n_1 \leq \min\{r, s\}$. The spaces $\mathcal{U}_{n_1,r}$ and $\mathcal{U}_{n_1,s}$ are orthogonal, because they are respectively subspaces of $\ell^2(S_r)$ and $\ell^2(S_s)$ which are orthogonal. It follows that $\mathcal{V}_{n_1} = \bigoplus_{r=n_1}^{\infty} \mathcal{U}_{n_1,r}$ is an orthogonal sum. Let moreover n_2 be an integer such that $n_2 > n_1$. The spaces $\mathcal{U}_{n_1,n_2}$ and $\mathcal{U}_{n_2,n_2}$ are orthogonal by definition of $\mathcal{U}_{n_2,n_2}$. By (4) of Proposition 4.1, the spaces $\mathcal{U}_{n_1,r} = H^{r-n_2}(\mathcal{U}_{n_1,n_2})$ and $\mathcal{U}_{n_2,r} = H^{r-n_2}(\mathcal{U}_{n_2,n_2})$ are orthogonal whenever $r \geq n_2$. It follows that $\mathcal{V}_{n_1} = \bigoplus_{r=n_1}^{\infty} \mathcal{U}_{n_1,r}$ and $\mathcal{V}_{n_2} = \bigoplus_{r=n_2}^{\infty} \mathcal{U}_{n_2,r}$ are orthogonal, and therefore that $\ell^2(V) = \bigoplus_{n=0}^{\infty} \mathcal{V}_r$ is an orthogonal sum.

By definition, each $\mathcal{V}_n$ is invariant by H. It remains to show that each $\mathcal{V}_n$ is also invariant by H^*, i.e., that $H^*(\mathcal{U}_{n,r}) \subset \mathcal{V}_n$ for all $r \geq n$.

Let $\xi \in \mathcal{U}_{n,r}$ for some n and r such that $0 \leq n \leq r$; we distinguish three cases.

Assume first that $r > n$. There exists $\eta \in \mathcal{U}_{n,n}$ such that $\xi = H^{r-n}\eta$. Then $H^*\xi = (H^*H)(H^{r-n-1}\eta) = d_{r-1}H^{r-n-1}\eta$ by (3) of Proposition 4.1, hence $H^*\xi \in \mathcal{U}_{n,r-1} \subset \mathcal{V}_n$.

Assume now that $r = n \geq 1$. Then $H^*\xi \in \ell^2(S_{n-1})$. We claim that $H^*\xi = 0$. Indeed, choose $\ell \in \{0, 1, \dots, n-1\}$ and $\zeta \in \mathcal{U}_{\ell,n-1}$. Then $H\zeta \in \mathcal{U}_{\ell,n}$ and $\xi \in \mathcal{U}_{n,n}$ are orthogonal (because $\ell < n$), so that $\langle H^*\xi \mid \zeta \rangle = \langle \xi \mid H\zeta \rangle = 0$; hence $H^*\xi$ is orthogonal to $\mathcal{U}_{\ell,n-1}$ for each $\ell \leq n-1$, i.e., $H^*\xi$ is orthogonal to $\ell^2(S_{n-1})$, i.e., $H^*\xi = 0$.

Assume finally that $r = n = 0$; then $H^*\xi = 0$. This shows that $H^*\xi \in \mathcal{V}_n$ in all cases. $\square$

The next proposition is now straightforward:

Proposition 4.3. *With the notation as above, we have*

(1) $\dim \ell^2(S_n) = |S_n| = \prod_{q=0}^{n-1} d_q$ *for all* $n \geq 0$*;*

(2) $\dim \mathcal{U}_{n,r} = \left(\prod_{q=0}^{n-2} d_q \right)(d_{n-1} - 1)$ *for all* $n \geq 2$ *and* $r \geq n$*; and* $\dim \mathcal{U}_{1,r} = d_0 - 1$ *for all* $r \geq 1$*; and* $\dim \mathcal{U}_{0,r} = 1$ *for all* $r \geq 0$*;*

(3) $\dim \mathcal{V}_n = \infty$ *for all* $n \geq 0$*.*

Let $n \geq 0$. Denote by $\ell^2(\mathbf{N}, \mathcal{U}_{n,n})$ the Hilbert space of sequences $(\xi_j)_{j \geq 0}$ of vectors in $\mathcal{U}_{n,n}$ such that $\sum\limits_{j=0}^{\infty} \|\xi_j\|^2 < \infty$. For all $j \geq 0$, by (3) of Proposition 4.1 and by definition of $\mathcal{U}_{n,n+j}$, the operator

$$\frac{1}{\sqrt{\prod_{q=n}^{n+j-1} d_q}} H^j : \mathcal{U}_{n,n} \to \mathcal{U}_{n,n+j}$$

is a surjective isometry.

Let $\xi \in \mathcal{V}_n$. For all $j \geq 0$, there exists $\xi_{n+j} \in \mathcal{U}_{n,n+j}$, and therefore $\chi_{n+j} \in \mathcal{U}_{n,n}$, such that

$$\xi = \left(\xi_{n+j}\right)_{j \geq 0} \text{ with } \xi_{n+j} = \frac{1}{\sqrt{\prod_{q=n}^{n+j-1} d_q}} H^j \chi_{n+j} \text{ for all } j \geq 0. \tag{4.6}$$

Note that $\|\xi_{n,j}\| = \|\chi_{n,j}\|$. We have shown:

Proposition 4.4. *Let the notation be as above. For any $n \geq 0$, the operator*

$$W_n : \mathcal{V}_n \to \ell^2(\mathbf{N},\mathcal{U}_{n,n}) \quad \textit{defined by} \quad W_n\big((\xi_{n+j})_{j\geq 0}\big) = (\chi_{n+j})_{j\geq 0}$$

is a surjective isometry, and $W_n^*\big((\chi_{n+j})_{j\geq 0}\big) = (\xi_{n+j})_{j\geq 0}$.

Let $n \geq 0$. We define the **weighted shift** $S_{\mathcal{U},n}$ on $\ell^2(\mathbf{N},\mathcal{U}_{n,n})$ by

$$\begin{aligned} S_{\mathcal{U},n}(\chi_n, \chi_{n+1}, \chi_{n+2}, \chi_{n+3}, \ldots) &= \\ &= (0, \sqrt{d_n}\chi_n, \sqrt{d_{n+1}}\chi_{n+1}, \sqrt{d_{n+2}}\chi_{n+2}, \ldots). \end{aligned}$$

The operator $S_{\mathcal{U},n}$ is the direct sum of $\dim(\mathcal{U}_{n,n})$ copies of the standard weighted shift S_n defined on the usual sequence space $\ell^2(\mathbf{N})$ by

$$S_n(\lambda_0, \lambda_1, \lambda_2, \lambda_3, \ldots) = (0, \sqrt{d_n}\lambda_0, \sqrt{d_{n+1}}\lambda_1, \sqrt{d_{n+2}}\lambda_2, \ldots). \tag{4.7}$$

Proposition 4.5. *With the notation as above, we have for all $n \geq 0$*

$$W_n H W_n^* = S_{\mathcal{U},n} \quad \textit{and} \quad W_n H^* W_n^* = S_{\mathcal{U},n}^*.$$

Proof. Let $\big(\chi_{n+j}\big)_{j\geq 0} \in \ell^2(\mathbf{N},\mathcal{U}_{n,n})$. The vector $W_n^*\big((\chi_{n+j})_{j\geq 0}\big)$ is the vector ξ of (4.6), so that

$$\begin{aligned} HW^*\big((\chi_{n+j})_{j\geq 0}\big) = H\big((\xi_{n+j})_{j\geq 0}\big) &= H\left(\left(\frac{\sqrt{d_{n+j}}}{\sqrt{\prod_{q=n}^{n+j} d_q}}\, H^j\chi_{n+j}\right)_{j\geq 0}\right) \\ &= (0, \eta_1, \eta_2, \ldots, \eta_k, \ldots) \end{aligned}$$

with

$$\eta_k = \sqrt{d_{n+k-1}}\frac{1}{\sqrt{\prod_{q=n}^{n+k-1} d_q}} H^{k-1}\chi_{n+k-1} = \sqrt{d_{n+k-1}}\xi_{n+k-1}$$

for all $k \geq 1$. Therefore

$$\begin{aligned} W_nHW_n^*\big((\chi_{n+j})_{j\geq 0}\big) &= W_n(0, \eta_1, \eta_2, \ldots, \eta_k, \ldots) \\ &= W_n(0, \sqrt{d_n}\xi_n, \sqrt{d_{n+1}}\xi_{n+1}, \sqrt{d_{n+2}}\xi_{n+2}, \ldots) \\ &= S_{\mathcal{U},n}(\chi_n, \chi_{n+1}, \chi_{n+2}, \chi_{n+3}, \ldots), \end{aligned}$$

hence $W_nHW_n^* = S_{\mathcal{U},n}$. Finally $W_nH^*W_n^* = (W_nHW_n^*)^* = S_{\mathcal{U},n}^*$. $\square$

For $n \geq 0$, we denote by

$$\delta_{*,n} \quad \text{the sequence} \quad (\sqrt{d_n}, \sqrt{d_{n+1}}, \dots, \sqrt{d_{n+j}}, \dots)$$

and we consider the infinite Jacobi matrix

$$J_{\delta_{*,n}} = \begin{pmatrix} 0 & \sqrt{d_n} & 0 & 0 & \cdots \\ \sqrt{d_n} & 0 & \sqrt{d_{n+1}} & 0 & \cdots \\ 0 & \sqrt{d_{n+1}} & 0 & \sqrt{d_{n+2}} & \cdots \\ 0 & 0 & \sqrt{d_{n+2}} & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \ddots \end{pmatrix}. \tag{4.8}$$

If we identify the operators S_n of (4.7) and S_n^* with their matrices with respect to the standard basis $(\delta_j)_{j \in \mathbf{N}}$ of $\ell^2(\mathbf{N})$, we have

$$J_{\delta_{*,n}} = S_n + S_n^*.$$

Here is a reformulation of part of the previous propositions.

Proposition 4.6. *Let $T = (V, E)$ be an infinite spherically symmetric tree with root v_0 and with sequence of branching degrees $(d_r)_{r \geq 0}$ such that $d_r \geq 2$ for all $r \geq 0$ and $\sup_r d_r < \infty$.*

The adjacency operator A_T of T is unitarily equivalent to a direct sum $\bigoplus_{n=0}^{\infty} m_n J_{\delta_{*,n}}$, where the multiplicities m_n are given by

$$\begin{aligned} m_n &= \dim \mathcal{U}_{n,n} = \Big(\prod_{q=0}^{n-2} d_q \Big) (d_{n-1} - 1) \quad \textit{for } n \geq 2 \\ m_1 &= \dim \mathcal{U}_{1,1} = d_0 - 1 \\ m_0 &= \dim \mathcal{U}_{0,0} = 1 \end{aligned}$$

and where the $J_{\delta_{*,n}}$'s are the Jacobi matrices of (4.8).

As a first particular case, consider an integer $d \geq 2$, the constant sequence $(d, d, d, \dots)$, and the **regular rooted tree** $T_d^{\mathrm{root}} = (V, E)$ **of branching degree** d; the relevant Jacobi matrix is the multiple $\sqrt{d}J$ of the free Jacobi matrix J of Section 3. By Proposition 3.1 for the marked spectrum of J and by Proposition 2.2, we obtain the marked spectrum of $\sqrt{d}J$:

(1) The norm of $\sqrt{d}J$ is $2\sqrt{d}$.
(2) The spectrum of $\sqrt{d}J$ is $[-2\sqrt{d}, 2\sqrt{d}]$.

(3) The vertex spectral measure of $\sqrt{d}J$ at δ_0 is

$$d\mu(x) = \frac{1}{2\pi d}\sqrt{4d - x^2}\, dx$$

for $x \in [-2\sqrt{d}, 2\sqrt{d}]$ (where dx stands for the Lebesgue measure).

(4) The vector δ_0 is cyclic for $\sqrt{d}J$ and $\sqrt{d}J$ is multiplicity-free.

By Proposition 4.6, the adjacency operator of T_d^{root} is the direct sum of infinitely many copies of $\sqrt{d}J$, and we obtain the following:

Proposition 4.7. *Let $d \geq 2$ and let $T_d^{\mathrm{root}} = (V, E)$ be the regular rooted tree of branching degree d. Let A_d^{root} denote the adjacency operator of T_d^{root}.*

(1) The norm of A_d^{root} is $2\sqrt{d}$.

(2) The spectrum of A_d^{root} is $[-2\sqrt{d}, 2\sqrt{d}]$.

(3) The vertex spectral measure at 0 is $d\mu(x) = \frac{1}{2\pi d}\sqrt{4d - x^2}\, dx$ for x in $\Sigma(A_d^{\mathrm{root}})$; it is a scalar-valued spectral measure for A_d^{root}.

(4) A_d^{root} has uniform infinite multiplicity.

Recall from the introduction that two graphs G, G' of bounded degree are **cospectral** if their adjacency operators have equal spectra, equivalent scalar-valued spectral measures, and spectral multiplicity functions which are equal almost everywhere.

Corollary 4.8. *For any integer $d \geq 2$, the lattice graph L_d and the regular rooted tree $T_{d^2}^{\mathrm{root}}$ are cospectral.*

Proof. This is an immediate consequence of Corollary 2.11 and of Propositions 3.4 and 4.7.

Note that the measure μ_d of Proposition 3.4 for L_d and the measure μ of Proposition 4.7 for $T_{d^2}^{\mathrm{root}}$ are not equal, but they are both equivalent to the Lebesgue measure on $[-d^2, d^2]$, and this is enough to apply Corollary 2.11. □

Example 4.9. Consider an integer $p \geq 2$ and a sequence of integers $d_* = (d_r)_{r \geq 0}$ such that $d_r \geq 2$ and $d_{p+r} = d_r$ for all $r \geq 0$. For $s \in \{0, 1, \ldots, p-1\}$, let T_s be the spherically symmetric rooted tree with sequence of branching degrees $d_{*,s} = (d_s, d_{s+1}, d_{s+2}, \ldots)$. When p is the smallest period of the sequence d_*, the trees $T_0, \ldots, T_{p-1}$ are pairwise non-isomorphic.

It follows from Proposition 4.6 that the p trees $T_0, \ldots, T_{p-1}$ are cospectral.

5. Regular trees

For any positive real number a, set

$$J_a = \begin{pmatrix} 0 & a & 0 & 0 & \cdots \\ a & 0 & 1 & 0 & \cdots \\ 0 & 1 & 0 & 1 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{5.1}$$

Note that J_1 is the free Jacobi matrix. Matrices J_{***} here and below are identified with the corresponding operators on the Hilbert space $\ell^2(\mathbf{N})$, with its canonical orthonormal basis.

Let d be an integer, $d \geq 3$; let $T_d = (V, E)$ be the **regular tree of degree** d. Choose one vertex $v_0 \in V$ to be the root of T_d. Then T_d is the spherically symmetric rooted tree with sequence of branching degrees $(d, d-1, d-1, d-1, \ldots)$ of which all terms are $d-1$ but the initial one which is d. The matrix $J_{\delta_{*,0}}$ of Proposition 4.6 is

$$\begin{aligned} J_{\sqrt{d},\sqrt{d-1}^\infty} &= \begin{pmatrix} 0 & \sqrt{d} & 0 & 0 & 0 & \cdots \\ \sqrt{d} & 0 & \sqrt{d-1} & 0 & 0 & \cdots \\ 0 & \sqrt{d-1} & 0 & \sqrt{d-1} & 0 & \cdots \\ 0 & 0 & \sqrt{d-1} & 0 & \sqrt{d-1} & \cdots \\ 0 & 0 & 0 & \sqrt{d-1} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \\ &= J_{\sqrt{d},\sqrt{d-1}^\infty} = \sqrt{d-1} J_a \quad \text{for} \quad a = \frac{\sqrt{d}}{\sqrt{d-1}}. \end{aligned} \tag{5.2}$$

Note that $1 \leq a \leq \sqrt{3/2}$, since $d \geq 3$. The other matrices $J_{\delta_{*n}}$ of Proposition 4.6, for $n \geq 1$, are all equal to $\sqrt{d-1} J_1$. For Proposition 5.3 below, we will need to know properties of the scalar-valued spectral measures defined by these matrices. This is straightforward and very standard for J_1, as already shown in Proposition 3.1, but we did not find a simple ad hoc argument for $J_{\sqrt{d}/\sqrt{d-1}}$, and we rather quote the following

Proposition 5.1. *Consider a real number a such that $0 < a \leq \sqrt{2}$ and the matrix J_a of (5.1), viewed as a self-adjoint operator acting on $\ell^2(\mathbf{N})$, with its canonical orthonormal basis $(\delta_n)_{n \geq 0}$.*

(1) The norm of J_a is 2.

(2) The spectrum of J_a is the interval $[-2, 2]$.

(3) The vector δ_0 is cyclic for the operator J_a.

(4) The vertex spectral measure of J_a is equivalent to the Lebesgue measure on $[-2, 2]$, and it is a scalar-valued spectral measure.

Proof for (1) to (3) and reference for (4). As in the proof of Proposition 3.7, we have $\Sigma_{\mathrm{ess}}(X + K) = \Sigma_{\mathrm{ess}}(X)$, so that $\Sigma_{\mathrm{ess}}(J_a) = [-2, 2]$; this holds for all $a \geq 0$. The eigenvalue equation $J_a \xi = \lambda \xi$ for $\xi = (\xi_n)_{n \geq 0} \in \ell^2(\mathbf{N})$ gives rise to a difference equation of second order with constant coefficients, and a routine computation shows that this equation has no solution in $\ell^2(\mathbf{N})$ when $0 < a^2 \leq 2$ (details for example in [14, Lemma 4.6]); it follows that $\Sigma(J_a) = \Sigma_{\mathrm{ess}}(J_a) = [-2, 2]$. This completes the proof of Claims (1) and (2). It is straightforward to check Claim (3).

Claim (4) is more delicate to prove, and we quote here a particular case of the result of [35] (particular because we impose diagonal coefficient $b_n = 0$ here, and because we exclude eigenvalues):

Let $(a_n)_{n \geq 0}$ be a sequence of positive real numbers such that

$$\lim_{n \to \infty} a_n = 1 \quad \textit{and} \quad \sum_{n=1}^{\infty} |a_{n+1} - a_n| < \infty.$$

Let μ be the measure associated to the sequence of orthonormal polynomials $(P_n)_{n \geq 0}$ defined by the recurrence formula

$$x P_n(x) = a_n P_{n+1}(x) + a_{n-1} P_{n-1}(x) \quad \textit{for} \quad n \geq 0$$

(with $a_{-1} = 0$, $P_{-1} = 0$, P_0 constant) and the normalisation $P_n(x) = \gamma_n x^n +$ lower order terms, $\gamma_n > 0$. Consider the operator J defined on the Hilbert space $\ell^2(\mathbf{N})$ with its canonical basis $(\delta_n)_{n \in \mathbf{N}}$ by the Jacobi matrix

$$\begin{pmatrix} 0 & a_0 & 0 & 0 & \cdots \\ a_0 & 0 & a_1 & 0 & \cdots \\ 0 & a_1 & 0 & a_2 & \cdots \\ 0 & 0 & a_2 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{5.3}$$

and assume that this operator does not have any eigenvalue. Let μ be the local spectral measure of J at δ_0, defined by $\int_{\Sigma(J)} f(x) d\mu(x) = \langle f(J) \delta_0 \mid \delta_0 \rangle$ for any function f continuous on the spectrum $\Sigma(J)$ of J.

Then $\Sigma(J) = [-2, 2]$ and $\mu = \rho\lambda$ for a function ρ which is continuous positive on $]-2, 2[$ and zero outside $[-2, 2]$ (where λ is the Lebesgue measure). In particular, μ is equivalent to λ on $[-2, 2]$.

Claim (4) follows. Rather than relying on [35], we could alternatively quote [48, Theorem III.11], which provides an explicit formula for the local spectral measure of J_a at the vector δ_0, or quote results related to that of [35], such as [21, Theorem 3] or [49, Theorem 8.18]. $\square$

By Corollary 2.11, we have the following consequence of Proposition 5.1, surprising for us:

Corollary 5.2. *For any $a \in]0, \sqrt{2}]$, the matrix J_a is unitarily equivalent to J_1.*

In contrast, for $a > \sqrt{2}$, the operator J_a has two simple eigenvalues $\pm\frac{a^2}{\sqrt{a^2-1}}$, and therefore is not unitarily equivalent to J_1. Let $d \geq 3$ and $a = \sqrt{d}/\sqrt{d-1}$; note that $a < \sqrt{2}$; since $J_{\sqrt{d}, \sqrt{d-1}^\infty} = \sqrt{d-1}J_a$, see (5.2), Proposition 5.1 implies: (1) The norm of $J_{\sqrt{d}, \sqrt{d-1}^\infty}$ is $2\sqrt{d-1}$. (2) The spectrum of $J_{\sqrt{d}, \sqrt{d-1}^\infty}$ is the interval $[-2\sqrt{d-1}, 2\sqrt{d-1}]$. (3) The vector δ_0 is cyclic for the operator $J_{\sqrt{d}, \sqrt{d-1}^\infty}$. (4) The vertex spectral measure of $J_{\sqrt{d}, \sqrt{d-1}^\infty}$ is equivalent to the Lebesgue measure on $[-2\sqrt{d-1}, 2\sqrt{d-1}]$; it is a scalar-valued spectral measure.

By Proposition 4.6, the adjacency operator A_d of T_d is the direct sum of one copy of $J_{\sqrt{d}, \sqrt{d-1}^\infty}$ and infinitely many copies of $\sqrt{d-1}J_1$, hence we obtain the following:

Proposition 5.3. *Let $d \geq 3$ and let $T_d = (V, E)$ be the regular tree of degree d. Let A_{T_d} be the adjacency operator T_d.*

(1) The norm of A_{T_d} is $2\sqrt{d-1}$.

(2) The spectrum of A_{T_d} is $[-2\sqrt{d-1}, 2\sqrt{d-1}]$.

(3) The vertex spectral measure at any vertex is equivalent to the Lebesgue measure on the spectrum of A_{T_d}; it is a scalar-valued spectral measure.

(4) A_{T_d} has uniform infinite multiplicity.

Corollary 5.4. *For any integer $d \geq 2$, the lattice graph L_d and the regular tree T_{d^2+1} are cospectral.*

Remark: the vertex spectral measures of T_d and T_d^{root} which appear here are equivalent to the Lebesgue measure on the appropriate interval. This is in sharp contrast with large families of spherically symmetric rooted trees, for which vertex spectral measures don't have absolutely continuous spectrum [10], [19].

6. An uncountable family of cospectral graphs

There are in [29] examples of uncountable families of pairwise non-isomorphic cospectral Schreier graphs. They are defined in terms of certain groups of automorphisms of infinite regular rooted trees called spinal groups, and the actions of these groups on the boundaries of the trees. We restrict here to the particular case of the Fabrykowski–Gupta group, which is the simplest of the spinal groups acting on rooted trees of branching degree ≥ 3, and we describe shortly one of these families as follows.

Consider the regular rooted tree $T = T_3^{\mathrm{root}}$ of branching degree 3, its boundary ∂T which is the Cantor space $\{0, 1, 2\}^{\mathbf{N}}$ of infinite sequences of $0, 1$ and 2 's, and the Bernoulli measure ν on ∂T which is a probability measure invariant by the automorphism group of T. The Fabrykowski–Gupta group Γ is the group of automorphisms of T generated by the symmetric set $S = \{a, a^{-1}, b, b^{-1}\}$, where a is the cyclic permutation of the three main branches of T just below the root, and where b is the automorphism of T usually defined recursively by $b = (a, 1, b)$, see for example [39, Subsection 8.2].

For $\xi \in \partial T$, let $\mathrm{Stab}_\xi(\Gamma)$ denote the stabilizer $\{g \in \Gamma : g\xi = \xi\}$. Let $\mathrm{Sc}_\xi = \mathrm{Sc}(\Gamma, \mathrm{Stab}_\xi(\Gamma), S)$ be the **Schreier graph** of the indicated triple, with vertex set the orbit $\Gamma\xi$ (i.e., the coset space $\Gamma/\mathrm{Stab}_\xi(\Gamma)$) and edges the pairs of the form $\{g\xi, sg\xi\}$ with $g \in \Gamma$ and $s \in S$. This graph may have loops (pairs with $g\xi = sg\xi$) and multiple edges (pairs $\{g\xi, sg\xi\}$ and $\{g\xi, s'g\xi\}$ with $s' \neq s$ and $sg\xi = s'g\xi$), but its adjacency operator A_ξ acting on $\ell^2(G\xi)$ can be naturally defined.

It is known that there exists a measurable subset $\mathcal{W}$ of ∂T of full measure, i.e., $\nu(\mathcal{W}) = 1$, such that for $\xi \in \mathcal{W}$ the adjacency operator A_ξ has the following properties:

- The closure of the set of eigenvalues of A_ξ, which is the spectrum of A_ξ, is the union of a Cantor subset of $\mathbf{R}$ of Lebesgue measure zero and of countably many points accumulating on this Cantor set; see [5, Theorem 3.6 and Corollary 4.13] and [29, Theorem 1.5].

– A_ξ has a pure point spectrum, more precisely there exists an orthonormal basis of $\ell^2(\Gamma\xi)$ of eigenvectors of A_ξ, moreover each eigenvector in this basis is a function of finite support on $\Gamma\xi$ [29, Theorem 1.8].

– The set of these eigenvalues and their multiplicities, which are all infinite, do not depend on ξ [29, Section 5].

Moreover, for $\xi \in \mathcal{W}$, the set of $\xi' \in \mathcal{W}$ for which $\mathrm{Sc}_{\xi'}$ is isomorphic to Sc_ξ has ν-measure 0 [39, Corollary 7.13].

In particular, there are uncountably many graphs Sc_ξ which are cospectral and pairwise non-isomorphic.

References

[1] Abrahamse, M.B.: Multiplication operators. Hilbert space operators (Proc. Conf., Calif. State Univ., Long Beach, Calif., 1977). Lectures Notes in Math., vol. **693**, pp. 17–36. Springer, Berlin (1978).

[2] Abrahamse, M.B., Kriete, T.L.: The spectral multiplicity of a multiplication operator. Indiana U. Math. Jour. **22**(9), 845–857 (1973).

[3] Allard, C., Froese, R.: A Mourre estimate for a Schrödinger operator on a binary tree. Rev. Math. Phys. **12**(12), 1655–1667 (2000).

[4] Arveson, W.: An invitation to C*-algebras. Graduate Texts in Mathematics, **39**. Springer (1976).

[5] Bartholdi, L., Grigorchuk, R.: On the spectrum of Hecke type operators related to some fractal groups. Proc. Steklov Inst. Math. **231**(4), 1–41 (2000).

[6] Bartholdi, L., Woess, W.: Spectral computations on lamplighter groups and Diestel–Leader graphs. J. Fourier Anal. Appl. **11**(2), 175–202 (2005). https://doi.org/10.1007/s00041-005-3079-0

[7] Biggs, N.: Algebraic graph theory. Second edition. Cambridge University Press (1993).

[8] Bogachev, V.I., Smolyanov, O.G.: Real and functional analysis. Moscow Lectures, vol. **4**. Springer (2020).

[9] Breuer, J.: Singular continuous spectrum for the Laplacian on certain sparse trees. Comm. Math. Phys. **269**(3), 851–857 (2007). https://doi.org/10.1007/s00220-006-0121-2

[10] Breuer, J., Frank, R.L.: Singular spectrum for radial trees. Rev. Math. Phys. **21**(7), 929–945 (2009). https://doi.org/10.1142/S0129055X09003773

[11] Brouwer, A.E., Haemers, W.H.: Spectra of graphs. Universitext. Springer, New York (2012).

[12] Brouwer, A.E. Neumaier, A.: The graphs with spectral radius between 2 and $\sqrt{2+\sqrt{5}}$. Linear Algebra Appl. **114–115**, 273–276 (1989). https://doi.org/10.1016/0024-3795(89)90466-7

[13] Brown, A., Pearcy, C.: Spectra of tensor products of operators. Proc. Amer. Math. Soc. **17**, 162–166 (1966).

[14] Bruchez, C., de la Harpe, P., Nagnibeda, T.: Spectral measures and dominant vertices in graphs of bounded degree. Journal of Operator Theory, to appear. Preprint at https://doi.org/10.48550/arXiv.2205.12819 (2022).

[15] Conway, J.B.: A course in functional analysis. Second Edition. Graduate Texts in Mathematics, vol. **96**. Springer (2007).

[16] Collatz, L., Sinogowitz, U.: Spektren endlicher Grafen. Abh. Math. Sem. Univ. Hamburg. **21**, 63–77 (1957). https://doi.org/10.1007/BF02941924

[17] Cvetković, D.M., Doob, M., Sachs, H.: Spectra of graphs. Theory and applications. Academic Press (1980).

[18] Cvetković, D.M., Gutman, I.: On the spectral structure of graphs having the maximal eigenvalue not greater than two. Publ. Inst. Math. (Beograd) (N.S.). **18**(32), 39–45 (1975).

[19] Damanik, D., Sukhtaiev, S.: Anderson localization for radial tree graphs with random branching numbers. J. Funct. Anal. **227**(2), 418–433 (2019). https://doi.org/10.1016/j.jfa.2018.11.007

[20] Dixmier, J.: Les algèbres d'opérateurs dans l'espace hilbertien (algèbres de von Neumann), deuxième édition, revue et augmentée. Gauthier-Villars (1969).

[21] Dombrowski, J., Nevai, P.: Orthogonal polynomials, measures and recurrence relations. SIAM J. Math. Anal. **17**(3), 752–759 (1986). https://doi.org/10.1137/0517054

[22] Douglas, R.G.: Banach algebra techniques in operator theory. Academic Press. Springer (1972).

[23] Dunford, N., Schwartz, J.T.: Linear operators. Part II: Spectral theory. Self adjoint operators in Hilbert space. Interscience (1963).

[24] Dudko, A., Grigorchuk, R.: On the question "Can one hear the shape of a group?" and a Hulanicki type problem for graphs. Israel J. Math. **237**(1), 53–74 (2020). https://doi.org/10.48550/arXiv.1809.04008

[25] Godsil, C.D.: Controllable subsets in graphs. Ann. Comb. **16**(4), 733–744 (2012). https://doi.org/10.1007/s00026-012-0156-3

[26] Godsil, C.D., Mohar, B.: Walk generating functions and spectral measures of infinite graphs. Linear Algebra Appl. **107**, 191–206 (1988). https://doi.org/10.1016/0024-3795(88)90245-5

[27] Godsil, C.D., Royle, C.D.: Algebraic graph theory. Graduate Texts in Mathematics, vol. **207**. Springer (2001).

[28] Gordon, C.S., Webb, D., Wolpert, S.: Isospectral plane domains and surfaces via Riemannian orbifolds. Invent. Math. **110**(1), 1–22 (1992). https://doi.org/10.1007/BF01231320 (See also: One cannot hear the shape of a drum. Bull. Amer. Math. Soc. (N.S.). **27**(1), 134–136 (1992). https://doi.org/10.1090/S0273-0979-1992-00289-6

[29] Grigorchuk, R., Nagnibeda, T., Pérez, A.: On spectra and spectral measures of Schreier and Cayley graphs. Inst. Math. Res. Not. IMRN. **2022**(15), 11957–12002 (2022). https://doi.org/10.1093/imrn/rnab234

[30] Grigorchuk, R., Zuk, A.: The lamplighter group as a group generated by a 2-state automaton, and its spectrum. Geom Dedicata. **87**, 209–244 (2001). https://doi.org/10.1023/A:1012061801279

[31] Haemers, W.H., Spence, E.: Enumeration of cospectral graphs. European J. Combin. **25**(2), 199–211 (2004). https://doi.org/10.1016/S0195-6698(03)00100-8

[32] Kac, M.: Can one hear the shape of a drum? Amer. Math. Monthly. **73**(4), part II, 1–23 (1966). https://doi.org/10.2307/2313748

[33] Kesten, H.: Symmetric random walks on groups. Trans. Amer. Math. Soc. **92**(2), 336–354 (1959).

[34] Kriete, T.L.: An elementary approach to the multiplicity theory of multiplication operators. Rocky Mountain J. Math. **16**(1), 23–32 (1986).

[35] Máté, A., Nevai, P.: Orthogonal polynomials and absolutely continuous measures. Approximation Theory. IV (College Station, Tex., 1983). Academic Press, 611–617 (1983).

[36] Mohar, B.: The spectrum of an infinite graph. Linear Algebra Appl. **48**, 245–256 (1982).

[37] Mohar, B., Woess, W.: A survey on spectra of infinite graphs. Bull. London Math. Soc. **21**(3), 209–234 (1989).

[38] Mowshowitz, A.: The group of a graph whose adjacency matrix has all distinct eigenvalues. In "Proof Techniques in Graph Theory" (Proc. Second Ann Arbor Graph Theory Conf., Ann Arbor, Mich., 1968), Academic Press, 109–110 (1969).

[39] Nagnibeda, T., Pérez, A.: Schreier graphs of spinal groups. Internat. J. Algebra Comput. **31**(8), 1191–1216 (2021). https://doi.org/10.1142/S0218196721400099

[40] O'Rourke, S., Touri, B.: On a conjecture of Godsil concerning controllable random graphs. SIAM J. Control Option. **54**(6), 3347–3378 (2016). https://doi.org/10.1137/15M1049622

[41] Romanov, R.V., Rudin, G.E.: Scattering on the Bruhat–Tits tree. I. Phys. Lett. A. **198**(2), 113–118 (1995). https://doi.org/10.1016/0375-9601(94)00997-4

[42] Rudin, W.: Real and complex analysis. McGraw-Hill Book Co. (1966).

[43] Schechter, M.: On the spectra of operators on tensor products. J. Functional Analysis. **4**, 95–99 (1969). https://doi.org/10.1016/0022-1236(69)90024-X

[44] Simon, B.: Operator theory. A comprehensive course in analysis, Part 4. Amer. Math. Soc. (2015).

[45] Solomyak, M.: On the spectrum of the Laplacian on regular metric trees. Waves Random Media. **14**(1), S155–S171 (2004). https://doi.org/10.1088/0959-7174/14/1/017

[46] Weyl, H.: Über die asymptotische Verteilung der Eigenwerte. Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse, vol. **1911**, pp. 110–117 (1911).

[47] Woo, R., Neumaier, A.: On graphs whose spectral radius is bounded by $\frac{3}{2}\sqrt{5}$. Graphs and Combinatorics. **23**, 713–726 (2007). https://doi.org/10.1007/s00373-007-0745-9

[48] Yafaev, D.R.: A point interaction for the discrete Schrödinger operator and generalized Chebyshev polynomials. J. Math. Phys. **58**(6), 063511 (2017). https://doi.org/10.1063/1.4986414

[49] Yafaev, D.R.: Spectral analysis of Jacobi operators and asymptotic behavior of orthogonal polynomials. Bulletin of Mathematical Sciences. **12**(3), 2250002 (2022). https://doi.org/10.1142/S1664360722500023

CONTACT INFORMATION

P. de la Harpe Section de mathématiques, Université de Genève, Uni Dufour, 24 rue du Général Dufour, Case postale 64, 1211 Genève 4, Suisse.
E-Mail: pierre.delaharpe@unige.ch

Received by the editors: 23.02.2024.