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Quasi-idempotents in certain transformation semigroups

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Abstract. Let P_n and T_n be the partial transformations semigroup and the (full) transformations semigroup on the set $X_n = \{1, \ldots, n\}$, respectively. In this paper, we first state the orbit structure of quasi-idempotents (non-idempotent element whose square is an idempotent) in P_n. Then, for $2 \leq r \leq n-1$, we find the quasi-idempotent ranks of the subsemigroup $PK(n,r) = \{\alpha \in P_n : \mathrm{h}(\alpha) \leq r\}$ of P_n, and the subsemigroup $K(n,r) = \{\alpha \in T_n : \mathrm{h}(\alpha) \leq r\}$ of T_n, where $\mathrm{h}(\alpha)$ denotes the cardinality of the image set of α.

Introduction

Let P_n and T_n be the partial transformations semigroup and the (full) transformations semigroup on the set $X_n = \{1, \ldots, n\}$, respectively. An element $\alpha \in P_n$ is called an *idempotent* if $\alpha^2 = \alpha$, and, it is well-known that, $\alpha \in P_n$ is an idempotent if and only if $x\alpha = x$ for each $x \in \mathrm{im}(\alpha)$. Now we borrow the concept of quasi-idempotent, as mentioned in [6], introduced by Garba and Imam. In like manner, an element $\alpha \in P_n$ is called a *quasi-idempotent* if

$$\alpha \neq \alpha^2 = \alpha^4,$$

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that is, α is a non-idempotent element whose square is an idempotent. We denote the set of all idempotents (all quasi-idempotents) in any subset U of any semigroup by $E(U)$ (by $Q(U)$).

For any $\alpha \in P_n$, the digraph Γ_α is defined by

$$\begin{aligned} V(\Gamma_\alpha) &= X_n \text{ and} \\ \overrightarrow{E}(\Gamma_\alpha) &= \{(u,v) \in X_n \times X_n : u \in \operatorname{dom}(\alpha),\, u\alpha = v\}. \end{aligned}$$

As emphasized in [4] that the digraph Γ_α decomposes into a disjoint union of connected components (connected subdigraphs), and the connected components of Γ_α are called *orbits* of α. Clearly, the orbit structure of α provides valuable information about the structure of α. Hence, first we investigate the orbit structure of any quasi-idempotent in P_n, and, as its special case, the orbit structure of any quasi-idempotent in T_n.

Let S be a semigroup. For any $\emptyset \neq A \subseteq S$, the smallest subsemigroup of S containing A is called the subsemigroup generated by A and denoted by $\langle A \rangle$. It is easy to see that $\langle A \rangle$ is the set of all finite products of elements of A. If $S = \langle A \rangle$, then A is called a generating set of S. Moreover, if there exists a finite subset $\emptyset \neq A \subseteq S$ such that $S = \langle A \rangle$, then S is called a finitely generated semigroup. In this case, there exists a unique positive integer defined as

$$\operatorname{rank}(S) = \min\{\, |A| : \langle A \rangle = S\},$$

and this integer is called the rank of S. Recently, some important results obtained by examining the similar terms: the quasi-idempotent generating set of a semigroup S, defined as a generating set of S consists entirely of quasi-idempotents, and the quasi-idempotent rank of S, defined as

$$\operatorname{qrank}(S) = \min\{\, |A| : \langle A \rangle = S,\, A \subseteq Q(S)\},$$

for various transformation semigroups (see, for examples, [2, 3, 8]).

Now, for any $\alpha \in P_n$, let $\operatorname{im}(\alpha)$ denotes the image set of α, and let $\operatorname{h}(\alpha)$ denotes the cardinality of $\operatorname{im}(\alpha)$, called the height of α, say $\operatorname{h}(\alpha) = |\operatorname{im}(\alpha)|$. Also, for $1 \leq r \leq n-1$, let

$$\begin{aligned} PK(n,r) &= \{\alpha \in P_n : \operatorname{h}(\alpha) \leq r\}, \\ K(n,r) &= \{\alpha \in T_n : \operatorname{h}(\alpha) \leq r\}. \end{aligned}$$

For $2 \leq r \leq n-1$, Howie and McFadden proved in [7] that the rank of the subsemigroup $K(n,r)$ of T_n is $S(n,r)$, and Garba proved in [5] that the rank of the subsemigroup $PK(n,r)$ of P_n is $S(n+1,r+1)$ where $S(k,t)$ denotes the Stirling number of the second kind. Hence, as a second aim of this article, we examine the quasi-idempotent ranks of $PK(n,r)$ and $K(n,r)$ for $2 \leq r \leq n-1$.

1. The orbit structure of quasi-idempotents in P_n and T_n

A digraph (in other words, a directed graph) Π consists of a non-empty finite set $V(\Pi)$ of elements called *vertices*, and a finite list $\overrightarrow{E}(\Pi) \subseteq V(\Pi) \times V(\Pi)$ of ordered pairs of elements of $V(\Pi)$ called *directed edges (arcs).* A directed edge $(u,v) \in \overrightarrow{E}(\Pi)$ is represented by $u \to v$. Here, $V(\Pi)$ is called *the vertex set* and $\overrightarrow{E}(\Pi)$ is called *the directed edge list* of Π. For k-many ($k \geq 2$) vertices $u_1, \dots, u_k \in V(\Pi)$, if $(u_1,u_2),(u_2,u_3),\dots,(u_{k-1},u_k) \in \overrightarrow{E}(\Pi)$, then $u_1 \to \cdots \to u_k$ is called *a directed path from u_1 to u_k of size k*, if also $(u_k,u_1) \in \overrightarrow{E}(\Pi)$, then the closed directed path $u_1 \to \cdots \to u_k \to u_1$ is called *a (directed) cycle of size k* (k-*cycle*) and denoted by $(u_1 \dots u_k)$. Moreover, for two vertices $u,v \in V(\Pi)$, we say u is *connected to v* in Π if either $(u,v) \in \overrightarrow{E}(\Pi)$ or there exists a directed path from u to v. The directed path $u_1 \to \cdots \to u_k$ is called a *chain* of size k (k-*chain*) if there is no directed edge in $\overrightarrow{E}(\Pi) \setminus \{(u_1,u_2),\dots,(u_{k-1},u_k)\}$ which contains any of u_i's for $1 \leq i \leq k$, and denoted by $[u_1 \dots u_k]$. In particular, for any $u \in V(\Pi)$, if $(u,u) \in \overrightarrow{E}(\Pi)$ then the cycle $u \to u$ is called a 1-*cycle* (*a fixed point* or *a loop*) and denoted by (u); and if there is no vertex $v \in V(\Pi)$ such that $(u,v) \in \overrightarrow{E}(\Pi)$ or $(v,u) \in \overrightarrow{E}(\Pi)$, then the *single point* u is called 1-*chain* and denoted by $[u]$.

Recall that two digraphs are said to be *disjoint* if their vertex sets are disjoint, and that, a digraph is called *connected* if its underlying graph is a connected graph. Also, recall that an arbitrary connected digraph whose underlying graph contains no cycle is called a *(directed) tree.* For any finite tree Υ, there exists at least one $u \in V(\Upsilon)$ such that there is no vertex $v \in V(\Upsilon)$ such that $(u,v) \in \overrightarrow{E}(\Upsilon)$, and in this case u is called *a root.* We define the size of a tree as the maximum size of all directed paths in the tree if there exist some directed paths, otherwise the tree consists of a root and we define the size of a root as 1. (For unexplained terms for graphs and digraphs we refer [9], for example.)

Recall that the orbits of $\alpha \in P_n$ are of three types: cycles, cycles with some trees attached, and trees with one root; and that the orbits of $\alpha \in T_n$ are of two types: cycles and cycles with some trees attached.

For any $\alpha \in P_n$, let

$$C(\alpha) = \{x \in \mathrm{dom}\,(\alpha) : x\alpha^k = x \text{ for some integer } k \geq 1\},$$

and it is clear that $C(\alpha)$ is the union of all vertex sets of all cycles in the digraph Γ_α.

Proposition 1. *A non-idempotent map $\alpha \in P_n$ is a quasi-idempotent if and only if $x\alpha^2 = x$ for each $x \in C(\alpha)$, and either $x\alpha \notin \mathrm{dom}\,(\alpha)$ or $x\alpha^2 \in C(\alpha)$ for each $x \in \mathrm{dom}\,(\alpha) \setminus C(\alpha)$.*

Proof. ($\Rightarrow$) Let a non-idempotent map $\alpha \in P_n$ be a quasi-idempotent, and let $x \in C(\alpha)$. If $x\alpha = x$ then the result is clear. Now let $x\alpha \neq x$. Then, we have

$$x = x\alpha^3 = x\alpha^5 = x\alpha^7 = \cdots$$

or

$$x = x\alpha^2 = x\alpha^4 = x\alpha^6 = \cdots$$

since $\alpha^2 = \alpha^4$. However, when $x\alpha^3 = x$, we have $x\alpha^2 = x\alpha^5 = x\alpha^3 = x$, and so $x\alpha = x\alpha^3 = x$ which is a contradiction. Thus, we have $x\alpha^2 = x$.

Now, consider any element $x \in \mathrm{dom}\,(\alpha) \setminus C(\alpha)$ such that $x\alpha \in \mathrm{dom}\,(\alpha)$. Then, also $x\alpha^2 \in \mathrm{dom}\,(\alpha)$, otherwise $\alpha^2 \neq \alpha^4$ which is a contradiction. Moreover, since $(x\alpha^2)\alpha^2 = x\alpha^4 = x\alpha^2$, we have $x\alpha^2 \in C(\alpha)$, as required.

($\Leftarrow$) Let a non-idempotent map $\alpha \in P_n$ has the properties given in the expression of the proposition. Then, for any $x \in C(\alpha)$ clearly $x\alpha^4 = x\alpha^2$, and, for any $x \in \mathrm{dom}\,(\alpha) \setminus C(\alpha)$, we have $x\alpha \notin \mathrm{dom}\,(\alpha)$ or $x\alpha^2 \in C(\alpha)$, and also $x\alpha^4 = (x\alpha^2)\alpha^2 = x\alpha^2$. Thus, $\alpha^2 = \alpha^4$, as required. □

As a consequence of Proposition 1, we can immetiately have the following corollary.

Corollary 1. *A non-idempotent map $\alpha \in P_n$ is a quasi-idempotent if and only if each one of orbits of α must be one of the following forms:*

(*i*) 1-*cycle or* 2-*cycle (without any tree attached);*

(*ii*) 1-*cycle or* 2-*cycle with some trees attached of size* 2 *or* 3*;*

(*iii*) *tree of size* 1 *or* 2 *with one root.*

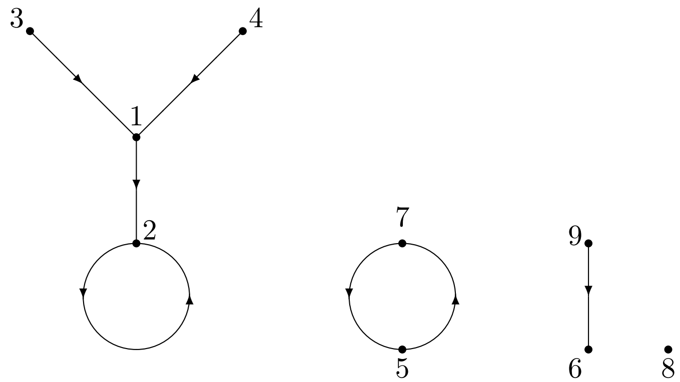
For any $\alpha \in P_n$, for ease of notations, we write $x\alpha = -$ for each $x \in X_n \setminus \mathrm{dom}(\alpha)$, and then α can be written as in the following tabular form:

$$\alpha = \begin{pmatrix} 1 & 2 & \cdots & n \\ 1\alpha & 2\alpha & \cdots & n\alpha \end{pmatrix} \in P_n.$$

With these notations, let

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 2 & 1 & 1 & 7 & - & 5 & - & 6 \end{pmatrix} \in Q(P_9).$$

Then the orbit structure of α is as follows:



Corollary 2. *A non-idempotent map* $\alpha \in T_n$ *is a quasi-idempotent if and only if* $x\alpha^2 = x$ *for each* $x \in C(\alpha)$*, and* $x\alpha^2 \in C(\alpha)$ *for each* $x \in X_n \setminus C(\alpha)$.

Corollary 3. *A non-idempotent map* $\alpha \in T_n$ *is a quasi-idempotent if and only if each one of orbits of* α *must be one of the following forms:*

(i) 1*-cycle or* 2*-cycle (without any tree attached);*

(ii) 1*-cycle or* 2*-cycle with some trees attached of size* 2 *or* 3*.*

2. Quasi-idempotent ranks of $PK(n,r)$ and $K(n,r)$

The *kernel* of $\alpha \in P_n$ is defined by

$$\ker(\alpha) = \{(x,y) \in X_n \times X_n : (x,y \in \mathrm{dom}(\alpha) \text{ and } x\alpha = y\alpha) \text{ or } (x,y \notin \mathrm{dom}(\alpha))\}.$$

It is well known that, for any $\alpha, \beta \in P_n$ (also in T_n), $\ker(\alpha) \subseteq \ker(\alpha\beta)$ and that $\ker(\alpha)$ is an equivalence relation on X_n where the equivalence classes of $\ker(\alpha)$ are all different pre-image sets of elements in $\mathrm{im}\,(\alpha)$ with $X_n \setminus \mathrm{dom}\,(\alpha)$. As in [1], we denote the set of all equivalence classes of $\ker(\alpha)$ except $X_n \setminus \mathrm{dom}\,(\alpha)$ by $\mathrm{kp}\,(\alpha)$, that is $\mathrm{kp}\,(\alpha) = \{y\alpha^{-1} : y \in \mathrm{im}\,(\alpha)\}$, and call *the kernel partition of* α. Notice that $\mathrm{kp}\,(\alpha)$ is a partition of $\mathrm{dom}\,(\alpha)$. Moreover, we denote the ordered pair $\big(\mathrm{kp}\,(\alpha), X_n \setminus \mathrm{dom}\,(\alpha)\big)$ by $\mathrm{ks}\,(\alpha)$, and call *the kernel structure of* α. For any $\alpha, \beta \in P_n$, notice that

$$\begin{aligned} \mathrm{ks}\,(\alpha) = \mathrm{ks}\,(\beta) \quad &\Leftrightarrow \quad \mathrm{kp}\,(\alpha) = \mathrm{kp}\,(\beta) \\ &\Leftrightarrow \quad \ker(\alpha) = \ker(\beta) \text{ and } \mathrm{dom}\,(\alpha) = \mathrm{dom}\,(\beta). \end{aligned}$$

For any $\alpha \in P_n$ with heigh k $(1 \le k \le n)$, it is easy to see that there exists a unique kernel partition $\{A_1, \ldots, A_k\}$ of α such that $\ker(\alpha) = \bigcup_{i=1}^{k+1} (A_i \times A_i)$ where $A_{k+1} = X_n \setminus \mathrm{dom}\,(\alpha)$ and a subset $\{a_1, \ldots, a_k\}$ of X_n such that $\mathrm{im}\,(\alpha) = \{a_1, \ldots, a_k\}$. Without loss of generality, let $A_i\alpha = a_i$ for each $1 \le i \le k$. Then α also can be written as in the following tabular form:

$$\alpha = \begin{pmatrix} A_1 & \cdots & A_k & A_{k+1} \\ a_1 & \cdots & a_k & - \end{pmatrix}.$$

Now, recall from [4, Theorem 4.5.1] that the characterization of the Green's equivalences on P_n (T_n) can be stated as follows:

(i) $\alpha\mathcal{R}\beta \Leftrightarrow \mathrm{ks}\,(\alpha) = \mathrm{ks}\,(\beta)$;

(ii) $\alpha\mathcal{L}\beta \Leftrightarrow \mathrm{im}\,(\alpha) = \mathrm{im}\,(\beta)$;

(iii) $\alpha\mathcal{D}\beta \Leftrightarrow \mathrm{h}\,(\alpha) = \mathrm{h}\,(\beta)$; and

(iv) $\alpha\mathcal{H}\beta \Leftrightarrow \mathrm{ks}\,(\alpha) = \mathrm{ks}\,(\beta)$ and $\mathrm{im}\,(\alpha) = \mathrm{im}\,(\beta)$

for any $\alpha, \beta \in P_n$ (T_n). We denote the $\mathcal{D}$-Green class of all elements in P_n (also in T_n) of height k by D_k for $0 \le k \le n$ (for $1 \le k \le n$).

Let U be a non-empty set and let $P = \{U_1, \ldots, U_k\}$ be a partition of U to k non-empty subset for $1 \le k \le |U|$. A *representative set* of a partition P, denoted by $R(P)$, is a set with the property that $|R(P)| = k$ and $|R(P) \cap U_i| = 1$ for each $1 \le i \le k$.

For any $1 \le k \le n$ and $A \subset X_n$ with cardinality at most $n-k$, let P_A be a partition of $X_n \setminus A$ to k subsets, and let I be a subset of X_n with cardinality k. Then we denote the $\mathcal{H}$-Green class of the partial transformations in P_n with kernel structure (P_A, A) and image set I by $H_I^{P_A}$. In particular, when we investigate T_n, since the kernel structure of any element in T_n has the form $(P_\emptyset, \emptyset)$, we can use the simplier notation P instead of $P_\emptyset$, and the notation H_I^P for the $\mathcal{H}$-Green class of the transformations in T_n with kernel partition P and image set I.

Proposition 2. *For $2 \leq r \leq n$ and $A \subset X_n$ with cardinality at most $n-r$, let P_A be a partition of $X_n \setminus A$ to r subsets, and let I be a subset of X_n with cardinality r. Then, we have $Q(H_I^{P_A}) \neq \emptyset$ in P_n.*

Proof. For $2 \leq r \leq n$, let $A \subset X_n$ with cardinality at most $n-r$, $P_A = \{A_1, \dots A_r\}$ be a partition of $X_n \setminus A$ to r subsets and let $I = \{a_1, \dots, a_r\}$ be a subsets of X_n with cardinality r.

First, suppose that $I \cap A = \emptyset$, that is $I \subseteq X_n \setminus A$. If I is a representative set of P_A, then, without loss of generality, we can suppose that $a_i \in A_i$ for each $1 \leq i \leq r$. Then, clearly

$$\alpha = \begin{pmatrix} A_1 & A_2 & A \\ a_2 & a_1 & - \end{pmatrix} \in Q(H_I^{P_A})$$

for $r = 2$, and

$$\alpha = \begin{pmatrix} A_1 & A_2 & A_3 & \cdots & A_r & A \\ a_2 & a_1 & a_3 & \cdots & a_r & - \end{pmatrix} \in Q(H_I^{P_A})$$

for $r \geq 3$. If I is not a representative set of P, then there exist distinct $k_1, \dots, k_l \in \{1, \dots, r\}$ such that $A_{k_i} \cap I \neq \emptyset$ for each $k_i \in \{k_1, \dots, k_l\}$ where $1 \leq l \leq r-1$, and $A_{t_j} \cap I = \emptyset$ for each $t_j \in \{t_1, \dots, t_s\} = \{1, \dots, r\} \setminus \{k_1, \dots, k_l\}$ where $1 \leq s = r - l$. We choose and fix unique $b_{k_i} \in A_{k_i} \cap I$ for each $1 \leq i \leq l$, and let $I \setminus \{b_{k_1}, \dots, b_{k_l}\} = \{b_{t_1}, \dots, b_{t_s}\}$. Then, we define $\alpha \in P_n$ as follows:

$$\alpha = \begin{pmatrix} A_{k_1} & \cdots & A_{k_l} & A_{t_1} & \cdots & A_{t_s} & A \\ b_{k_1} & \cdots & b_{k_l} & b_{t_1} & \cdots & b_{t_s} & - \end{pmatrix}.$$

It is clear that α is not an idempotent since $s \geq 1$, and that $\alpha \neq \alpha^2 = \alpha^4$. Hence, we have $\alpha \in Q(H_I^{P_A})$, as required.

Next, suppose that $I \cap A \neq \emptyset$. If $I \subseteq A$, then clearly

$$\alpha = \begin{pmatrix} A_1 & A_2 & \cdots & A_r & A \\ a_1 & a_2 & \cdots & a_r & - \end{pmatrix} \in Q(H_I^{P_A}),$$

otherwise, with the same notations given above,

$$\alpha = \begin{pmatrix} A_{k_1} & \cdots & A_{k_l} & A_{t_1} & \cdots & A_{t_s} & A \\ b_{k_1} & \cdots & b_{k_l} & b_{t_1} & \cdots & b_{t_s} & - \end{pmatrix} \in Q(H_I^{P_A}),$$

as required. □

Now we state a similar result for T_n whose proof is the special case of the proof of Proposition 2 for $A = \emptyset$.

Proposition 3. *For $2 \leq r \leq n$, let P be a partition of X_n to r subsets and let I be a subset of X_n with cardinality r. Then, we have $Q(H_I^P) \neq \emptyset$ in T_n.*

As a clear consequence of Proposition 2 (Proposition 3) that, each $\mathcal{L}$-class L and each $\mathcal{R}$-class R on P_n (T_n) contains at least one quasi-idempotent.

In [1], Ayık and Bugay first determined the digraph Γ_X for any $\emptyset \neq X \subseteq D_r$ in P_n (T_n) as follows:

$$\begin{aligned} V(\Gamma_X) &= X \text{ and} \\ \overrightarrow{E}(\Gamma_X) &= \{(\alpha, \beta) \in X \times X : \alpha\beta \in D_r\}, \end{aligned}$$

and show that, for any $\alpha, \beta \in D_r$, also $\alpha\beta \in D_r$ if and only if $\operatorname{im}(\alpha)$ is a representative set of $\operatorname{kp}(\beta)$. Then, they stated the following two main results that we will use in the proof of the main two results of this article.

Theorem 1 ([1, Theorem 4])**.** *Let X be a subset of the $\mathcal{D}$-Green class D_r of P_n for $2 \leq r \leq n-1$. Then X is a generating set of $PK(n, r)$ if and only if, for each idempotent ξ in D_r, there exist $\alpha, \beta \in X$ such that $\operatorname{ks}(\alpha) = \operatorname{ks}(\xi)$ $\operatorname{im}(\beta) = \operatorname{im}(\xi)$, and α is connected to β in the digraph Γ_X (or equivalently, X is a generating set of $PK(n, r)$ if and only if, for each $A \subset X_n$ with cardinality at most $n - r$, for each partition P_A of $X_n \setminus A$ to r subsets, and for each representative set $R(P_A)$ of P_A, there exist $\alpha, \beta \in X$ such that $\operatorname{ks}(\alpha) = (P_A, A)$, $\operatorname{im}(\beta) = R(P_A)$ and that α is connected to β in the digraph Γ_X).*

Theorem 2 ([1, Theorem 10])**.** *Let X be a subset of the $\mathcal{D}$-Green class D_r of T_n for $2 \leq r \leq n-1$. Then X is a generating set of $K(n, r)$ if and only if, for each idempotent ξ in D_r there exist $\alpha, \beta \in X$ such that $\ker(\alpha) = \ker(\xi)$, $\operatorname{im}(\beta) = \operatorname{im}(\xi)$ and α is connected to β in the digraph Γ_X (or equivalently, X is a generating set of $K(n, r)$ if and only if, for each partition $\mathcal{A}$ of X_n, and for each representative set $R(\mathcal{A})$ of $\mathcal{A}$, there exist $\alpha, \beta \in X$ such that $\operatorname{kp}(\alpha) = \mathcal{A}$, $\operatorname{im}(\beta) = R(\mathcal{A})$ and that α is connected to β in the digraph Γ_X).*

Theorem 3. *For $2 \leq r \leq n-1$, $\operatorname{qrank}(PK_{n,r}) = \operatorname{rank}(PK_{n,r}) = S(n+1, r+1)$.*

Proof. For $2 \leq r \leq n-1$, let $I_1, \ldots, I_m$ with $\binom{n}{r} = m$ be all subsets of X_n of cardinality r. Then, as shown in [1, Lemma 6], for each $1 \leq i \leq m$, there exists a subset $A_{i_{r+1}}$ of X_n and a partition $P_{A_{i_{r+1}}} = \{A_{i_1}, \ldots, A_{i_r}\}$

of $X_n \setminus A_{i_{r+1}}$ such that I_i is a representative set of $P_{A_{i_{r+1}}}$, and that $P_{A_{i_{r+1}}} \neq P_{A_{j_{r+1}}}$ if $1 \leq i \neq j \leq m$. From Proposition 2, there exists $\alpha_i \in Q(H_{I_i}^{P_{A_{i+1_{r+1}}}})$ for each $1 \leq i \leq m$. Notice that

(i) $\mathrm{im}\,(\alpha_i) = I_i$ for each $1 \leq i \leq m$;

(ii) $\mathrm{ks}\,(\alpha_i) \neq \mathrm{ks}\,(\alpha_j)$ if $1 \leq i \neq j \leq m$; and

(iii) I_i is a representative set of $\mathrm{kp}\,(\alpha_{i+1})$ for each $1 \leq i \leq m-1$, and I_m is a representative set of $\mathrm{kp}\,(\alpha_1)$.

Let $R_1, \ldots, R_t$ be all the $\mathcal{R}$-Green classes in D_r where $\binom{n}{r} = m$ and $S(n+1, r+1) = t$. Without loss of generality, we may suppose that $\alpha_i \in R_i$ for $1 \leq i \leq m$. Then we take an arbitrary $\alpha_{m+j} \in Q(R_{m+j})$ for each $1 \leq j \leq t-m$, and consider the set

$$X = \{\alpha_1, \ldots, \alpha_m, \alpha_{m+1}, \ldots \alpha_t\}.$$

It is easy to see that $\alpha_1 \to \alpha_2 \to \cdots \to \alpha_m \to \alpha_1$ is a cycle on the digraph Γ_X, and that, for each $1 \leq j \leq t-m$, there exists a directed edge from α_j to at least one element on the cycle $\alpha_1 \to \alpha_2 \to \cdots \to \alpha_m \to \alpha_1$. Therefore, from Theorem 1, X is a quasi-idempotent generating set of $PK(n,r)$, and so the result follows from the fact $\mathrm{rank}\,(PK_{n,r}) = S(n+1, r+1)$. □

Theorem 4. *For* $2 \leq r \leq n-1$, $\mathrm{qrank}\,(K_{n,r}) = \mathrm{rank}\,(K_{n,r}) = S(n,r)$.

Proof. For $2 \leq r \leq n-1$, let $I_1, \ldots, I_m$ with $\binom{n}{r} = m$ be all subsets of X_n of cardinality r. Then, as shown in [1, Lemma 12], there exist m different partitions $P_1, \ldots, P_m$ of X_n to r subsets with the property that I_i is a representative set of P_i for each $1 \leq i \leq m$. From Proposition 3, there exists $\alpha_i \in Q(H_{I_i}^{P_{i+1}})$ for each $1 \leq i \leq m$. Notice that

(i) $\mathrm{im}\,(\alpha_i) = I_i$ for each $1 \leq i \leq m$;

(ii) $\mathrm{kp}\,(\alpha_i) \neq \mathrm{kp}\,(\alpha_j)$ if $1 \leq i \neq j \leq m$; and

(iii) I_i is a representative set of $\mathrm{kp}\,(\alpha_{i+1})$ for each $1 \leq i \leq m-1$, and I_m is a representative set of $\mathrm{kp}\,(\alpha_1)$.

Let $R_1, \ldots, R_t$ be a list of all $\mathcal{R}$-Green classes in D_r where $\binom{n}{r} = m$ and $S(n,r) = t$. Without loss of generality, suppose that $\alpha_i \in R_i$ for

$1 \leq i \leq m$. Then we take an arbitrary $\alpha_{m+j} \in Q(R_{m+j})$ for each $1 \leq j \leq t-m$, and consider the set

$$X = \{\alpha_1, \ldots, \alpha_m, \alpha_{m+1}, \ldots \alpha_t\}.$$

Then similarly, from Theorem 2, X is a quasi-idempotent generating set of $K(n,r)$, and so the result follows from the fact rank $(K_{n,r}) = S(n,r)$. □

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