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RESEARCH ARTICLE

Sandwich semigroups and Brandt semigroups

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ABSTRACT. In this paper we study a connection between variants of semigroups and Brandt semigroups. We find necessary conditions under which a variant of a semigroup is a Brandt semigroup. For variants of Rees matrix semigroups we studied a structure of a sandwich matrix. We proved that if semigroup does not contain a bicyclic subsemigroup, then any variant of this semigroup is not a Brandt semigroup. Thus a variant of a finite semigroup is not a Brandt semigroup.

Introduction and preliminaries

For an arbitrary but fixed element $a \in S$ of a semigroup $(S, \cdot)$ we can define a new operation $*_a$, by the next equality

$$x *_a y = x \cdot a \cdot y, \quad \text{for any elements } x,\ y \in S.$$

The operation $*_a$ is called a *sandwich-multiplication*, and the semigroup $(S, *_a)$ is called a *variant* or a *sandwich-semigroup* with the *sandwich element a*.

The concept of the variant of a semigroup first was introduced in 1960 by Ljapin [1] for semigroups of transformations. Further variants of other classes of semigroups were studied by various authors. For example, Hickey in [2] studied a general properties of variants, and Chase in [3]

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studied variants of binary relations semigroup. Research of regular semigroups was provided in [4] by Khan and Lawson. Study of subsemigroups of variants was made by Mazorchuk and Tsyaputa in [5]. In [6] Dolinka and East considered variants of finite full transformation semigroup.

Gutik and Maksymyk studied variants of a bicyclic monoid in [7] and variants of a bicyclic extendes semigroup in [8]. Variants of a policyclic monoid were studied by Givens, Rosin and Linton in [10] and by Khylynskyi in [9]. In [11] we studied variants of Rees matrix semigroup.

In the paper [12] necessary and sufficiency conditions for variants of a commutative lattices with zero to be isomorphic is obtained. Variants of a lattice of partitions of countable set were studied in [13]. In [14] the automorphism group of a variant of the lattice of partitions of finite set was described. Automorphism groups for variants of some other semigroup classes were studied in [15].

One of the naturally arising questions in research of variants is to define which semigroups are variants. It is not trivial only for semigroups without a unit, since a semigroup with unite e is a self-variant with a sandwich element e.

In this paper we study the question is there exist a variant of a semigroup which is a Brandt semigroup.

Let G be a group and let $G^0 = G \cup \{0\}$ be the group with zero obtained from G by the adjunction of a zero element 0. Let I and J be arbitrary sets. By *Rees $I \times J$ matrix over G^0* we mean $I \times J$ matrix over G^0 having at most one non-zero element. If $g \in G$ and in the matrix A it placed at kl position we denote such Rees matrix as $A_{kl}(g)$ or $[g]_{kl}$

Let $P = (p_{ji})_{i\in I,\, j\in J}$ be an arbitrary but fixed $J \times I$ matrix over G^0.

On the set of all Rees $I \times J$ matrices over G^0 we define a binary operation $\circ$ as follows:

$$A \circ B = A \cdot P \cdot B.$$

The operation $\circ$ is associative. Thus the set of all Rees $I \times J$ matrices over the group G^0 is a semigroup with respect to the binary operation $\circ$. We call it *Rees $I \times J$ matrix semigroup over the group G^0 with sandwich matrix P* and denote it $\mathcal{M}^0(G; I, J; P)$.

A semigroup is called *regular* if for each element $a \in S$ there exists an element $x \in S$ such that $axa = a$.

A semigroup without zero is called *simple* if it has no proper ideals. A semigroup S with zero is called *0-simple* if $\{0\}$ and S are its only ideals, and $S^2 \neq \{0\}$.

Let E be a set of idempotents of a semigroup S. For idempotents

$e,\ f \in E$ we set $e \leq f$ if $ef = fe = e$. Such defined $\leq$ is a partial ordering of E. If S contains a zero element 0, then $0 \leq e$ for every $e \in E$. An idempotent element f of S is called *primitive* if $f \neq 0$ and if $e \leq f$ implies $e = 0$ or $e = f$.

A semigroup is called *completely simple* [*completely 0-simple*] if it is simple [0-simple] and has a primitive idempotent.

The *bicyclic semigroup* is the semigroup $\mathscr{C}(p, q)$ with identity element generated by two symbols p and q subject to the single generating relation $pq = 1$, thus $\mathscr{C}(p, q) = \langle p,\ q \mid pq = 1\rangle$.

A semigroup S with zero is called a *Brandt semigroup* if $eSf \neq 0$ for any non-zero idempotents e, f, and for any $a \neq 0$ there exists unique element e, such that $ea = a$, unique element f, such that $af = a$, and unique element a', such that $a'a = f$.

Theorem 1 ([16]). *The following three conditions on a semigroup S with zero are equivalent.*

- (i) *S is a Brandt semigroup.*
- (ii) *S is a completely 0-simple semigroup.*
- (iii) *S is isomorphic with a regular Rees $I \times I$ matrix semigroup $\mathcal{M}^0(G; I, I; E)$ over a group with zero G^0 and with the $I \times I$ identity matrix E as sandwich matrix.*

By the equivalence of (i) and (ii) in Theorem 1 it is obvious that to study the Brandt semigroup we are interested in study of completely 0-simple semigroups. Thus we collected already known results which we would use further.

Proposition 1 ([2]). *Let a variant $(S, *_a)$ be a 0-simple semigroup. Then S is a 0-simple semigroup.*

Proposition 2 ([17]). *Every finite 0-simple semigroup is a complete 0-simple.*

Proposition 3 ([18]). *A 0-simple semigroup with a non-zero idempotent is completely 0-simple if and only if it does not contain a bicyclic subsemigroup.*

Further we need the famous Rees Theorem 2 to state a connection of completely 0-simple semigroups and Rees matrix semigroups.

Theorem 2 (Rees). *A semigroup is completely 0-simple if and only if it is isomorphic with a regular Rees matrix semigroup over a group with zero.*

1. Sandwich semigroups which are Brandt semigroups

In this section we determine which properties a variant needs to have to be isomorphic to a Brand semigroup.

Proposition 4. *Let a variant $(S, *_a)$ be isomorphic to a Brandt semigroup. Then the semigroup S is 0-simple.*

Proof. Let the variant $(S, *_a)$ be a Brandt semigroup, then by Theorem 1 it is an inverse completely 0-simple semigroup. Thus by Proposition 1 the semigroup S is a 0-simple semigroup. □

Proposition 5. *Let a variant $(S, *_a)$ be a finite Brandt semigroup. Then S is a finite completely 0-simple semigroup.*

Proof. Since the variant $(S, *_a)$ is isomorphic to the Brandt semigroup then by Proposition 4 semigroup S is 0-simple. Considering that the semigroup S is now finite 0-simple then Proposition 2 proves that S is completely 0-simple. □

Next we state a useful corollary which follows from Proposition 1 and Proposition 2.

Corollary 1. *Let a variant $(S, *_a)$ be a finite 0-simple semigroup. Then a semigroup S is finite completely 0-simple.*

Proof. Let $(S, *_a)$ be a 0-simple semigroup, then S is a 0-simple by Proposition 1. Further from the finiteness of S by Proposition 2, it follows that the semigroup S is complete 0-simple. □

2. Variants of Rees matrix semigroup

Our goal is to define properties of a semigroup which variants can be isomorphic to Brandt semigroups. Thus from Section 1 it is evident that we should study variants of 0-simple semigroups. In this section we study properties of variants of completely 0-simple semigroups.

Proposition 6. *The variant of the Rees semigroup $\mathcal{M}^0(G^0; I, J; P)$ generated by arbitrary non-zero Rees matrix A_{ij} is a Rees matrix semigroup with the sandwich matrix $Q = PA_{ij}P$.*

Proof. We consider the variant $(\mathcal{M}^0(G; I, J; P), *_{A_{ij}})$. A multiplication in this variant is defined as follows. For any Rees matrices X_{kl} and Y_{uv} we have that

$$X_{kl} *_{A_{ij}} Y_{uv} = X_{kl} \circ A_{ij} \circ Y_{uv},$$

here $\circ$ is a multiplication in the semigroup $\mathcal{M}^0(G^0; I, J; P)$. Then

$$X_{kl} *_{A_{ij}} Y_{uv} = X_{kl} P A_{ij} P Y_{uv} = X_{kl} (P A_{ij} P) Y_{uv}.$$

Hence the variant $(\mathcal{M}^0(G^0; I, J; P), *_{A_{ij}})$ coincides with a Rees matrix semigroup with the sandwich matrix $Q = P A_{ij} P$. □

Further we study which structure can have the sandwich matrix Q.

Lemma 1 ([16, Lem. 3.1])**.** *The Rees $I \times J$ matrix semigroup $\mathcal{M}^0(G; I, J; P)$ over a group with zero G^0, and with sandwich matrix P, is regular if and only if each row and each column of P contains a non-zero entry.*

Proposition 7. *Let the matrix Q have a zero at the position lk. Then or all column k or all row l or at the same time column k and row l have only zero entries.*

Proof. Let e_{ij} be a non-zero entry of the matrix A_{ij}. Let the element q_{lk} of the matrix Q be a zero entry. Thus by Proposition 6 we have that $q_{lk} = p_{li} e_{ij} p_{jk} = 0$. Last equality holds only in three next cases. If $p_{jk} = 0$ then in the matrix Q all entries of the column k are zeros. If $p_{li} = 0$ then in the matrix Q all entries of row l are zeros. If $p_{jk} = 0$ and $p_{li} = 0$ then both column k and row l contains only zeros. □

Recall that $I \times I'$ matrix U over a group with zero G^0 is called *invertible* if each row and each column of U contains exactly one non-zero element of G^0. This clearly implies that $|I| = |I'|$.

Also if ω is a homomorphism of G^0 into a group with zero $(G')^0$, and $P = (p_{kl})$ is any $J \times I$ matrix over G^0, then by $\omega(P)$ we mean the $J \times I$ matrix $(\omega(p_{kl}))_{k \in J, l \in I}$.

Proposition 8 ([16, Cor. 3.12])**.** *Two regular Rees matrix semigroups $\mathcal{M}(G; I, J; P)$ and $\mathcal{M}((G'); I', J'; P')$ are isomorphic if and only if there exists an isomorphism ω of G^0 onto $(G')^0$, an invertible $I \times I'$ matrix U, and an invertible $J \times J'$ matrix V, such that $\omega(P) = V P' U$.*

Immediately Proposition 7 and Lemma 1 imply the following corollary.

Corollary 2. *The variant* $\left(\mathcal{M}^0(G;I,J;P),*_{A_{ij}}\right)$ *generated by an arbitrary non-zero Rees matrix* A_{ij} *is regular if and only if the matrix* $Q = PA_{ij}P$ *does not have zero entries.*

Proposition 9. *There is no such variant* $(\mathcal{M}^0(G;I,J;P),*_{A_{ij}})$ *of the Rees matrix semigroup that is isomorphic to the Rees matrix semigroup* $\mathcal{M}^0(H;K,K;E)$ *with the unite sandwich matrix* E.

Proof. Taking into account Propositions 6 and 8 we check if semigroups $\mathcal{M}^0(G;I,J;PA_{ij}P)$ and $\mathcal{M}^0(H;K,K;E)$ are isomorphic. We denote $Q = PA_{ij}P$.

If $\mathcal{M}^0(G;I,J;PA_{ij}P) \cong \mathcal{M}^0(H;K,K;E)$ then there exists such isomorphism $\omega : G^0 \to H^0$ and such invertible $I \times K$ matrix U and $J \times K$ matrix V, such that $\omega(E) = VQU$. Since V and U are invertible, then each row and each column of these matrices contain precisely one non zero element. Hence the matrix VQU contains the same number of non-zero elements as the matrix Q. Since for each non-zero element $q_{ij} \in Q$ there exists exactly one element $v_{ki} \in V$ in i-th column of the matrix V and there exists exactly one element $u_{jt} \in U$ in j-row of the matrix U.

Zero entries are mapped to zero entries, mean $\omega(0_{ij}) = 0_{ij}$ and non-zero entries are mapped to non-zeros. Since E is diagonal, we see that the matrix VQU is diagonal and diagonal entries are non-zeros. Since the matrix VQU have zero entries out of the diagonal, it follows that the matrix $Q = PA_{ij}P$ have to contain zero entries too. But then by Proposition 7 the matrix Q have to contain zero rows or columns.

From the other hand a multiplication of an arbitrary matrix M by an invertible matrix by the left [right] side corresponds to a multiplication of rows [columns] of the matrix M by non-zero elements and their permutation. Hence there are zero rows or columns in the matrix VQU. This is a contradiction because the matrix VQU is diagonal with non-zero diagonal entries. The obtained contradiction proves that semigroups $\mathcal{M}^0(G;I,J;PA_{ij}P)$ and $\mathcal{M}^0(H;K,K;E)$ are not isomorphic. □

Theorem 3. *Let* S *be a semigroup which does not contain a bicyclic subsemigroup. Then for any* $a \in S$ *the variant* $(S,*_a)$ *is not a Brandt semigroup.*

Proof. Let S be a semogroup which does not contain a bicyclic subsemigroup. Let the variant $(S,*_a)$ be isomorphic to a Brandt semigroup. Then by Theorem 1 (ii) the variant $(S,*_a)$ is a completely 0-simple inverse semigroup. Thus it contains some primitive idempotent $f \in (S,*_a)$.

Thus $f *_a f = f = faf \neq 0$. Then it is obvious that $af \neq 0$ and $fa \neq 0$. By multiplying equality $f *_a f = f$ by the element $a \in S$ from the right and from the left side, we get $fafa = fa$ and $afaf = af$ respectively. Hence elements af and fa are non-zero idempotents in the semigroup S.

By the other hand since $(S, *_a)$ is completely 0-simple, then by Proposition 1 the semigroup S is 0-simple.

Thus we proved that the semigroup S is a 0-simple and contains a non-zero idempotent. Since S does not contain a bicyclic semigroup by the theorem statement, then by the Proposition 3 we have that S is completely 0-simple. Hence by Rees Theorem 2 the semigroup S is isomorphic to the regular Rees matrix semigroup $\mathcal{M}^0(G; I, J; P)$.

By the Proposition 9 the variant $(\mathcal{M}^0(G; I, J; P), *_{A_{ij}})$ is not isomorphic to the Rees matrix semigroup $\mathcal{M}^0(G; I, I; E)$ with an identity sandwich matrix E. But from the Theorem 1 (iii) by Rees matrix semigroup with identity matrix over a group with zero all Brand semigroups are described.

Hence the variant $(S, *_a)$ is not isomorphic to a Brandt semigroup. This completes the proof. □

Since a finite semigroup could not contain a bicyclic subsemigroup, then by Theorem 3 we obtain the next corollary.

Theorem 4. *A finite Brandt semigroup is not a variant of a finite semigroup.*

Proposition 10. *A variant $(\mathscr{C}(p,q), *_{q^m p^k})$ of a bicyclic semigroup $\mathscr{C}(p,q) = \langle p,\, q \mid pq = 1\rangle$ is not a Brandt semigroup.*

Proof. By proposition from [19] the set of idempotents in the variant $(\mathscr{C}(p,q), *_{q^m p^k})$ have the form $\{q^{k+i}p^{m+i} \mid i \geq 0\}$ and these idempotents form an infinite decreasing chain with respect to natural partial order on the set of idempotents. Thus the variant $(\mathscr{C}(p,q), *_{q^m p^k})$ do not contain any primitive idempotent. Hence the variant is not a completely 0-simple semigroup. Then by Theorem 1 this variant is not a Brandt semigroup. □

A question, could a semigroup which contains a bicyclic subsemigroup be Brand semigroup, remains opened.

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