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Anti-tori in quaternionic lattices over $\mathbb{F}_q(t)$

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Abstract. An anti-torus in a CAT(0) group is a subgroup $\langle a, b\rangle$, where a and b do not have commuting powers. We study anti-tori in quaternionic lattices Γ_τ over the field $\mathbb{F}_q(t)$ introduced by Stix-Vdovina (2017). We determine when every pair of generators of Γ_τ generates an anti-torus, and establish the existence of $a, b \in \Gamma_\tau$ such that the subgroup $\langle a^{p^n}, b^{p^n}\rangle$ is not abelian and not free for all $n \geq 0$. Explicit examples of matrices $a, b \in SL_3(\mathbb{F}_q(t))$ with this property are given.

Introduction

The celebrated result of Tits (1972) states that every finitely generated linear group G satisfies the following alternative: every subgroup of G is either virtually solvable or contains a nonabelian free group. Other classes of groups that enjoy the Tits' alternative include: hyperbolic groups, mapping class groups of compact surfaces, CAT(0) cubical groups, and certain classes of Artin groups.

Torsion-free hyperbolic groups satisfy a strong form of this alternative: for $a, b \in G$, either $\langle a, b\rangle$ is abelian or $\langle a^n, b^n\rangle$ is free for sufficiently large n. In [1, Question 2.7], Wise asked whether the analogous statement holds for CAT(0) groups: does there exist $n \geq 1$ such that $\langle a^n, b^n\rangle$ is either abelian or free? A positive answer is known for Coxeter groups, non-exceptional mapping class groups, and some classes of Artin groups.

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A CAT(0) group that does not satisfy this alternative was recently constructed in [3]. Now the question is which CAT(0) groups satisfy the Wise's alternative.

An important open case are the fundamental groups of complete VH-complexes X [9], for which the universal cover $\widetilde{X}$ is the product of two regular trees $T_n \times T_m$. Then $\pi_1(X)$ is a lattice in $\mathrm{Aut}(T_n) \times \mathrm{Aut}(T_m)$ and admits a group factorization $\pi_1(X) = \langle A \rangle \cdot \langle B \rangle$ corresponding to the vertical-horizontal structure of X. An anti-torus in $\pi_1(X)$ is a subgroup $\langle a, b \rangle$ for $a \in \langle A \rangle$, $b \in \langle B \rangle$ such that $\langle a^n, b^m \rangle$ is non-abelian for every $n, m \geq 1$. The notion of an anti-torus appeared to describe an interesting phenomenon: $\widetilde{X}$ could contain an isometrically embedded plane with an axis for $a, b \in \pi_1(X)$, while a and b do not have powers that commute. An anti-torus for a certain VH-complex was used by Wise to construct the first example of a non-residually finite CAT(0) group. It is an open question whether there exists a free anti-torus (see [9, Problem 10.8]).

Quaternionic lattices provide influential examples of lattices in the product of two trees. There are two beautiful constructions: the groups $\Gamma_{p,l}$ introduced by Mozes [5] in zero characteristic and the groups Γ_τ introduced by Stix-Vdovina [7] in prime characteristic. In [6], Rattaggi studied the anti-tori in $\Gamma_{p,l}$, and established their relation to non-commuting Hamilton quaternions. In this paper, we study anti-tori in the groups $\Gamma_\tau = \langle A, B_\tau \rangle$, which are arithmetic lattices over the field $\mathbb{F}_q(t)$ of prime characteristic p and $1 \neq \tau \in \mathbb{F}_q^*$.

Theorem 1. *A subgroup $\langle a, b \rangle$ is an anti-torus in Γ_τ for every $a \in A$ and $b \in B_\tau$ if and only if $\frac{\tau-1}{\tau}$ is a non-square in $\mathbb{F}_q$.*

Theorem 2. *Let $\frac{1-\tau}{\tau}$ be a non-square in $\mathbb{F}_q$. There exist $a \in A$ and $b \in B_\tau$ such that the group $\langle a^{p^n}, b^{p^n} \rangle$ is not abelian and not free for all $n \geq 0$.*

In the last section, we compute explicit examples of free and non-free subgroups of the form $\langle a^{p^n}, b^{p^n} \rangle$ in quaternion algebras over $\mathbb{F}_p(t)$ and in the group $SL_3(\mathbb{F}_p(t))$.

1. Quaternion algebras and groups Γ_τ

Let us define quaternionic lattices introduced in [7] (we preserve the original notations, except for the notation of algebra basis).

Let q be a power of an odd prime p. Let $\mathbb{F}_q$ be a finite field of order q, and $\mathbb{F}_q(t)$ the function field over $\mathbb{F}_q$. Fix a non-square $c \in \mathbb{F}_q^*$ and a parameter $1 \neq \tau \in \mathbb{F}_q^*$.

Let $D = (c, t(t-1)|K)$ be the quaternion algebra over the field $K := \mathbb{F}_q(t)$ with basis $1, \mathsf{i}, \mathsf{j}, \mathsf{k}$ and multiplication

$$\mathsf{i}^2 = c,\ \mathsf{j}^2 = t(t-1),\ \mathsf{k}^2 = -ct(t-1), \quad \text{and} \quad \mathsf{k} = \mathsf{ij} = -\mathsf{ji}.$$

The algebra D does not depend on the non-square c up to isomorphism.

For $a = a_0 + a_1\mathsf{i} + a_2\mathsf{j} + a_3\mathsf{k} \in D$, the $Re(a) = a_0$ is the real part of a and $Im(a) = a - Re(a)$ its imaginary part. The conjugate of a is $\overline{a} = Re(a) - Im(a)$. The reduced norm on D is the map $\mathrm{Nrd} : D \to K$, $\mathrm{Nrd}(a) = a\overline{a}$. The group D^* of invertible elements consists of elements $a \in D$ with $\mathrm{Nrd}(a) \neq 0$.

The subset $\mathbb{F}_q[\mathsf{i}] \subset D$ is a field of order q^2, a quadratic extension of $\mathbb{F}_q$. The norm map of the extension $\mathbb{F}_q \subset \mathbb{F}_q[\mathsf{i}]$ is

$$N : \mathbb{F}_q[\mathsf{i}]^* \to \mathbb{F}_q^*, \quad N(\xi) = \xi \cdot \overline{\xi} = a^2 - cb^2, \quad \text{where} \quad \xi = a + b\mathsf{i} \in \mathbb{F}_q[\mathsf{i}].$$

We denote $x_\xi := ct - \xi\mathsf{k} = ct - cb\mathsf{j} - a\mathsf{k} \in D^*$. For $\delta \in \mathbb{F}_q^*$, define the subsets of the groups $\mathbb{F}_q[\mathsf{i}]^*$ and D^*:

$$N_\delta = \{\xi \in \mathbb{F}_q[\mathsf{i}]^* : N(\xi) = \delta\} \quad \text{and} \quad X_\delta = \{x_\xi \in D^* : \xi \in N_\delta\}.$$

Note that $|N_\delta| = |X_\delta| = q+1$ and $x_\xi^{-1} = x_{-\xi} \in X_\delta$, $-\xi \in N_\delta$ for $\xi \in N_\delta$. We put $M_\tau := N_{\frac{c\tau}{1-\tau}}$, $Y_\tau := X_{\frac{c\tau}{1-\tau}}$, and use notation $y_\eta := x_\eta$ for $\eta \in M_\tau$.

Let $\phi : D^* \to D^*/K^*$ be the quotient map. For better readability, we denote separately $a_\xi = \phi(x_\xi)$ for $\xi \in N_{-c}$ and $b_\eta = \phi(y_\eta)$ for $\eta \in M_\tau$. The group $\Gamma_\tau = \langle A, B_\tau \rangle$ is a subgroup of the group D^*/K^* generated by

$$A = \{a_\xi : \xi \in N_{-c}\} \quad \text{and} \quad B_\tau = \{b_\eta : \eta \in M_\tau\}.$$

It is proved in [7] that for every $(\xi, \eta) \in N_{-c} \times M_\tau$ there exists a unique $(\mu, \lambda) \in N_{-c} \times M_\tau$ such that $a_\xi b_\eta = b_\lambda a_\mu$, where (μ, λ) is a unique solution of the system

$$\xi\overline{\eta} = \lambda\overline{\mu} \quad \text{and} \quad \xi + \eta = \lambda + \mu. \tag{1}$$

Theorem 3 ([7]). *The group Γ_τ is a torsion-free arithmetic lattice with finite presentation*

$$\begin{aligned} \Gamma_\tau &= \langle A, B_\tau \,|\, a_\xi a_{-\xi}, b_\eta b_{-\eta},\ \textit{and}\ a_\xi b_\eta = b_\lambda a_\mu \ \ \textit{iff (1) holds} \rangle, \\ &= \langle A \rangle \cdot \langle B_\tau \rangle, \quad \langle A \rangle \cap \langle B_\tau \rangle = E \ \ \textit{and} \ \ \langle A \rangle \cong \langle B_\tau \rangle \cong F_{\frac{q+1}{2}}, \end{aligned}$$

where F_n denotes the free group of rank n.

In particular, the group Γ_τ admits the following ab-normal forms:

$$\forall\ g \in \Gamma_\tau \quad \exists!\ a,d \in \langle A\rangle\ \exists!\ b,c \in \langle B_\tau\rangle : \quad g = ab = cd. \tag{2}$$

We need the following properties of anti-/commuting quaternions in D^* and their images in D^*/K^*. (They are analogs of the properties of Hamilton quaternions proved in [6]).

Lemma 1. *1. The relation $ab = -ba$ does not hold for $a, b \in D^*$ with $Re(a), Re(b) \neq 0$.*

2. Two quaternions $a, b \in D^* \setminus K^*$ commute if and only if $Im(a)$ and $Im(b)$ are K-proportional.

Proof. Let $a = a_0 + a_1\mathsf{i} + a_2\mathsf{j} + a_3\mathsf{k}$ and $b = b_0 + b_1\mathsf{i} + b_2\mathsf{j} + b_3\mathsf{k}$. Since $char(K) \neq 2$, the relation $ab = -ba$ is equivalent to the system

$$a_0b_0 + a_1b_1\mathsf{i}^2 + a_2b_2\mathsf{j}^2 + a_3b_3\mathsf{k}^2 = 0,$$
$$a_0b_1 + a_1b_0 = 0, \quad a_0b_2 + a_2b_0 = 0, \quad a_0b_3 + a_3b_0 = 0.$$

By solving the latter equations for b_1, b_2, b_3 and plugging in the first one, we get

$$a_0b_0 - \frac{a_1^2b_0}{a_0}\mathsf{i}^2 - \frac{a_2^2b_0}{a_0}\mathsf{j}^2 - \frac{a_3^2b_0}{a_0}\mathsf{k}^2 = \frac{b_0}{a_0}(a_0^2 - a_1^2\mathsf{i}^2 - a_2^2\mathsf{j}^2 - a_3^2\mathsf{k}^2) = \frac{b_0}{a_0}N(a) = 0.$$

Since $N(a) \neq 0$ for $a \in D^*$, we get $b_0 = 0$.

The relation $ab = ba$ is equivalent to

$$a_2b_3 = a_3b_2,\ a_3b_1 = a_1b_3,\ a_1b_2 = a_2b_1 \quad \Leftrightarrow \quad \frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}.$$

□

Lemma 2. *Let $\phi : D^* \to D^*/K^*$ be the quotient map.*

1. *The quaternions $a, b \in D^*$ with $Re(a), Re(b) \neq 0$ commute if and only if $\phi(a), \phi(b)$ commute.*

2. *The quaternions $a_1, \ldots, a_n \in D^*$ generate a free subgroup of rank n for $n \geq 2$ if and only if $\phi(a_1), \ldots, \phi(a_n) \in D^*/K^*$ generate a free subgroup of rank n.*

Proof. If a, b commute, then $\phi(a), \phi(b)$ commute. Conversely, assume $\phi(a)\phi(b) = \phi(b)\phi(a)$. Then $ab = kba$ for $k \in K^*$. By taking the reduced norm, we get $\mathrm{Nrd}(a)\mathrm{Nrd}(b) = k^2\mathrm{Nrd}(b)\mathrm{Nrd}(a)$. Hence $k^2 = 1$ and $k = \pm 1$. The case $k = -1$ is impossible by Lemma 1. Hence $k = 1$ and $ab = ba$.

If $\langle \phi(a_1), \ldots, \phi(a_n)\rangle \cong F_n$, then clearly $\langle a_1, \ldots, a_n\rangle \cong F_n$. Conversely, let $\langle a_1, \ldots, a_n\rangle \cong F_n$. Since the center of F_n is trivial for $n \geq 2$, $\langle a_1, \ldots, a_n\rangle \cap K^* = E$. Then ϕ restricted to $\langle a_1, \ldots, a_n\rangle$ is injective, and $\langle \phi(a_1), \ldots, \phi(a_n)\rangle \cong F_n$. □

Corollary 1. *The subgroups $\langle X_\tau\rangle$ of D^* are free groups of rank $\frac{q+1}{2}$.*

2. Anti-tori and non-free subgroups in the groups Γ_τ

An *anti-torus* in the group Γ_τ is a subgroup $\langle a, b\rangle$ for $a \in \langle A\rangle$, $b \in \langle B_\tau\rangle$ that do not have commuting non-trivial powers, i.e., $a^n b^m \neq b^m a^n$ for all $n, m \in \mathbb{Z}\backslash\{0\}$. Similar to [6], we relate anti-tori in Γ_τ and non-commuting quaternions in D^*.

Lemma 3. *Let $a \in \langle A\rangle$ and $b \in \langle B_\tau\rangle$. If $a^n b^m = b^m a^n$ for some $n, m \in \mathbb{Z} \setminus \{0\}$, then $ab = ba$.*

Proof. Let $a = \phi(u)$ and $b = \phi(v)$ for $u, v \in D^*$, note that $Re(u), Re(v) \neq 0$. Then $a^n b^m = b^m a^n$ implies $u^n v^m = u^m v^n$ by Lemma 2. Then u^n, v^m, u, v have K-proportional imaginary parts by Lemma 1. Hence $uv = uv$ and $ab = ba$. □

Lemma 4. *There exist $\xi \in N_{-c}$ and $\eta \in M_\tau$ such that $a_\xi b_\eta = b_\eta a_\xi$ if and only if $\frac{\tau-1}{\tau}$ is a square in $\mathbb{F}_q$.*

Proof. The relation $a_\xi b_\eta = b_\eta a_\xi$ is equivalent to $\xi\overline{\eta} = \eta\overline{\xi}$ by Equation (1). Put $\xi = u\eta$ for $u \in \mathbb{F}_q[\mathrm{i}]$. Then $u = \overline{u}$. Therefore, $u \in \mathbb{F}_q$ and

$$N(u) = \frac{N(\xi)}{N(\eta)} = \frac{\tau - 1}{\tau} = u^2.$$

Hence, the equation admits a solution if and only if $\frac{\tau-1}{\tau}$ is a square. □

Theorem 1 is a consequence of the following statement.

Proposition 1. *Assume $\frac{\tau-1}{\tau}$ is a non-square in $\mathbb{F}_q$. Then for every nontrivial $a \in \langle A\rangle$ and every $b \in B_\tau$ (or $a \in A$ and $b \in \langle B_\tau\rangle$) the subgroup $\langle a, b\rangle$ is an anti-torus in Γ_τ.*

Proof. If $a = a_1 a_2 \dots a_n \in \langle A \rangle$ and $b \in B_\tau$ have commuting powers, then they commute by Lemma 3. Then

$$ab = a_1 a_2 \dots a_n \cdot b = b \cdot a_1 a_2 \dots a_n = a_1' b_1 \cdot a_2 \dots a_n = \\ = a_1' a_2' b_2 \dots a_n = \dots = a_1' a_2' \dots a_n' \cdot b_n.$$

The uniqueness of the ab-normal forms (2) implies: $a_i = a_i'$, $b_i = b$, and b commutes with each a_i. Contradiction with Lemma 4. □

Remark 1. There always exist nontrivial elements $a \in \langle A \rangle$ and $b \in \langle B_\tau \rangle$ that commute. Indeed, take any cycle in the defining relations of the form:

$$a_1 b_1 = b_2 a_2,\ a_2 b_2 = a_3 b_3,\ \dots,\ a_n b_n = a_1 b_1.$$

Then $a_n \dots a_2 a_1 \cdot b_1 b_2 \dots b_n = b_1 b_2 \dots b_n \cdot a_n \dots a_2 a_1$.

We use the next lemma to indicate non-free subgroups in D^*.

Lemma 5. *1. If τ is a non-square in $\mathbb{F}_q$, then for every $\eta \in M_\tau$ there exist $\xi, \mu \in N_{-c}$ such that $a_\xi b_\eta = b_{-\eta} a_\mu$.*

2. If $1 - \tau$ is a non-square in $\mathbb{F}_q$, then for every $\xi \in N_{-c}$ there exist $\eta, \lambda \in M_\tau$ such that $a_\xi b_\eta = b_\lambda a_{-\xi}$.

Proof. The relation $a_\xi b_\eta = b_{-\eta} a_\mu$ is equivalent to the system

$$\begin{cases} \xi + \eta &= -\eta + \mu \\ \xi\overline{\eta} &= -\eta\overline{\mu} \end{cases} \Rightarrow \begin{cases} \mu &= \xi + 2\eta \\ \xi\overline{\eta} &= -\eta\overline{\mu} = -\eta(\overline{\xi} + 2\overline{\eta}) \end{cases}.$$

Put $\xi = u\eta$ for $u \in \mathbb{F}_q[\mathsf{i}]$. Then the last equation gives $u + \overline{u} = -2$. Hence $u = -1 + \beta\mathsf{i}$ for $\beta \in \mathbb{F}_q$, and we compute its norm:

$$N(u) = \frac{N(\xi)}{N(\eta)} = \frac{\tau - 1}{\tau} = 1 - c\beta^2 \quad \Rightarrow \quad c\tau\beta^2 = 1.$$

Since c is a non-square, the last equation has a solution if and only if τ is a non-square. It remains to check the norm of $\xi = u\eta$ and $\mu = -\overline{\xi}\eta/\overline{\eta}$: for every $\eta \in M_\tau$,

$$N(\xi) = N(u)N(\eta) = -c \quad \text{and} \quad N(\mu) = N(-\overline{\xi}\eta/\overline{\eta}) = N(\xi).$$

The item 2) is obtained similarly. □

Corollary 2. *Let $\frac{1-\tau}{\tau}$ be a non-square in $\mathbb{F}_q$. Then either $a_\xi b_\eta = b_{-\eta} a_\mu$ or $a_\xi b_\eta = b_\lambda a_{-\xi}$ holds for some $\xi, \mu \in N_{-c}$ and $\eta, \lambda \in M_\tau$.*

Remark 2. The conditions on τ in Proposition 1 and Corollary 2 hold simultaneously when -1 is a square in $\mathbb{F}_q$, that is, when either $p \equiv 1 (\mathrm{mod}\, 4)$ or $q = p^{2n}$.

Lemma 6. *Let $\eta, \lambda \in M_\tau$ be such that $\lambda \neq \pm\eta, \pm\overline{\eta}$. Then the relation $a_\xi b_\eta = b_\lambda a_\mu$ holds for at most one pair $\xi, \mu \in N_{-c}$.*

Proof. From Equation (1) we get $\xi\overline{\eta} - \overline{\xi}\lambda = \lambda(\overline{\eta} - \overline{\lambda})$. Write $\xi = x + y\mathsf{i}$, $\eta = a_1 + b_1\mathsf{i}$, $\lambda = a_2 + b_2\mathsf{i}$, and $\lambda(\overline{\eta} - \overline{\lambda}) = a_3 + b_3\mathsf{i}$ for $x, y, a_i, b_i \in \mathbb{F}_q$, and plug in the last equation:

$$\begin{aligned} \xi\overline{\eta} - \overline{\xi}\lambda &= ((a_1 - a_2)x + c(b_1 + b_2)y) + ((b_1 - b_2)x + c(a_1 + a_2)y)\mathsf{i} = \\ &= a_3 + b_3\mathsf{i}. \end{aligned}$$

We get a linear system over the field $\mathbb{F}_q$:

$$\begin{cases} (a_1 - a_2)x + c(b_1 + b_2)y & = a_3 \\ (b_1 - b_2)x + c(a_1 + a_2)y & = b_3 \end{cases}.$$

Its determinant is equal to $\Delta = (a_1^2 - a_2^2) - c(b_1^2 - b_2^2)$. Since $N(\eta) = N(\lambda)$, we have $a_1^2 - a_2^2 = b_1^2 - b_2^2$. Then $\Delta = 0$ only when $a_1 = \pm a_2$ and $b_1 = \pm b_2$, what is equivalent to the condition $\lambda = \pm\eta, \pm\overline{\eta}$. □

Theorem 4. *Let $\frac{1-\tau}{\tau}$ be a non-square in $\mathbb{F}_q$. There exist $a \in A$ and $b \in B_\tau$ such that $\langle a^{p^n}, b^{p^n} \rangle$ is not abelian and not free for all $n \geq 0$.*

Proof. By Lemma 5, there exists a relation of the form $ab = b^{-1}c$ for some $a, c \in A$ and $b \in B_\tau$. Then the uniqueness of the ab-normal form implies that a and b do not commute. Hence $\langle a^n, b^n \rangle$ is non-abelian for every $n \in \mathbb{N}$ by Lemma 3.

It is proved in [2] that the map $x \mapsto x^{q^2}$ on the generators $x \in A \cup B_\tau$ extends to an injective endomorphism of Γ_τ. Therefore, $a^n b^n = b^{-n} c^n$ for $n = q^2$. By plugging $c = bab$, we get a relation $b^n a^n b^n = (bab)^n$, and the group $\langle a, b \rangle$ is not free. Then $\langle a^n, b^n \rangle$ is not free for all n of the form $n = q^{2k}$, $k \geq 0$. Since $\langle a^{p^k}, b^{p^k} \rangle$ contains a subgroup $\langle a^n, b^n \rangle$ for $n = q^{2k}$, and subgroups of free groups are free, $\langle a^{p^k}, b^{p^k} \rangle$ is not free for every $k \geq 0$. □

Corollary 3. *Let $\frac{1-\tau}{\tau}$ be a non-square in $\mathbb{F}_q$. There are quaternions $x \in X_{-c}$ and $y \in Y_\tau$ such that $\langle x^{p^n}, y^{p^n} \rangle$ is not abelian and not free for all $n \geq 0$.*

We do not know if the statement of Theorem 4 holds for any powers of a, b.

Question 1. Is $\langle a^n, b^n\rangle$ a free group for some $a \in A$, $b \in B_\tau$ and $n \geq 1$?

If $a \in A$ and $b \in B_\tau$ commute, then $\langle a, b\rangle$ has infinite index in Γ_τ. In all examples below, we have checked that $\langle a, b\rangle$ has finite index in Γ_τ for all $a \in A$ and $b \in B_\tau$ that do not commute. It follows that $\langle a^{p^n}, b^{p^n}\rangle$ is not abelian and not free for all $n \geq 0$. We do not know if this holds for all groups Γ_τ.

Question 2. Is the index of $\langle a, b\rangle$ finite in Γ_τ for all $a \in A$ and $b \in B_\tau$ that do not commute?

3. Examples

We compute a few explicit examples of free and non-free subgroups in D^* and D^*/K^*. Also, for the quaternion algebra $D = (a, b|K)$, there is an embedding $\psi : D^*/K^* \to SL_3(K)$ given by mapping $x = x_0 + x_1\mathsf{i} + x_2\mathsf{j} + x_3\mathsf{k}$ to the 3×3-matrix

$$\frac{1}{\mathrm{Nrd}(x)}\begin{pmatrix} x_0^2 - ax_1^2 + bx_2^2 - abx_3^2 & 2b(x_0x_3 - x_1x_2) & 2b(ax_1x_3 - x_0x_2) \\ -2a(x_0x_3 + x_1x_2) & x_0^2 + ax_1^2 - bx_2^2 - abx_3^2 & 2a(x_0x_1 + bx_2x_3) \\ -2(x_0x_2 + ax_1x_3) & 2(x_0x_1 - bx_2x_3) & x_0^2 + ax_1^2 + bx_2^2 + abx_3^2 \end{pmatrix}.$$

(See Prop. 4.5.10 in [8]). This allows us to produce explicit free and non-free subgroups in $SL_3(\mathbb{F}_q(t))$.

Example 1. Let $q = 3$ and $\tau = c = 2 \in \mathbb{F}_3$. Then

$$N_{-c} = \{\pm 1, \pm\mathsf{i}\}, \qquad M_\tau = \{\pm 1 \pm \mathsf{i}\},$$
$$A = \{2t \pm \mathsf{j}, 2t \pm \mathsf{k}\}, \qquad B_\tau = \{2t \pm \mathsf{j} \pm \mathsf{k}\}.$$

We get free subgroups in D^* and D^*/K^*:

$$\langle t + \mathsf{j}, t + \mathsf{k}\rangle \cong \langle t + \mathsf{j} + \mathsf{k}, t + \mathsf{j} - \mathsf{k}\rangle \cong F_2.$$

Here $\frac{\tau-1}{\tau} = 2$ is a non-square in $\mathbb{F}_3$. Then $\langle a, b\rangle$ is an anti-torus in Γ_τ for all $a \in A$, $b \in B_\tau$ by Proposition 1. Moreover, we have directly checked that $\langle a^i, b^j\rangle$ has finite index in Γ_τ for all $a \in A$, $b \in B_\tau$ and $i, j = 1, 2$. Therefore, the following subgroups are not free:

$$\langle (t \pm \mathsf{k})^n, (t \pm \mathsf{j} \pm \mathsf{k})^n\rangle, \langle (t \pm \mathsf{j})^n, (t \pm \mathsf{j} \pm \mathsf{k})^n\rangle \not\cong F_2,\ n = 3^k, 2 \cdot 3^k,\ k \in \mathbb{N}.$$

By applying the embedding ψ, we get explicit free and non-free subgroups in $SL_3(\mathbb{F}_3(t))$ for $\psi(t+\mathsf{j})$, $\psi(t+\mathsf{k})$, $\psi(t+\mathsf{j}+\mathsf{k})$:

$$\langle \begin{pmatrix} -1-t & 0 & t^2-t \\ 0 & 1 & 0 \\ 1 & 0 & -1-t \end{pmatrix}, \begin{pmatrix} -1-t & t-t^2 & 0 \\ -1 & -1-t & 0 \\ 0 & 0 & 1 \end{pmatrix} \rangle \cong F_2,$$
$$\langle \begin{pmatrix} -1-t & 0 & t^2-t \\ 0 & 1 & 0 \\ 1 & 0 & -1-t \end{pmatrix}, \frac{1}{t+1} \begin{pmatrix} -1 & t^2-t & t-t^2 \\ 1 & -t & 1-t \\ -1 & 1-t & -t \end{pmatrix} \rangle \ncong F_2.$$

Example 2. Let $q=5$ and $c=2$, $\tau=3\in\mathbb{F}_5$. Then

$$N_{-c} = \{\pm\mathsf{i}, \pm 1 \pm 2\mathsf{i}\}, \qquad M_\tau = \{\pm 2\mathsf{i}, \pm 2 \pm \mathsf{i}\},$$
$$A = \{2t \pm 2\mathsf{j}, 2t \pm \mathsf{j} \pm \mathsf{k}\}, \qquad B_\tau = \{2t \pm \mathsf{j}, 2t \pm 2\mathsf{j} \pm 2\mathsf{k}\}.$$

We get free subgroups in D^* and D^*/K^*:

$$\langle t+\mathsf{j}, t+2\mathsf{j}\pm 2\mathsf{k}\rangle \cong \langle t+2\mathsf{j}, t+\mathsf{j}\pm\mathsf{k}\rangle \cong F_2.$$

Here $\frac{\tau-1}{\tau}=4$ is a square in $\mathbb{F}_5$, and indeed we have commuting pairs of quaternions $2t\pm 2\mathsf{j}$, $2t\pm 4\mathsf{j}$ and $2t\pm(\mathsf{j}+\mathsf{k}), 2t\pm 2(\mathsf{j}+\mathsf{k})$. We have directly checked that $\langle a,b\rangle$ has finite index in Γ_τ for all $a\in A$, $b\in B_\tau$ that do not commute. Therefore, the following subgroups are not free:

$$\langle (t\pm\mathsf{j})^n, (t\pm 2\mathsf{j})^n\rangle, \langle (t\pm\mathsf{j})^n, (t\pm\mathsf{j}\pm\mathsf{k})^n\rangle \ncong F_2, \quad \text{for } n=5^k,\ k\in\mathbb{N},$$
$$\langle (t\pm 2\mathsf{j}\pm 2\mathsf{k})^n, (t\pm 2\mathsf{j})^n\rangle, \langle (t\pm 2\mathsf{j}\pm 2\mathsf{k})^n, (t\pm\mathsf{j}\pm\mathsf{k})^n\rangle \ncong F_2,$$

and some subgroups in $SL_3(\mathbb{F}_5(t))$ for $\psi(t+2\mathsf{j})$, $\psi(t+\mathsf{j}+\mathsf{k})$, $\psi(t+\mathsf{j})$:

$$\langle \frac{1}{2t-1} \begin{pmatrix} 1 & 0 & t^2-t \\ 0 & 2t-1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \frac{1}{2t-1} \begin{pmatrix} 1 & 2t^2+3t & 3t^2+2t \\ 1 & 3t+3 & 1-t \\ 3 & 3t+2 & 2-t \end{pmatrix} \rangle \cong F_2,$$
$$\langle \begin{pmatrix} 2t-1 & 0 & 3t^2+2t \\ 0 & 1 & 0 \\ 3 & 0 & 2t-1 \end{pmatrix}, \frac{1}{2t-1} \begin{pmatrix} 1 & 2t^2+3t & 3t^2+2t \\ 1 & 3t+3 & 1-t \\ 3 & 3t+2 & 2-t \end{pmatrix} \rangle \ncong F_2.$$

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