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Characterization of commuting graphs of finite groups having small genus

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Abstract. In this paper we first show that among all double-toroidal and triple-toroidal finite graphs only $K_8 \sqcup 9K_1$, $K_8 \sqcup 5K_2$, $K_8 \sqcup 3K_4$, $K_8 \sqcup 9K_3$, $K_8 \sqcup 9(K_1 \vee 3K_2)$, $3K_6$ and $3K_6 \sqcup 4K_4 \sqcup 6K_2$ can be realized as commuting graphs of finite groups, where $\sqcup$ and $\vee$ stand for disjoint union and join of graphs respectively. As consequences of our results we also show that for any finite non-abelian group G if the commuting graph of G (denoted by $\Gamma_c(G)$) is double-toroidal or triple-toroidal then $\Gamma_c(G)$ and its complement satisfy Hansen-Vukičević Conjecture and E-LE conjecture. In the process we find a non-complete graph, namely the non-commuting graph of the group $(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes Q_8$, that is hyperenergetic. This gives a new counter example to a conjecture of Gutman regarding hyperenergetic graphs.

Introduction

Finite groups are being characterized through various graphs defined on it for a long time now. A survey on graphs defined on groups can be

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found in [4]. One such graph defined on groups is the commuting graph. The commuting graph of a finite group was originated from the works of Brauer and Fowler in [3]. Let G be a finite non-abelian group with center $Z(G)$. The commuting graph of G is a simple undirected graph whose vertex set is $G \setminus Z(G)$ and two vertices x and y are adjacent if $xy = yx$. It is denoted by $\Gamma_c(G)$. The complement of this graph is the non-commuting graph of G, denoted by $\Gamma_{nc}(G)$. The study of non-commuting graph of a finite non-abelian group gets popularity because of a question posed by Erdös in the year 1975 which was answered by Neumann in 1976 [18].

The genus of a graph Γ is the smallest non-negative integer n such that the graph can be embedded on the surface obtained by attaching n handles to a sphere. It is denoted by $\gamma(\Gamma)$. The graphs which have genus zero are called planar graphs, those which have genus one are called toroidal graphs, those which have genus two are called double-toroidal graphs and those which have genus three are called triple-toroidal graphs. Classification of finite non-abelian groups whose commuting graphs are planar or toroidal can be found in [1] and [6] (also see [9, Theorem 3.3]). Recently, finite non-abelian groups such that their commuting graphs are double-toroidal or triple-toroidal are classified in [20]. In this paper, we consider finite non-abelian groups whose commuting graphs are double or triple-toroidal and realize their commuting graphs. As such we show that among all double-toroidal and triple-toroidal finite graphs only $K_8 \sqcup 9K_1$, $K_8 \sqcup 5K_2$, $K_8 \sqcup 3K_4$, $K_8 \sqcup 9K_3$, $K_8 \sqcup 9(K_1 \vee 3K_2)$, $3K_6$ and $3K_6 \sqcup 4K_4 \sqcup 6K_2$ can be realized as commuting graphs of finite groups, where $\sqcup$ and $\vee$ stand for disjoint union and join of graphs respectively. We also compute first and second Zagreb indices of $\Gamma_c(G)$ and $\Gamma_{nc}(G)$ and show that they satisfy Hansen-Vukičević conjecture if $\Gamma_c(G)$ is double-toroidal or triple-toroidal. Further, we show that these graphs also satisfy E-LE conjecture.

Let Γ be a simple undirected graph with vertex set $v(\Gamma)$ and edge set $e(\Gamma)$. The first and second Zagreb indices of Γ, denoted by $M_1(\Gamma)$ and $M_2(\Gamma)$ respectively, are defined as

$$M_1(\Gamma) = \sum_{v \in v(\Gamma)} \deg(v)^2 \text{ and } M_2(\Gamma) = \sum_{uv \in e(\Gamma)} \deg(u)\deg(v),$$

where $\deg(v)$ is the number of edges incident on v (called degree of v). Zagreb indices of graphs were introduced by Gutman and Trinajstić [13] in 1972 to examine the dependence of total π-electron energy on molecular structure. As noted in [19], Zagreb indices are also used in studying molecular complexity, chirality, ZE-isomerism and heterosystems etc. Later on, general mathematical properties of these indices are also studied by many mathematicians. A survey on mathematical properties of Zagreb indices can be found in [14]. Comparing first and second Zagreb indices, Hansen and Vukičević [15] posed the following conjecture in 2007.

Conjecture 1 (Hansen-Vukičević Conjecture)**.** For any simple finite graph Γ,

$$\frac{M_2(\Gamma)}{|e(\Gamma)|} \geq \frac{M_1(\Gamma)}{|v(\Gamma)|}. \tag{1}$$

It was shown in [15] that the conjecture is not true if $\Gamma = K_{1,5} \sqcup K_3$. However, Hansen and Vukičević [15] showed that Conjecture 1 holds for chemical graphs. In [22], it was shown that the conjecture holds for trees with equality in (1) when Γ is a star graph. In [16], it was shown that the conjecture holds for connected unicyclic graphs with equality when the graph is a cycle. However, the search of graphs validating or invalidating Conjecture 1 is not completed yet. Recently, Das et al. [7] have obtained various finite non-abelian groups such that their commuting graphs satisfy Hansen-Vukičević Conjecture. It was also shown that $\Gamma_c(G)$ satisfies Hansen-Vukičević Conjecture if $\Gamma_c(G)$ is planar or toroidal.

Let $A(\Gamma)$ and $D(\Gamma)$ denote the adjacency matrix and degree matrix of Γ respectively. The set of eigenvalues of $A(\Gamma)$ along with their multiplicities is called the spectrum of Γ. The Laplacian matrix and signless Laplacian matrix of Γ are given by $L(\Gamma) := D(\Gamma) - A(\Gamma)$ and $Q(\Gamma) := D(\Gamma) + A(\Gamma)$ respectively. The Laplacian spectrum and signless Laplacian spectrum are the set of eigenvalues of $L(\Gamma)$ and $Q(\Gamma)$ along with their multiplicities respectively. Let $v(\Gamma) := \{v_i : i = 1, 2, \dots, n\}$. The common neighbourhood of two distinct vertices v_i and v_j, denoted by $C(v_i, v_j)$, is the set of all vertices other than v_i and v_j which are adjacent to both v_i and v_j. The common neighbourhood matrix of Γ, denoted by $CN(\Gamma)$, is defined as

$$(CN(\Gamma))_{i,j} = \begin{cases} |C(v_i, v_j)|, & \text{if } i \neq j, \\ 0, & \text{if } i = j. \end{cases}$$

The common neighbourhood spectrum of Γ is the set of all eigenvalues of $CN(\Gamma)$ along with their multiplicities. We write Spec(Γ), L-spec(Γ), Q-spec(Γ) and CN-spec(Γ) to denote the spectrum, Laplacian spectrum, signless Laplacian spectrum and common neighbourhood spectrum of Γ respectively.

The energy, $E(\Gamma)$ and common neighbourhood energy, $E_{CN}(\Gamma)$ of Γ are the sum of the absolute values of the elements of Spec(Γ) and CN-spec(Γ) respectively. The Laplacian energy, $LE(\Gamma)$ and signless Laplacian energy, $LE^+(\Gamma)$ of Γ are defined as

$$LE(\Gamma) = \sum_{\lambda \in \text{L-spec}(\Gamma)} \left| \lambda - \frac{2m}{n} \right| \quad \text{and} \quad LE^+(\Gamma) = \sum_{\mu \in \text{Q-spec}(\Gamma)} \left| \mu - \frac{2m}{n} \right|,$$

where $m = |e(\Gamma)|$. It is well-known that $E(K_n) = LE(K_n) = LE^+(K_n) = 2(n-1)$ and $E_{CN}(K_n) = 2(n-1)(n-2)$. A graph Γ with $|v(\Gamma)| = n$ is called hyperenergetic if $E(\Gamma) > E(K_n)$. It is called hypoenergetic if $E(\Gamma) < n$. Similarly, Γ is called L-hyperenergetic if $LE(\Gamma) > LE(K_n)$, Q-hyperenergetic if $LE^+(\Gamma) > LE^+(K_n)$ and CN-hyperenergetic if $E_{CN}(\Gamma) > E_{CN}(K_n)$.

Gutman et al. [12] conjectured that $E(\Gamma) \leq LE(\Gamma)$ which is known as E-LE conjecture. Gutman [11] also conjectured that "$\mathcal{G}$ is not hyperenergetic if $\mathcal{G} \ncong K_{|v(\mathcal{G})|}$". Note that both the conjectures were disproved. However, it is still unknown whether the commuting or non-commuting graphs of finite groups satisfy E-LE conjecture. In this paper, we show that $\Gamma_c(G)$ and $\Gamma_{nc}(G)$ satisfy E-LE conjecture if $\Gamma_c(G)$ is double-toroidal or triple-toroidal. Further, we find a non-complete graph, namely the non-commuting graph of the group $(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes Q_8$, that is hyperenergetic. This gives a new counter example to the above mentioned conjecture of Gutman. We shall also determine whether $\Gamma_c(G)$ and $\Gamma_{nc}(G)$ are hypoenergetic, hyperenergetic, L-hyperenergetic, Q-hyperenergetic and CN-hyperenergetic if $\Gamma_c(G)$ is double-toroidal or triple-toroidal. It is worth mentioning that the universal adjacency eigenvalues/eigenpairs of the commuting and non-commuting graphs for most of the groups considered in this paper can be obtained from [2].

1. Realization of commuting graph

In this section, we determine all finite planar, toroidal, double-toroidal and triple-toroidal graphs that can be realized as commuting graphs of finite groups. Using [1, Theorem 2.2], [9, Theorem 3.3] and commuting graphs of various finite non-abelian groups considered in [9] we have the following theorem.

Theorem 1. (a) *Among all the planar finite graphs only* $K_2 \sqcup 3K_1$, $3K_2$, $K_4 \sqcup 5K_1, K_4 \sqcup 3K_2$, $3K_4, K_3 \sqcup 4K_2$, $5K_3 \sqcup 10K_2 \sqcup 6K_4$, $3K_2 \sqcup 4K_4$, $K_4 \sqcup 5K_3$ *and* $7K_2 \sqcup D$ *can be realized as commuting graphs of finite groups, where* D *is the graph obtained from* $4K_3$ *after three vertex contractions as shown in Figure 1.*

(b) *Among all the toroidal finite graphs only* $K_6 \sqcup 7K_1$, $K_6 \sqcup 4K_2$, $K_6 \sqcup 3K_3$, $K_6 \sqcup 4K_4$ *and* $K_6 \sqcup 7K_2$ *can be realized as commuting graphs of finite groups.*

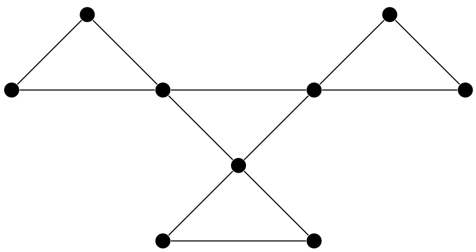

Figure 1: Graph after three vertex contractions in $4K_3$

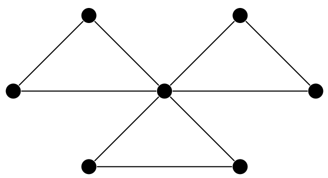

Figure 2: $K_1 \vee 3K_2$

The following two results from [20] are useful in determining all finite double-toroidal and triple-toroidal graphs that can be realized as commuting graphs of finite groups.

Theorem 2 ([20, Theorem 3.6])**.** *Let* G *be a finite non-abelian group. Then the commuting graph of* G *is double-toroidal if and only if* G *is isomorphic to one of the following groups:*

(a) $D_{18}, D_{20}, Q_{20}, S_3 \times \mathbb{Z}_2 \times \mathbb{Z}_2, S_3 \times \mathbb{Z}_4$,

(b) $(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_2 \cong \langle x, y, z : x^3 = y^3 = z^2 = [x, y] = 1,$
$$x^z = x^{-1}, y^z = y^{-1}\rangle,$$

(c) $\mathbb{Z}_3 \rtimes \mathbb{Z}_8 \cong \langle x, y : x^8 = y^3 = 1, y^x = y^{-1} \rangle$,

(d) $(\mathbb{Z}_3 \rtimes \mathbb{Z}_4) \times \mathbb{Z}_2 \cong \langle x, y, z : x^4 = y^3 = z^2 = 1,$
$$xyx^{-1} = y^{-1}, xz = zx, yz = zy \rangle,$$

(e) $(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_4 \cong \langle x, y : x^4 = y^3 = (yx^2)^2 = [x^{-1}yx, y] = 1 \rangle$,

(f) $(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes Q_8 \cong \langle x, y, z : x^4 = y^4 = z^3 = 1, y^x = y^{-1},$
$$z^{y^2} = z^{-1}, z^{x^2} = z^{-1}, x^{-1}zx^{-1} = (zy)^2 \rangle.$$

Theorem 3 ([20, Theorem 3.7])**.** *Let G be a finite non-abelian group. Then the commuting graph of G is triple-toroidal if and only if G is isomorphic to either*

(a) $GL(2, 3), D_8 \times \mathbb{Z}_3, Q_8 \times \mathbb{Z}_3$,

(b) $SL(2, 3) \circ \mathbb{Z}_2 \cong \langle x, y, z : y^3 = z^4 = 1, x^2 = z^2, y^x = y^{-1},$
$$y^{-1}zy^{-1}z^{-1}y^{-1}z = xz^{-1}xy^{-1}zy = 1 \rangle.$$

Now we realize the structures of $\Gamma_c(G)$ if $\Gamma_c(G)$ is double-toroidal or triple-toroidal.

Theorem 4. *Let G be a finite non-abelian group. If $\Gamma_c(G)$ is double-toroidal then $\Gamma_c(G)$ is isomorphic to $K_8 \sqcup 9K_1$, $K_8 \sqcup 5K_2$, $K_8 \sqcup 3K_4$, $K_8 \sqcup 9K_3$ or $K_8 \sqcup 9(K_1 \vee 3K_2)$.*

Proof. From Theorem 2, we have $\Gamma_c(G)$ is double-toroidal if and only if G is isomorphic to either $D_{18}, D_{20}, Q_{20}, S_3 \times \mathbb{Z}_2 \times \mathbb{Z}_2, S_3 \times \mathbb{Z}_4$, $(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_2$, $\mathbb{Z}_3 \rtimes \mathbb{Z}_8$, $(\mathbb{Z}_3 \rtimes \mathbb{Z}_4) \times \mathbb{Z}_2$, $(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_4$ or $(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes Q_8$.

Let G be any of the groups D_{18} and $(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_2$. Then G is an AC-group. The centralizers of the non-central elements of G are of size 9 and 2. There is exactly one centralizer of size 9 and nine distinct centralizers of size 2. Thus $\Gamma_c(G) \cong K_8 \sqcup 9K_1$.

Let G be any of the groups D_{20} and Q_{20}. Then G is an AC-group, $|Z(G)| = 2$ and it has one centralizer of size 10 and 5 distinct centralizers of size 4. Thus $\Gamma_c(G) \cong K_8 \sqcup 5K_2$.

Let G be any of the groups $S_3 \times \mathbb{Z}_2 \times \mathbb{Z}_2, S_3 \times \mathbb{Z}_4, \mathbb{Z}_3 \rtimes \mathbb{Z}_8$ and $(\mathbb{Z}_3 \rtimes \mathbb{Z}_4) \times \mathbb{Z}_2$. Then G is an AC-group, $|Z(G)| = 4$ and it has three centralizer of size 8 and one centralizer of size 12. Thus $\Gamma_c(G) \cong K_8 \sqcup 3K_4$.

If $G = (\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_4$, then G is an AC-group, $|Z(G)| = 1$ and it has one centralizer of size 9 and 9 centralizers of size 4. Thus $\Gamma_c(G) \cong K_8 \sqcup 9K_3$.

Let $G = (\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes Q_8$. The group G consist of one sylow 3-subgroup of order 9 and 9 sylow 2-subgroups of order 8. The sylow 2-subgroups of G are isomorphic to Q_8 and the sylow 3-subgroup is isomorphic to $\mathbb{Z}_3 \times \mathbb{Z}_3$. The intersection of any two of these subgroups is trivial. Thus G is exactly the union of these subgroups. Let L be any of these subgroups and $x \in L, x \neq 1$. Then $C_G(x) \subseteq L$. Thus the commuting graph of G consist of 10 components. One of the component is $\Gamma_c(G)[H]$, where $H \cup \{1\}$ is the sylow 3-subgroup of G. The other 9 components are $\Gamma_c(G)[K_i]$, where $K_i \cup \{1\}, i = 1, 2, \ldots, 9$, are the sylow 2-subgroups of G. It can be seen that $\Gamma_c(G)[H] \cong K_8$ and $\Gamma_c(G)[K_i] \cong K_1 \vee 3K_2$ for $i = 1, 2, \ldots, 9$. Thus $\Gamma_c(G) \cong K_8 \sqcup 9(K_1 \vee 3K_2)$. □

Theorem 5. *Let G be a finite non-abelian group. If $\Gamma_c(G)$ is triple-toroidal, then $\Gamma_c(G)$ is isomorphic to $3K_6$ or $3K_6 \sqcup 4K_4 \sqcup 6K_2$.*

Proof. From Theorem 3, we have $\Gamma_c(G)$ is triple-toroidal if and only if G is isomorphic to $GL(2,3), D_8 \times \mathbb{Z}_3, Q_8 \times \mathbb{Z}_3$ or $C_2 \circ S_4$.

If $G = D_8 \times \mathbb{Z}_3$ or $Q_8 \times \mathbb{Z}_3$, then G is an AC-group, $|Z(G)| = 6$ and has three distinct centralizers of size 12. Therefore, $\Gamma_c(G) = 3K_6$.

If $G = GL(2,3)$ or $C_2 \circ S_4$, then G is an AC-group, $|Z(G)| = 2$ and it has three centralizers of size 8, four centralizers of size 6 and six centralizers of size 4. Thus, $\Gamma_c(G) = 3K_6 \sqcup 4K_4 \sqcup 6K_2$. □

We conclude this section with the following corollary.

Corollary 1. (a) *Among all the double-toroidal finite graphs only $K_8 \sqcup 9K_1$, $K_8 \sqcup 5K_2$, $K_8 \sqcup 3K_4$, $K_8 \sqcup 9K_3$ or $K_8 \sqcup 9(K_1 \vee 3K_2)$ can be realized as commuting graphs of finite groups.*

(b) *Among all the triple-toroidal finite graphs only $3K_6$ and $3K_6 \sqcup 4K_4 \sqcup 6K_2$ can be realized as commuting graphs of finite groups.*

2. Some consequences

In this section we show that for any finite non-abelian group G if $\Gamma_c(G)$ is double-toroidal or $\Gamma_c(G)$ is triple-toroidal then $\Gamma_c(G)$ and $\Gamma_{nc}(G)$ satisfy Hansen-Vukičević Conjecture and E-LE conjecture. The following result is useful in our study.

Theorem 6 ([8, page 575] and [5, Lemma 3])**.** *For any graph Γ and its complement $\overline{\Gamma}$,*

$$M_1(\overline{\Gamma}) = |v(\Gamma)|(|v(\Gamma)| - 1)^2 - 4|e(\Gamma)|(|v(\Gamma)| - 1) + M_1(\Gamma) \quad \textit{and}$$

$$M_2(\overline{\Gamma}) = \frac{|v(\Gamma)|(|v(\Gamma)|-1)^3}{2} + 2|e(\Gamma)|^2 - 3|e(\Gamma)|(|v(\Gamma)|-1)^2 + \left(|v(\Gamma)| - \frac{3}{2}\right) M_1(\Gamma) - M_2(\Gamma).$$

In [7], it was shown that $\Gamma_c(G)$ satisfies Hansen-Vukičević Conjecture if $\Gamma_c(G)$ is planar or toroidal. The following theorem shows that if $\Gamma_c(G)$ is double-toroidal then $\Gamma_c(G)$ and $\Gamma_{nc}(G)$ satisfy Hansen-Vukičević Conjecture.

Theorem 7. *Let G be a finite non-abelian group. If $\Gamma_c(G)$ is double-toroidal then $\frac{M_2(\Gamma(G))}{|e(\Gamma(G))|} \geq \frac{M_1(\Gamma(G))}{|v(\Gamma(G))|}$, where $\Gamma(G) = \Gamma_c(G)$ or $\Gamma_{nc}(G)$.*

Proof. From Theorem 4, we have that $\Gamma_c(G)$ is isomorphic to $K_8 \sqcup 3K_4$, $K_8 \sqcup 9K_1$, $K_8 \sqcup 5K_2$, $K_8 \sqcup 9K_3$ or $K_8 \sqcup 9(K_1 \vee 3K_2)$. If $\Gamma_c(G) \cong K_8 \sqcup 3K_4$, then $|v(\Gamma_c(G))| = 20, |e(\Gamma_c(G))| = 46, |e(\Gamma_{nc}(G))| = \binom{20}{2} - 46 = 144$. Using Theorem 2.1 of [7] and Theorem 6, we have $M_1(\Gamma_c(G)) = 500$, $M_2(\Gamma_c(G)) = 1534, M_1(\Gamma_{nc}(G)) = 4224$ and $M_2(\Gamma_{nc}(G)) = 30720$. Therefore

$$\frac{M_2(\Gamma_c(G))}{|e(\Gamma_c(G))|} = \frac{767}{23} > 25 = \frac{M_1(\Gamma_c(G))}{|v(\Gamma_c(G))|}$$

and

$$\frac{M_2(\Gamma_{nc}(G))}{|e(\Gamma_{nc}(G))|} = \frac{640}{3} > \frac{1056}{5} = \frac{M_1(\Gamma_{nc}(G))}{|v(\Gamma_{nc}(G))|}.$$

If $\Gamma_c(G) \cong K_8 \sqcup 9K_1$, then $|v(\Gamma_c(G))| = 17, |e(\Gamma_c(G))| = 28$, $|e(\Gamma_{nc}(G))| = \binom{17}{2} - 28 = 108$. Using Theorem 2.1 of [7] and Theorem 6, we have $M_1(\Gamma_c(G)) = 392$, $M_2(\Gamma_c(G)) = 1372$, $M_1(\Gamma_{nc}(G)) = 2952$ and $M_2(\Gamma_{nc}(G)) = 19584$. Therefore

$$\frac{M_2(\Gamma_c(G))}{|e(\Gamma_c(G))|} = 49 > \frac{392}{17} = \frac{M_1(\Gamma_c(G))}{|v(\Gamma_c(G))|}$$

and

$$\frac{M_2(\Gamma_{nc}(G))}{|e(\Gamma_{nc}(G))|} = \frac{1632}{9} > \frac{2952}{17} = \frac{M_1(\Gamma_{nc}(G))}{|v(\Gamma_{nc}(G))|}.$$

If $\Gamma_c(G) \cong K_8 \sqcup 5K_2$, then $|v(\Gamma_c(G))| = 18, |e(\Gamma_c(G))| = 33$, $|e(\Gamma_{nc}(G))| = \binom{18}{2} - 33 = 120$. Using Theorem 2.1 of [7] and Theorem 6, we have $M_1(\Gamma_c(G)) = 402, M_2(\Gamma_c(G)) = 1377$, $M_1(\Gamma_{nc}(G)) = 3360$ and $M_2(\Gamma_{nc}(G)) = 23040$. Therefore

$$\frac{M_2(\Gamma_c(G))}{|e(\Gamma_c(G))|} = \frac{67}{3} > \frac{459}{11} = \frac{M_1(\Gamma_c(G))}{|v(\Gamma_c(G))|}$$

and

$$\frac{M_2(\Gamma_{nc}(G))}{|e(\Gamma_{nc}(G))|} = 192 > \frac{560}{3} = \frac{M_1(\Gamma_{nc}(G))}{|v(\Gamma_{nc}(G))|}.$$

If $\Gamma_c(G) \cong K_8 \sqcup 9K_3$, then $|v(\Gamma_c(G))| = 35, |e(\Gamma_c(G))| = 55$, $|e(\Gamma_{nc}(G))| = \binom{35}{2} - 55 = 540$. Using Theorem 2.1 of [7] and Theorem 6, we have $M_1(\Gamma_c(G)) = 500, M_2(\Gamma_c(G)) = 1480$, $M_1(\Gamma_{nc}(G)) = 33480$ and $M_2(\Gamma_{nc}(G)) = 518400$. Therefore

$$\frac{M_2(\Gamma_c(G))}{|e(\Gamma_c(G))|} = \frac{1480}{55} > \frac{500}{35} = \frac{M_1(\Gamma_c(G))}{|v(\Gamma_c(G))|}$$

and

$$\frac{M_2(\Gamma_{nc}(G))}{|e(\Gamma_{nc}(G))|} = 960 > \frac{33480}{35} = \frac{M_1(\Gamma_{nc}(G))}{|v(\Gamma_{nc}(G))|}.$$

If $\Gamma_c(G) \cong K_8 \sqcup 9(K_1 \vee 3K_2)$, then $|v(\Gamma_c(G))| = 71, |e(\Gamma_c(G))| = 109$, $|e(\Gamma_{nc}(G))| = \binom{71}{2} - 109 = 2376$, $M_1(\Gamma_c(G)) = 932$ and $M_2(\Gamma_c(G)) = 2128$. Using Theorem 6 we have $M_1(\Gamma_{nc}(G)) = 318312$ and $M_2(\Gamma_{nc}(G)) = 10660608$. Therefore

$$\frac{M_2(\Gamma_c(G))}{|e(\Gamma_c(G))|} = \frac{2128}{109} > \frac{932}{71} = \frac{M_1(\Gamma_c(G))}{|v(\Gamma_c(G))|}$$

and

$$\frac{M_2(\Gamma_{nc}(G))}{|e(\Gamma_{nc}(G))|} = \frac{10660608}{2376} > \frac{318312}{71} = \frac{M_1(\Gamma_{nc}(G))}{|v(\Gamma_{nc}(G))|}.$$

Hence, the result follows. $\square$

The following theorem shows that if $\Gamma_c(G)$ is triple-toroidal then $\Gamma_c(G)$ and $\Gamma_{nc}(G)$ satisfy Hansen-Vukičević Conjecture.

Theorem 8. *Let G be a finite non-abelian group. If $\Gamma_c(G)$ is triple-toroidal, then $\frac{M_2(\Gamma(G))}{|e(\Gamma(G))|} \geq \frac{M_1(\Gamma(G))}{|v(\Gamma(G))|}$, where $\Gamma(G) = \Gamma_c(G)$ or $\Gamma_{nc}(G)$.*

Proof. From Theorem 5, we have that $\Gamma_c(G)$ is isomorphic to $6K_2 \sqcup 3K_6 \sqcup 4K_4$ or $3K_6$. If $\Gamma_c(G) \cong 6K_2 \sqcup 3K_6 \sqcup 4K_4$, then $|v(\Gamma_c(G))| = 46$, $|e(\Gamma_c(G))| = 75, |e(\Gamma_{nc}(G))| = \binom{46}{2} - 75 = 960$. Using Theorem 2.1 of [7] and Theorem 6, we have $M_1(\Gamma_c(G)) = 606, M_2(\Gamma_c(G)) = 1347$, $M_1(\Gamma_{nc}(G)) = 80256$ and $M_2(\Gamma_{nc}(G)) = 1677120$. Therefore

$$\frac{M_2(\Gamma_c(G))}{|e(\Gamma_c(G))|} = \frac{449}{25} > \frac{303}{23} = \frac{M_1(\Gamma_c(G))}{|v(\Gamma_c(G))|}$$

and
$$\frac{M_2(\Gamma_{nc}(G))}{|e(\Gamma_{nc}(G))|} = 1747 > \frac{40128}{23} = \frac{M_1(\Gamma_{nc}(G))}{|v(\Gamma_{nc}(G))|}.$$
If $\Gamma_c(G) \cong 3K_6$, then $|v(\Gamma_c(G))| = 18$, $|e(\Gamma_c(G))| = 45$, $|e(\Gamma_{nc}(G))| = \binom{18}{2} - 45 = 108$. Using Theorem 2.1 of [7] and Theorem 6, we have $M_1(\Gamma_c(G)) = 450$, $M_2(\Gamma_c(G)) = 1125$, $M_1(\Gamma_{nc}(G)) = 2592$ and $M_2(\Gamma_{nc}(G)) = 15552$. Therefore
$$\frac{M_2(\Gamma_c(G))}{|e(\Gamma_c(G))|} = 25 = \frac{M_1(\Gamma_c(G))}{|v(\Gamma_c(G))|}$$
and
$$\frac{M_2(\Gamma_{nc}(G))}{|e(\Gamma_{nc}(G))|} = 144 = \frac{M_1(\Gamma_{nc}(G))}{|v(\Gamma_{nc}(G))|}.$$
Hence, the result follows. □

Using results from [10,17,21], we have the following characterizations of $\Gamma_c(G)$ and $\Gamma_{nc}(G)$ if $\Gamma_c(G)$ is planar or toroidal.

Theorem 9. *Let G be a finite non-abelian group such that $\Gamma_c(G)$ is planar. Then*

(a) *$\Gamma_c(G)$ is neither hyperenergetic, L-hyperenergetic nor CN-hyperenergetic.*

(b) *$\Gamma_c(G)$ is hypoenergetic only when $G \cong D_6$ or D_{10}.*

(c) *$\Gamma_c(G)$ is Q-hyperenergetic only when $G \cong A_4$.*

(d) *$E(\Gamma_c(G)) < LE(\Gamma_c(G)) < LE^+(\Gamma_c(G))$ when $G \cong A_4$ or S_4; $LE^+(\Gamma_c(G)) < E(\Gamma_c(G)) < LE(\Gamma_c(G))$ when $G \cong A_5, SL(2,3)$ or $S_z(2)$ and $E(\Gamma_c(G)) \leq LE^+(\Gamma_c(G)) \leq LE(\Gamma_c(G))$ otherwise.*

(e) *$\Gamma_{nc}(G)$ is neither hypoenergetic nor CN-hyperenergetic.*

(f) *$\Gamma_{nc}(G)$ is hyperenergetic only when $G \cong S_4$.*

(g) *$\Gamma_{nc}(G)$ is L-hyperenergetic when $G \cong D_6, D_{10}, D_{12}, Q_{12}, A_4, A_5, S_4, SL(2,3)$ or $Sz(2)$.*

(h) *$\Gamma_{nc}(G)$ is Q-hyperenergetic when $G \cong D_{10}, D_{12}, Q_{12}, A_4, A_5, S_4$ or $SL(2,3)$.*

(i) $E(\Gamma_{nc}(G)) \leq LE^{+}(\Gamma_{nc}(G)) \leq LE(\Gamma_{nc}(G))$ *but*

$$E(\Gamma_{nc}(S_4)) < LE(\Gamma_{nc}(S_4)) < LE^{+}(\Gamma_{nc}(S_4)).$$

Theorem 10. *Let G be a finite non-abelian group such that $\Gamma_c(G)$ is toroidal. Then*

(a) $\Gamma_c(G)$ *is neither hypoenergetic, hyperenergetic nor CN-hyperenergetic.*

(b) $\Gamma_c(G)$ *is L-hyperenergetic and Q-hyperenergetic when $G \cong D_{14}$, D_{16}, Q_{16}, QD_{16} or $D_6 \times \mathbb{Z}_3$.*

(c) $E(\Gamma_c(G)) < LE(\Gamma_c(G)) < LE^{+}(\Gamma_c(G))$ *when $G \cong D_6 \times \mathbb{Z}_3$ or $A_4 \times \mathbb{Z}_2$ and $E(\Gamma_c(G)) < LE^{+}(\Gamma_c(G)) < LE(\Gamma_c(G))$ otherwise.*

(d) $\Gamma_{nc}(G)$ *is neither hypoenergetic, hyperenergetic nor CN-hyperenergetic but is L-hyperenergetic as well as Q-hyperenergetic.*

(e) $E(\Gamma_{nc}(G)) \leq LE^{+}(\Gamma_{nc}(G)) \leq LE(\Gamma_{nc}(G))$ *but*

$$E(\Gamma_{nc}(A_4\times\mathbb{Z}_2)) < LE(\Gamma_{nc}(A_4\times\mathbb{Z}_2)) < LE^{+}(\Gamma_{nc}(A_4\times\mathbb{Z}_2)).$$

From Theorems 9-10, it follows that $\Gamma_c(G)$ and $\Gamma_{nc}(G)$ satisfy E-LE conjecture if $\Gamma_c(G)$ is planar or toroidal. In the following theorems we show that $\Gamma_c(G)$ and $\Gamma_{nc}(G)$ satisfy E-LE conjecture if $\Gamma_c(G)$ is double-toroidal or triple-toroidal.

Theorem 11. *Let G be a finite non-abelian group such that $\Gamma_c(G)$ is double-toroidal. Then*

(a) $\Gamma_c(G)$ *is neither hyperenergetic nor CN-hyperenergetic.*

(b) $\Gamma_c(G)$ *is not L-hyperenergetic only when $G \cong (\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_4$ or $(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes Q_8$.*

(c) $\Gamma_c(G)$ *is Q-hyperenergetic.*

(d) $\Gamma_c(G)$ *is hypoenergetic only when $G \cong D_{18}$ or $(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_2$.*

(e) $E(\Gamma_c(G)) < LE(\Gamma_c(G)) < LE^{+}(\Gamma_c(G))$ *only when $G \cong (\mathbb{Z}_3\times\mathbb{Z}_3)\rtimes \mathbb{Z}_4$ or $(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes Q_8$ and $E(\Gamma_c(G)) < LE^{+}(\Gamma_c(G)) < LE(\Gamma_c(G))$ otherwise.*

Proof. From Theorem 4, we have that $\Gamma_c(G)$ is isomorphic to $K_8 \sqcup 3K_4$, $K_8 \sqcup 9K_1$, $K_8 \sqcup 5K_2$, $K_8 \sqcup 9K_3$ or $K_8 \sqcup 9(K_1 \vee 3K_2)$.

If $\Gamma_c(G) \cong K_8 \sqcup 3K_4$, then $\text{Spec}(\Gamma_c(G)) = \{(-1)^{16}, (7)^1, (3)^3\}$ and so $E(\Gamma_c(G)) = 16 + 7 + 9 = 32$. We also have $\text{L-spec}(\Gamma_c(G)) = \{(0)^4, (8)^7, (4)^9\}$ and $\text{Q-spec}(\Gamma_c(G)) = \{(14)^1, (6)^{10}, (2)^9\}$. Here, $\frac{2|e(\Gamma_c(G))|}{|v(\Gamma_c(G))|} = \frac{23}{5}$ and $|0 - \frac{23}{5}| = \frac{23}{5}, |8 - \frac{23}{5}| = \frac{17}{5}, |4 - \frac{23}{5}| = \frac{3}{5}$. Therefore

$$LE(\Gamma_c(G)) = 4 \cdot \tfrac{23}{5} + 7 \cdot \tfrac{17}{5} + 9 \cdot \tfrac{3}{5} = \tfrac{238}{5}.$$

Similarly, $|14 - \frac{23}{5}| = \frac{47}{5}, |6 - \frac{23}{5}| = \frac{7}{5}, |2 - \frac{23}{5}| = \frac{13}{5}$ and hence

$$LE^+(\Gamma_c(G)) = \tfrac{47}{5} + 10 \cdot \tfrac{7}{5} + 9 \cdot \tfrac{13}{5} = \tfrac{234}{5}.$$

Further, $\text{CN-spec}(\Gamma_c(G)) = \{(-6)^7, (42)^1, (-2)^9, (6)^3\}$ and so $E_{CN}(\Gamma_c(G)) = 120$. We have

$$\begin{gathered} |v(\Gamma_c(G))| = 20 < 32 = E(\Gamma_c(G)), \\ E(K_{20}) = 2(20-1) = 38 > 32 = E(\Gamma_c(G)) \text{ and} \\ E_{CN}(K_{20}) = 2(20-1)(20-2) = 684 > 120 = E_{CN}(\Gamma_c(G)) \end{gathered}$$

Thus, $\Gamma_c(G)$ *is neither hypoenergetic, hyperenergetic nor CN-hyperenergetic.* Also,

$$\begin{gathered} LE(K_{20}) = 2(20-1) = 38 < \tfrac{238}{5} = LE(\Gamma_c(G)) \text{ and} \\ LE^+(K_{20}) = 2(20-1) = 38 < \tfrac{234}{5} = LE^+(\Gamma_c(G)). \end{gathered}$$

Therefore, $\Gamma_c(G)$ *is L-hyperenergetic as well as Q-hyperenergetic.* Further

$$E(\Gamma_c(G)) = 32 < \tfrac{234}{5} = LE^+(\Gamma_c(G)) < \tfrac{238}{5} = LE(\Gamma_c(G)).$$

If $\Gamma_c(G) \cong K_8 \sqcup 9K_1$, then $\text{Spec}(\Gamma_c(G)) = \{(-1)^7, (7)^1, (0)^9\}$ and so $E(\Gamma_c(G)) = 7 + 7 = 14$. We also have $\text{L-spec}(\Gamma_c(G)) = \{(0)^{10}, (8)^7\}$ and $\text{Q-spec}(\Gamma_c(G)) = \{(14)^1, (6)^7, (0)^9\}$. Here, $\frac{2|e(\Gamma_c(G))|}{|v(\Gamma_c(G))|} = \frac{56}{17}$ and $|0 - \frac{56}{17}| = \frac{56}{17}, |8 - \frac{56}{17}| = \frac{80}{17}$. Therefore

$$LE(\Gamma_c(G)) = 10 \cdot \tfrac{56}{17} + 7 \cdot \tfrac{80}{17} = \tfrac{1120}{17}.$$

Similarly, $|14 - \frac{56}{17}| = \frac{182}{17}, |6 - \frac{56}{17}| = \frac{46}{17}, |0 - \frac{56}{17}| = \frac{56}{17}$ and hence

$$LE^+(\Gamma_c(G)) = \tfrac{182}{17} + 7 \cdot \tfrac{46}{17} + 9 \cdot \tfrac{56}{17} = \tfrac{1008}{17}.$$

Further, $\text{CN-spec}(\Gamma_c(G)) = \{(-6)^7, (42)^1, (0)^9\}$ and so $E_{CN}(\Gamma_c(G)) = 84$. We have

$$\begin{gathered}|v(\Gamma_c(G))| = 17 > 14 = E(\Gamma_c(G)),\\ E(K_{17}) = 2(17-1) = 32 > 14 = E(\Gamma_c(G)) \text{ and}\\ E_{CN}(K_{17}) = 2(17-1)(17-2) = 480 > 84 = E_{CN}(\Gamma_c(G)).\end{gathered}$$

Thus, $\Gamma_c(G)$ *is hypoenergetic but neither hyperenergetic nor CN-hyperenergetic*. Also

$$\begin{gathered}LE(K_{17}) = 2(17-1) = 32 < \tfrac{1120}{17} = LE(\Gamma_c(G)) \text{ and}\\ LE^+(K_{17}) = 2(17-1) = 32 < \tfrac{1008}{17} = LE^+(\Gamma_c(G)).\end{gathered}$$

Therefore, $\Gamma_c(G)$ *is L-hyperenergetic as well as Q-hyperenergetic*. Further,

$$E(\Gamma_c(G)) = 14 < \tfrac{1008}{17} = LE^+(\Gamma_c(G)) < \tfrac{1120}{17} = LE(\Gamma_c(G)).$$

If $\Gamma_c(G) \cong K_8 \sqcup 5K_2$, then $\operatorname{Spec}(\Gamma_c(G)) = \left\{(-1)^{12}, (7)^1, (1)^5\right\}$ and so $E(\Gamma_c(G)) = 12 + 7 + 5 = 24$. We also have L-spec$(\Gamma_c(G)) = \left\{(0)^6, (8)^7, (2)^5\right\}$ and Q-spec$(\Gamma_c(G)) = \left\{(14)^1, (6)^7, (2)^5, (0)^5\right\}$. Here, $\frac{2|e(\Gamma_c(G))|}{|v(\Gamma_c(G))|} = \frac{11}{3}$ and $|0 - \frac{11}{3}| = \frac{11}{3}$, $|8 - \frac{11}{3}| = \frac{13}{3}$, $|2 - \frac{11}{3}| = \frac{5}{3}$. Therefore

$$LE(\Gamma_c(G)) = 6 \cdot \tfrac{11}{3} + 7 \cdot \tfrac{13}{3} + 5 \cdot \tfrac{5}{3} = \tfrac{182}{3}.$$

Similarly, $|14 - \frac{11}{3}| = \frac{31}{3}$, $|6 - \frac{11}{3}| = \frac{7}{3}$, $|2 - \frac{11}{3}| = \frac{5}{3}$, $|0 - \frac{11}{3}| = \frac{11}{3}$ and hence

$$LE^+(\Gamma_c(G)) = \tfrac{31}{3} + 7 \cdot \tfrac{7}{3} + 5 \cdot \tfrac{5}{3} + 5 \cdot \tfrac{11}{3} = \tfrac{160}{3}.$$

Further, CN-spec$(\Gamma_c(G)) = \left\{(-6)^7, (42)^1, (0)^{10}\right\}$ and so $E_{CN}(\Gamma_c(G)) = 84$. We have

$$\begin{gathered}|v(\Gamma_c(G))| = 18 < 24 = E(\Gamma_c(G)),\\ E(K_{18}) = 2(18-1) = 34 > 24 = E(\Gamma_c(G)) \text{ and}\\ E_{CN}(K_{18}) = 2(18-1)(18-2) = 544 > 84 = E_{CN}(\Gamma_c(G)).\end{gathered}$$

Thus, $\Gamma_c(G)$ *is neither hypoenergetic, hyperenergetic nor CN-hyperenergetic*. Also

$$\begin{gathered}LE(K_{18}) = 2(18-1) = 34 < \tfrac{182}{3} = LE(\Gamma_c(G)) \text{ and}\\ LE^+(K_{18}) = 2(18-1) = 34 < \tfrac{160}{3} = LE^+(\Gamma_c(G)).\end{gathered}$$

Therefore, $\Gamma_c(G)$ *is L-hyperenergetic as well as Q-hyperenergetic*. Further

$$E(\Gamma_c(G)) = 24 < \tfrac{160}{3} = LE^+(\Gamma_c(G)) < \tfrac{182}{3} = LE(\Gamma_c(G)).$$

If $\Gamma_c(G) \cong K_8 \sqcup 9K_3$, then $\mathrm{Spec}(\Gamma_c(G)) = \left\{(-1)^{25}, (7)^1, (2)^9\right\}$ and so $E(\Gamma_c(G)) = 25 + 7 + 18 = 50$. We also have $\text{L-spec}(\Gamma_c(G)) = \left\{(0)^{10}, (8)^7, (3)^{18}\right\}$ and $\text{Q-spec}(\Gamma_c(G)) = \left\{(14)^1, (6)^7, (4)^9, (1)^{18}\right\}$. Here, $\frac{2|e(\Gamma_c(G))|}{|v(\Gamma_c(G))|} = \frac{22}{7}$ and $|0 - \frac{22}{7}| = \frac{22}{7}$, $|8 - \frac{22}{7}| = \frac{34}{7}$, $|3 - \frac{22}{7}| = \frac{1}{7}$. Therefore,

$$LE(\Gamma_c(G)) = 10 \cdot \tfrac{22}{7} + 7 \cdot \tfrac{34}{7} + 18 \cdot \tfrac{1}{7} = 68.$$

Similarly, $|14 - \frac{22}{7}| = \frac{76}{7}$, $|6 - \frac{22}{7}| = \frac{20}{7}$, $|4 - \frac{22}{7}| = \frac{6}{7}$, $|1 - \frac{22}{7}| = \frac{15}{7}$ and hence

$$LE^+(\Gamma_c(G)) = \tfrac{76}{7} + 7 \cdot \tfrac{20}{7} + 9 \cdot \tfrac{6}{7} + 18 \cdot \tfrac{15}{7} = \tfrac{540}{7}.$$

Further, $\text{CN-spec}(\Gamma_c(G)) = \left\{(-6)^7, (42)^1, (-1)^{18}, (2)^9\right\}$ and so $E_{CN}(\Gamma_c(G)) = 120$. We have

$$|v(\Gamma_c(G))| = 35 < 50 = E(\Gamma_c(G)),$$
$$E(K_{35}) = 2(35-1) = 68 > 50 = E(\Gamma_c(G)) \text{ and}$$
$$E_{CN}(K_{35}) = 2(35-1)(35-2) = 2244 > 120 = E_{CN}(\Gamma_c(G)).$$

Thus, $\Gamma_c(G)$ *is neither hypoenergetic, hyperenergetic nor CN-hyperenergetic*. Also, $LE(K_{35}) = 2(35-1) = 68 = LE(\Gamma_c(G))$ and $LE^+(K_{35}) = 2(35-1) = 68 < \frac{540}{7} = LE^+(\Gamma_c(G))$. Therefore, $\Gamma_c(G)$ *is not L-hyperenergetic but Q-hyperenergetic*. Further,

$$E(\Gamma_c(G)) = 50 < 68 = LE(\Gamma_c(G)) < \tfrac{540}{7} = LE^+(\Gamma_c(G)).$$

If $\Gamma_c(G) \cong K_8 \sqcup 9(K_1 \vee 3K_2)$, then $\mathrm{Spec}(\Gamma_c(G)) = \left\{(-1)^{34}, (7)^1, (-2)^9, (1)^{18}, (3)^9\right\}$ and so $E(\Gamma_c(G)) = 34 + 34 + 18 + 18 = 104$. We also have $\text{L-spec}(\Gamma_c(G)) = \left\{(0)^{10}, (8)^7, (3)^{27}, (1)^{18}, (7)^9\right\}$ and $\text{Q-spec}(\Gamma_c(G)) = \left\{(14)^1, (6)^7, (3)^{18}, (1)^{27}, \left(\frac{9+\sqrt{33}}{2}\right)^9, \left(\frac{9-\sqrt{33}}{2}\right)^9\right\}$. Here, $\frac{2|e(\Gamma_c(G))|}{|v(\Gamma_c(G))|} = \frac{218}{71}$ and $|0 - \frac{218}{71}| = \frac{218}{71}$, $|8 - \frac{218}{71}| = \frac{350}{71}$, $|3 - \frac{218}{71}| = \frac{5}{71}$, $|1 - \frac{218}{71}| = \frac{147}{71}$, $|7 - \frac{218}{71}| = \frac{279}{71}$. Therefore,

$$LE(\Gamma_c(G)) = 10 \cdot \tfrac{218}{71} + 7 \cdot \tfrac{350}{71} + 18 \cdot \tfrac{147}{71} + 27 \cdot \tfrac{5}{71} + 9 \cdot \tfrac{279}{71} = \tfrac{9922}{71}.$$

Similarly, $|14 - \frac{218}{71}| = \frac{776}{71}$, $|6 - \frac{218}{71}| = \frac{208}{71}$, $|\frac{9+\sqrt{33}}{2} - \frac{218}{71}| \approx \frac{610.86}{71}$, $|\frac{9-\sqrt{33}}{2} - \frac{218}{71}| \approx \frac{204.86}{71}$ and hence

$$LE^+(\Gamma_c(G)) \approx \tfrac{776}{71} + 7 \cdot \tfrac{208}{71} + 27 \cdot \tfrac{147}{71} + 18 \cdot \tfrac{5}{71} + 9 \cdot \tfrac{610.86}{71} + 9 \cdot \tfrac{204.86}{71} \approx \tfrac{13632.48}{71}.$$

Further, $\text{CN-spec}(\Gamma_c(G)) = \left\{(-6)^7, (42)^1, (-1)^{54}, (6)^9\right\}$ and so $E_{CN}(\Gamma_c(G)) = 192$. We have

$$|v(\Gamma_c(G))| = 71 < 104 = E(\Gamma_c(G)),$$
$$E(K_{71}) = 2(71-1) = 140 > 104 = E(\Gamma_c(G)) \text{ and}$$
$$E_{CN}(K_{71}) = 2(71-1)(71-2) = 9660 > 192 = E_{CN}(\Gamma_c(G)).$$

Thus, $\Gamma_c(G)$ *is neither hypoenergetic, hyperenergetic nor CN-hyperenergetic*. Also, $LE(K_{71}) = 2(71-1) = 140 > \frac{9922}{71} = LE(\Gamma_c(G))$ and $LE^+(K_{71}) = 2(71-1) = 140 < \frac{13632.48}{71} \approx LE^+(\Gamma_c(G))$. Therefore, $\Gamma_c(G)$ *is not L-hyperenergetic but Q-hyperenergetic.* Further,

$$E(\Gamma_c(G)) = 104 < \frac{9922}{71} = LE(\Gamma_c(G)) < \frac{13632.48}{71} = LE^+(\Gamma_c(G)).$$

Hence, the result follows. □

Theorem 12. *Let G be a finite non-abelian group such that $\Gamma_c(G)$ is triple-toroidal. Then*

(a) $\Gamma_c(G)$ *is neither hypoenergetic, hyperenergetic, CN-hyperenergetic nor Q-hyperenergetic.*

(b) $\Gamma_c(G)$ *is L-hyperenergetic only when $G \cong GL(2,3)$.*

(c) $E(\Gamma_c(G)) \leq LE^+(\Gamma_c(G)) \leq LE(\Gamma_c(G))$.

Proof. From Theorem 5, we have that $\Gamma_c(G)$ is isomorphic to $6K_2 \sqcup 3K_6 \sqcup 4K_4$ or $3K_6$.

If $\Gamma_c(G) \cong 6K_2 \sqcup 3K_6 \sqcup 4K_4$, then $\text{Spec}(\Gamma_c(G)) = \{(-1)^{33}, (1)^6, (5)^3, (3)^4\}$ and so $E(\Gamma_c(G)) = 33+6+15+12 = 66$. We also have $\text{L-spec}(\Gamma_c(G)) = \{(0)^{13}, (2)^6, (6)^{15}, (4)^{12}\}$ and $\text{Q-spec}(\Gamma_c(G)) = \{(0)^6, (10)^3, (4)^{15}, (6)^4, (2)^{18}\}$. Here, $\frac{2|e(\Gamma_c(G))|}{|v(\Gamma_c(G))|} = \frac{75}{23}$ and $|0 - \frac{75}{23}| = \frac{75}{23}$, $|2 - \frac{75}{23}| = \frac{29}{23}$, $|6 - \frac{75}{23}| = \frac{63}{23}$, $|4 - \frac{75}{23}| = \frac{17}{23}$. Therefore,

$$LE(\Gamma_c(G)) = 13 \cdot \frac{75}{23} + 6 \cdot \frac{29}{23} + 15 \cdot \frac{63}{23} + 12 \cdot \frac{17}{23} = \frac{2298}{23}.$$

Similarly, $|0 - \frac{75}{23}| = \frac{75}{23}, |10 - \frac{75}{23}| = \frac{155}{23}, |4 - \frac{75}{23}| = \frac{17}{23}, |6 - \frac{75}{23}| = \frac{63}{23}$, $|2 - \frac{75}{23}| = \frac{29}{23}$ and hence

$$LE^+(\Gamma_c(G)) = 6 \cdot \frac{75}{23} + 3 \cdot \frac{155}{23} + 15 \cdot \frac{17}{23} + 4 \cdot \frac{63}{23} + 18 \cdot \frac{29}{23} = \frac{1944}{23}.$$

Further, $\text{CN-spec}(\Gamma_c(G)) = \{(0)^{12}, (-4)^{15}, (20)^3, (-2)^{12}, (6)^4\}$ and so $E_{CN}(\Gamma_c(G)) = 168$. We have

$$|v(\Gamma_c(G))| = 46 < 66 = E(\Gamma_c(G)),$$
$$E(K_{46}) = 2(46-1) = 90 > 66 = E(\Gamma_c(G)) \text{ and}$$
$$E_{CN}(K_{46}) = 2(46-1)(46-2) = 3960 > 168 = E_{CN}(\Gamma_c(G)).$$

Thus, $\Gamma_c(G)$ *is neither hypoenergetic, hyperenergetic nor CN-hyperenergetic.* Also, $LE(K_{46}) = 2(46-1) = 90 < \frac{2298}{23} = LE(\Gamma_c(G))$ and $LE^+(K_{46}) = 2(46-1) = 90 > \frac{1944}{23} = LE^+(\Gamma_c(G))$. Therefore, $\Gamma_c(G)$ *is L-hyperenergetic but not Q-hyperenergetic.* Further,

$$E(\Gamma_c(G)) = 66 < \tfrac{1944}{23} = LE^+(\Gamma_c(G)) < \tfrac{2298}{23} = LE(\Gamma_c(G)).$$

If $\Gamma_c(G) \cong 3K_6$, then $\text{Spec}(\Gamma_c(G)) = \{(-1)^{15}, (5)^3\}$ and so $E(\Gamma_c(G)) = 15 + 15 = 30$. We also have $\text{L-spec}(\Gamma_c(G)) = \{(0)^3, (6)^{15}\}$ and $\text{Q-spec}(\Gamma_c(G)) = \{(10)^3, (4)^{15}\}$. Here, $\frac{2|e(\Gamma_c(G))|}{|v(\Gamma_c(G))|} = 5$ and $|0-5| = 5$, $|6-5| = 1$. Therefore, $LE(\Gamma_c(G)) = 3 \cdot 5 + 15 \cdot 1 = 30$. Similarly, $|10-5| = 5, |4-5| = 1$ and hence $LE^+(\Gamma_c(G)) = 3 \cdot 5 + 15 \cdot 1 = 30$. Further, $\text{CN-spec}(\Gamma_c(G)) = \{(-4)^{15}, (20)^3\}$ and so $E_{CN}(\Gamma_c(G)) = 120$. We have

$$\begin{gathered} |v(\Gamma_c(G))| = 18 < 30 = E(\Gamma_c(G)), \\ E(K_{18}) = 2(18-1) = 34 > 30 = E(\Gamma_c(G)) \text{ and} \\ E_{CN}(K_{18}) = 2(18-1)(18-2) = 544 > 120 = E_{CN}(\Gamma_c(G)). \end{gathered}$$

Thus, $\Gamma_c(G)$ *is neither hypoenergetic, hyperenergetic nor CN-hyperenergetic.* Also, $LE(K_{18}) = 2(18-1) = 34 > 30 = LE(\Gamma_c(G))$ and $LE^+(K_{18}) = 2(18-1) = 34 > 30 = LE^+(\Gamma_c(G))$. Therefore, $\Gamma_c(G)$ *is neither L-hyperenergetic nor Q-hyperenergetic.* Further,

$$E(\Gamma_c(G)) = 30 = LE^+(\Gamma_c(G)) = LE(\Gamma_c(G)).$$

Hence, the result follows. □

Theorem 13. *Let G be a finite non-abelian group such that $\Gamma_c(G)$ is double-toroidal. Then*

(a) $\Gamma_{nc}(G)$ *is neither hypoenergetic nor CN-hyperenergetic.*

(b) $\Gamma_{nc}(G)$ *is hyperenergetic only when* $G \cong (\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes Q_8$.

(c) $\Gamma_{nc}(G)$ *is L-hyperenergetic and Q-hyperenergetic.*

(d) $E(\Gamma_{nc}(G)) < LE(\Gamma_{nc}(G)) < LE^+(\Gamma_{nc}(G))$ *only when* $G \cong (\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_4$ *and* $E(\Gamma_{nc}(G)) < LE^+(\Gamma_{nc}(G)) < LE(\Gamma_{nc}(G))$ *otherwise.*

Proof. From Theorem 4, we have that $\Gamma_c(G)$ is isomorphic to $K_8 \sqcup 3K_4$, $K_8 \sqcup 9K_1$, $K_8 \sqcup 5K_2$, $K_8 \sqcup 9K_3$ or $K_8 \sqcup 9(K_1 \vee 3K_2)$.

If $\Gamma_c(G) \cong K_8 \sqcup 3K_4$, then $\text{Spec}(\Gamma_{nc}(G)) = \{(0)^{16}, (-4)^2, (4+\sqrt{112})^1, (4-\sqrt{112})^1\}$ and so $E(\Gamma_{nc}(G)) = 8 + 2\sqrt{112}$. We also have $\text{L-spec}(\Gamma_{nc}(G)) = \{(0)^1, (16)^9, (12)^7, (20)^3\}$ and $\text{Q-spec}(\Gamma_{nc}(G)) = \{(12)^9, (16)^9, (18+\sqrt{132})^1,$

$(18-\sqrt{132})^1\}$. Here, $\frac{2|e(\Gamma_{nc}(G))|}{|v(\Gamma_{nc}(G))|} = \frac{72}{5}$ and $|0-\frac{72}{5}| = \frac{72}{5}, |16-\frac{72}{5}| = \frac{8}{5}$, $|12-\frac{72}{5}| = \frac{12}{5}, |20-\frac{72}{5}| = \frac{28}{5}$. Therefore,

$$LE(\Gamma_{nc}(G)) = \tfrac{72}{5} + 9\cdot\tfrac{8}{5} + 7\cdot\tfrac{12}{5} + 3\cdot\tfrac{28}{5} = \tfrac{312}{5}.$$

Similarly, $|12-\frac{72}{5}| = \frac{12}{5}$, $|16-\frac{72}{5}| = \frac{7}{5}$, $|18+\sqrt{132}-\frac{72}{5}| = \frac{18+5\sqrt{132}}{5}$, $|18-\sqrt{132}-\frac{72}{5}| = \frac{5\sqrt{132}-18}{5}$ and hence

$$LE^+(\Gamma_{nc}(G)) = 9\cdot\tfrac{12}{5} + 9\cdot\tfrac{8}{5} + \tfrac{18+5\sqrt{132}}{5} + \tfrac{5\sqrt{132}-18}{5} = 36+2\sqrt{132}.$$

Further, CN-spec$(\Gamma_{nc}(G)) = \{2(57+\sqrt{1761})^1, 2(57-\sqrt{1761})^1, (-16)^9, (-12)^7, (0)^2\}$ and so $E_{CN}(\Gamma_{nc}(G)) = 456$. We have

$$\begin{gathered}|v(\Gamma_{nc}(G))| = 20 < 8+2\sqrt{112} = E(\Gamma_{nc}(G)),\\ E(K_{20}) = 2(20-1) = 38 > 8+2\sqrt{112} = E(\Gamma_{nc}(G)) \text{ and}\\ E_{CN}(K_{20}) = 2(20-1)(20-2) = 684 > 456 = E_{CN}(\Gamma_{nc}(G)).\end{gathered}$$

Thus, $\Gamma_{nc}(G)$ *is neither hypoenergetic, hyperenergetic nor CN-hyperenergetic*. Also, $LE(K_{20}) = 2(20-1) = 38 < \frac{312}{5} = LE(\Gamma_{nc}(G))$ and $LE^+(K_{20}) = 2(20-1) = 38 < 36+2\sqrt{132} = LE^+(\Gamma_{nc}(G))$. Therefore, $\Gamma_{nc}(G)$ *is L-hyperenergetic as well as Q-hyperenergetic*. Further,

$$\begin{aligned}E(\Gamma_{nc}(G)) = 8+2\sqrt{112} &< 36+2\sqrt{132}\\ &= LE^+(\Gamma_{nc}(G)) < \frac{312}{5} = LE(\Gamma_{nc}(G)).\end{aligned}$$

If $\Gamma_c(G) \cong K_8 \sqcup 9K_1$, then Spec$(\Gamma_{nc}(G)) = \{(0)^7, (-1)^8, (4+\sqrt{88})^1, (4-\sqrt{88})^1\}$ and so $E(\Gamma_{nc}(G)) = 8+2\sqrt{88}$. We also have L-spec$(\Gamma_{nc}(G))$ $= \{(0)^1, (9)^7, (17)^9\}$ and Q-spec$(\Gamma_{nc}(G)) = \left\{(9)^7, (15)^8, \left(\frac{33+\sqrt{513}}{2}\right)^1, \left(\frac{33-\sqrt{513}}{2}\right)^1\right\}$. Here, $\frac{2|e(\Gamma_{nc}(G))|}{|v(\Gamma_{nc}(G))|} = \frac{216}{17}$ and $|0-\frac{216}{17}| = \frac{216}{17}$, $|9-\frac{216}{17}| = \frac{63}{17}$, $|17-\frac{216}{17}| = \frac{73}{17}$. Therefore,

$$LE(\Gamma_{nc}(G)) = \tfrac{216}{17} + 7\cdot\tfrac{63}{17} + 9\cdot\tfrac{73}{17} = \tfrac{1314}{17}.$$

Similarly, $|9-\frac{216}{17}| = \frac{63}{17}$, $|15-\frac{216}{17}| = \frac{39}{17}$, $|\frac{33+\sqrt{513}}{2}-\frac{216}{17}| = \frac{129+17\sqrt{513}}{34}$, $|\frac{33-\sqrt{513}}{2}-\frac{216}{17}| = \frac{17\sqrt{513}-129}{34}$ and hence

$$LE^+(\Gamma_{nc}(G)) = 7\cdot\tfrac{63}{17} + 8\cdot\tfrac{39}{17} + \tfrac{129+17\sqrt{513}}{34} + \tfrac{17\sqrt{513}-129}{34} = \tfrac{753+17\sqrt{513}}{17}.$$

Further,

$$\text{CN-spec}(\Gamma_{nc}(G)) = \left\{\tfrac{3}{2}(61+\sqrt{2049})^1,\ \tfrac{3}{2}(61-\sqrt{2049})^1, (-15)^8, (-9)^7\right\}$$

and so $E_{CN}(\Gamma_{nc}(G)) = 366$. We have

$$|v(\Gamma_{nc}(G))| = 17 < 8 + 2\sqrt{88} = E(\Gamma_{nc}(G)),$$
$$E(K_{17}) = 2(17-1) = 32 > 8 + 2\sqrt{88} = E(\Gamma_{nc}(G)) \text{ and}$$
$$E_{CN}(K_{17}) = 2(17-1)(17-2) = 480 > 366 = E_{CN}(\Gamma_{nc}(G)).$$

Thus, $\Gamma_{nc}(G)$ *is neither hypoenergetic, hyperenergetic nor CN-hyperenergetic*. Also, $LE(K_{17}) = 2(17-1) = 32 < \frac{1314}{17} = LE(\Gamma_{nc}(G))$ and $LE^+(K_{17}) = 2(17-1) = 32 < \frac{753+17\sqrt{513}}{17} = LE^+(\Gamma_{nc}(G))$. Therefore, $\Gamma_{nc}(G)$ *is L-hyperenergetic as well as Q-hyperenergetic*. Further,

$$\begin{aligned} E(\Gamma_{nc}(G)) = 8 + 2\sqrt{88} <& \frac{753 + 17\sqrt{513}}{17} \\ &= LE^+(\Gamma_{nc}(G)) < \frac{1314}{17} = LE(\Gamma_{nc}(G)). \end{aligned}$$

If $\Gamma_c(G) \cong K_8 \sqcup 5K_2$, then $\text{Spec}(\Gamma_{nc}(G)) = \{(0)^{12}, (-2)^6, (4+\sqrt{96})^1, (4-\sqrt{96})^1\}$ and so $E(\Gamma_{nc}(G)) = 12 + 2\sqrt{96}$. We also have $\text{L-spec}(\Gamma_{nc}(G)) = \{(0)^1, (16)^5, (10)^7, (18)^5\}$ and $\text{Q-spec}(\Gamma_{nc}(G)) = \{(10)^7, (16)^5, (14)^4, (17+\sqrt{129})^1, (17-\sqrt{129})^1\}$. Here, $\frac{2|e(\Gamma_{nc}(G))|}{|v(\Gamma_{nc}(G))|} = \frac{40}{3}$ and $|0 - \frac{40}{3}| = \frac{40}{3}$, $|16 - \frac{40}{3}| = \frac{8}{3}, |10 - \frac{40}{3}| = \frac{10}{3}, |18 - \frac{40}{3}| = \frac{14}{3}$. Therefore,

$$LE(\Gamma_{nc}(G)) = \tfrac{40}{3} + 5 \cdot \tfrac{8}{3} + 7 \cdot \tfrac{10}{3} + 5 \cdot \tfrac{14}{3} = \tfrac{220}{3}.$$

Similarly, $|10 - \frac{40}{3}| = \frac{10}{3}$, $|16 - \frac{40}{3}| = \frac{8}{3}$, $|14 - \frac{40}{3}| = \frac{2}{3}$, $|17 + \sqrt{129} - \frac{40}{3}| = \frac{11+3\sqrt{129}}{3}$, $|17 - \sqrt{129} - \frac{40}{3}| = \frac{3\sqrt{129}-11}{3}$ and hence

$$LE^+(\Gamma_{nc}(G)) = 7 \cdot \tfrac{10}{3} + 5 \cdot \tfrac{8}{3} + 4 \cdot \tfrac{2}{3} + \tfrac{11+3\sqrt{129}}{3} + \tfrac{3\sqrt{129}-11}{3} = \tfrac{118+6\sqrt{129}}{3}.$$

Further, $\text{CN-spec}(\Gamma_{nc}(G)) = \{(99+\sqrt{5961})^1, (99-\sqrt{5961})^1, (-16)^5, (-2)^4, (-10)^7\}$ and so $E_{CN}(\Gamma_{nc}(G)) = 356$. We have

$$|v(\Gamma_{nc}(G))| = 18 < 12 + 2\sqrt{96} = E(\Gamma_{nc}(G)),$$
$$E(K_{18}) = 2(18-1) = 34 > 12 + 2\sqrt{96} = E(\Gamma_{nc}(G)) \text{ and}$$
$$E_{CN}(K_{18}) = 2(18-1)(18-2) = 544 > 356 = E_{CN}(\Gamma_{nc}(G)).$$

Thus, $\Gamma_{nc}(G)$ *is neither hypoenergetic, hyperenergetic nor CN-hyperenergetic*. Also, $LE(K_{18}) = 2(18-1) = 34 < \frac{220}{3} = LE(\Gamma_{nc}(G))$ and $LE^+(K_{18}) = 2(18-1) = 34 < \frac{118+6\sqrt{129}}{3} = LE^+(\Gamma_{nc}(G))$. Therefore,

$\Gamma_{nc}(G)$ *is L-hyperenergetic as well as Q-hyperenergetic.* Further,

$$E(\Gamma_{nc}(G)) = 12 + 2\sqrt{96} < \frac{118 + 6\sqrt{129}}{3} = LE^+(\Gamma_{nc}(G)) < \frac{220}{3} = LE(\Gamma_{nc}(G)).$$

If $\Gamma_c(G) \cong K_8 \sqcup 9K_3$, then $\text{Spec}(\Gamma_{nc}(G)) = \Big\{(0)^{25}, (-3)^8, \left(12 + 6\sqrt{10}\right)^1, \left(12 - 6\sqrt{10}\right)^1\Big\}$ and so $E(\Gamma_{nc}(G)) = 24 + 12\sqrt{10}$. We also have

$$\text{L-spec}(\Gamma_{nc}(G)) = \left\{(0)^1, (27)^7, (32)^{18}, (35)^9\right\} \text{ and}$$

$$\text{Q-spec}(\Gamma_{nc}(G)) = \left\{(27)^7, (29)^8, (32)^{18}, \left(\tfrac{83+\sqrt{12073}}{2}\right)^1, \left(\tfrac{83-\sqrt{12073}}{2}\right)^1\right\}.$$

Here, $\frac{2|e(\Gamma_{nc}(G))|}{|v(\Gamma_{nc}(G))|} = \frac{216}{7}$ and $|0 - \frac{216}{7}| = \frac{216}{7}$, $|32 - \frac{216}{7}| = \frac{8}{7}$, $|27 - \frac{216}{7}| = \frac{27}{7}$, $|35 - \frac{216}{7}| = \frac{29}{7}$. Therefore,

$$LE(\Gamma_{nc}(G)) = \tfrac{216}{7} + 18 \cdot \tfrac{8}{7} + 7 \cdot \tfrac{27}{7} + 9 \cdot \tfrac{29}{7} = \tfrac{810}{7}.$$

Similarly, $|29 - \frac{216}{7}| = \frac{13}{7}$, $|\frac{83+\sqrt{12073}}{2} - \frac{216}{7}| \approx \frac{918.14}{14}$, $|\frac{83-\sqrt{12073}}{2} - \frac{216}{7}| \approx \frac{620.14}{14}$ and hence

$$LE^+(\Gamma_{nc}(G)) = 7 \cdot \tfrac{27}{7} + 18 \cdot \tfrac{8}{7} + 8 \cdot \tfrac{13}{7} + \tfrac{918.14}{14} + \tfrac{620.14}{14} \approx \tfrac{2412.28}{14}.$$

Further, $\text{CN-spec}(\Gamma_{nc}(G)) = \Big\{\left(\frac{949+\sqrt{823705}}{2}\right)^1, \left(\frac{949-\sqrt{823705}}{2}\right)^1, (-32)^{18}, (-27)^7, (-23)^8\Big\}$ and so $E_{CN}(\Gamma_{nc}(G)) = 1898$. We have

$$|v(\Gamma_{nc}(G))| = 35 < 24 + 12\sqrt{10} = E(\Gamma_{nc}(G)),$$
$$E(K_{35}) = 2(35-1) = 68 > 24 + 12\sqrt{10} = E(\Gamma_{nc}(G)) \text{ and}$$
$$E_{CN}(K_{35}) = 2(35-1)(35-2) = 2244 > 1898 = E_{CN}(\Gamma_{nc}(G)).$$

Thus, $\Gamma_{nc}(G)$ *is neither hypoenergetic, hyperenergetic nor CN-hyperenergetic.* Also, $LE(K_{35}) = 2(35-1) = 68 < \frac{810}{7} = LE(\Gamma_{nc}(G))$ and $LE^+(K_{35}) = 2(35-1) = 68 < \frac{2412.28}{14} = LE^+(\Gamma_{nc}(G))$. Therefore, $\Gamma_{nc}(G)$ *is L-hyperenergetic as well as Q-hyperenergetic.* Further,

$$E(\Gamma_{nc}(G)) = 24 + 12\sqrt{10} < \frac{810}{7} = LE(\Gamma_{nc}(G)) < \frac{2412.28}{14} = LE^+(\Gamma_{nc}(G)).$$

If $\Gamma_c(G) \cong K_8 \sqcup 9(K_1 \vee 3K_2)$, then $\text{Spec}(\Gamma_{nc}(G)) = \left\{(0)^{34}, (-2)^{18}, (-4)^8, (1)^8, (x_1)^1, (x_2)^1, (x_3)^1\right\}$, where x_1, x_2 and x_3 are roots of the equation

$x^3 - 60x^2 - 472x + 288 = 0$. Since $x_1 \approx 66.98, x_2 \approx -7.55, x_3 \approx 0.569$, we have

$$E(\Gamma_{nc}(G)) = 8 + 36 + 32 + 66.98 + 7.55 + 0.569 \approx 151.09.$$

We also have L-spec$(\Gamma_{nc}(G)) = \big\{(71)^7, (70)^{16}, (68)^{27}, (64)^7, (63)^7, (y_1)^1, (y_2)^1, (y_3)^1, (z_1)^1, (z_2)^1, (z_3)^1, (z_4)^1\big\}$, where y_1, y_2 and y_3 are roots of the equation $x^3 - 205x^2 + 13994x - 318088 = 0$ and z_1, z_2, z_3 and z_4 are roots of the equation $x^4 - 205x^3 + 14010x^2 - 320232x + 71680 = 0$ and Q-spec$(\Gamma_{nc}(G)) = \bigg\{(68)^{27}, (66)^{18}, (63)^7, \left(\frac{129+\sqrt{33}}{2}\right)^8, \left(\frac{129-\sqrt{33}}{2}\right)^8, (l_1)^1, (l_2)^1, (l_3)^1\bigg\}$, where l_1, l_2 and l_3 are roots of the equation $x^3 - 255x^2 + 19848x - 487296 = 0$. Here, $\frac{2|e(\Gamma_{nc}(G))|}{|v(\Gamma_{nc}(G))|} = \frac{4752}{71}$ and $|71 - \frac{4752}{71}| = \frac{289}{71}$, $|70 - \frac{4752}{71}| = \frac{218}{71}$, $|68 - \frac{4752}{71}| = \frac{76}{71}$, $|64 - \frac{4752}{71}| = \frac{208}{71}$, $|63 - \frac{4752}{71}| = \frac{279}{71}$. Since $y_1 \approx 71.63, y_2 \approx 69.07, y_3 \approx 64.20, z_1 \approx 71.49, z_2 \approx 69.15, z_3 \approx 64.21$ and $z_4 \approx 0.226$, we have $|y_1 - \frac{4752}{71}| \approx \frac{333.73}{71}, |y_2 - \frac{4752}{71}| \approx \frac{151.97}{71}, |y_3 - \frac{4752}{71}| \approx \frac{193.8}{71}, |z_1 - \frac{4752}{71}| \approx \frac{323.79}{71}, |z_2 - \frac{4752}{71}| \approx \frac{157.65}{71}, |z_3 - \frac{4752}{71}| \approx \frac{193.09}{71}$ and $|z_4 - \frac{4752}{71}| \approx \frac{4736.38}{71}$. Therefore,

$$\begin{aligned} LE(\Gamma_{nc}(G)) &\approx 7 \cdot \frac{289}{71} + 16 \cdot \frac{218}{71} + 27 \cdot \frac{76}{71} + 7 \cdot \frac{208}{71} + 7 \cdot \frac{279}{71} + \frac{333.73}{71} \\ &\quad + \frac{151.97}{71} + \frac{193.8}{71} + \frac{323.79}{71} + \frac{157.65}{71} + \frac{193.09}{71} + \frac{4736.38}{71} \\ &\approx \frac{17062.41}{71}. \end{aligned}$$

Similarly, $|66 - \frac{4752}{71}| = \frac{66}{71}, |\frac{129+\sqrt{33}}{2} - \frac{4752}{71}| \approx \frac{62.86}{142}, |\frac{129-\sqrt{33}}{2} - \frac{4752}{71}| \approx \frac{752.86}{142}$. Since $l_1 \approx 134.06, l_2 \approx 65.11$ and $l_3 \approx 55.82$, we have $|y_1 - \frac{4752}{71}| \approx \frac{4766.26}{71}, |y_2 - \frac{4752}{71}| \approx \frac{129.19}{71}, |y_3 - \frac{4752}{71}| \approx \frac{788.78}{71}$ and hence

$$\begin{aligned} LE^+(\Gamma_{nc}(G)) &\approx 27 \cdot \frac{76}{71} + 18 \cdot \frac{66}{71} + 7 \cdot \frac{279}{71} + 8 \cdot \frac{62.86}{142} + 8 \cdot \frac{752.86}{142} \\ &\quad + \frac{4766.26}{71} + \frac{129.19}{71} + \frac{788.78}{71} \approx \frac{28280.22}{142}. \end{aligned}$$

Further, CN-spec$(\Gamma_{nc}(G)) = \bigg\{(-68)^{27}, (-64)^{18}, (-63)^7, \left(\frac{-115-\sqrt{217}}{2}\right)^8, \left(\frac{-115+\sqrt{217}}{2}\right)^8, (m_1)^1, (m_2)^1, (m_3)^1\bigg\}$, where m_1, m_2 and m_3 are roots of the equation $x^3 - 4349x^2 - 311676x - 1809504 = 0$. Since $m_1 \approx 4419.69$, $m_2 \approx -64.86$ and $m_3 \approx -6.37$ we have $E_{CN}(\Gamma_{nc}(G)) \approx 8839.83$. We have

$$|v(\Gamma_{nc}(G))| = 71 < 151.09 = E(\Gamma_{nc}(G)),$$
$$E(K_{71}) = 2(71-1) = 140 < 151.09 = E(\Gamma_{nc}(G)) \text{ and}$$
$$E_{CN}(K_{71}) = 2(71-1)(71-2) = 9660 > 8839.83 = E_{CN}(\Gamma_{nc}(G)).$$

Thus, $\Gamma_{nc}(G)$ *is hyperenergetic but neither hypoenergetic nor CN-hyperenergetic.* Also, $LE(K_{71}) = 2(71-1) = 140 < \frac{17062.41}{71} \approx LE(\Gamma_{nc}(G))$ and $LE^+(K_{71}) = 2(71-1) = 140 < \frac{28280.22}{142} \approx LE^+(\Gamma_{nc}(G))$. Therefore, $\Gamma_{nc}(G)$ *is L-hyperenergetic as well as Q-hyperenergetic.* Further,

$$E(\Gamma_{nc}(G)) \approx 151.09 < \frac{28280.22}{142} = LE^+(\Gamma_{nc}(G)) < \frac{17062.41}{71} \approx LE(\Gamma_{nc}(G)).$$

Hence, the result follows. □

Theorem 14. *Let G be a finite non-abelian group such that $\Gamma_c(G)$ is triple-toroidal. Then*

(a) $\Gamma_{nc}(G)$ *is neither hypoenergetic, hyperenergetic nor CN-hyperenergetic.*

(b) $\Gamma_{nc}(G)$ *is L-hyperenergetic as well as Q-hyperenergetic only when $G \cong GL(2,3)$.*

(c) $E(\Gamma_{nc}(G)) \leq LE(\Gamma_{nc}(G)) \leq LE^+(\Gamma_{nc}(G))$.

Proof. From Theorem 5, we have that $\Gamma_c(G)$ is isomorphic to $6K_2 \sqcup 3K_6 \sqcup 4K_4$ or $3K_6$.

If $\Gamma_c(G) \cong 6K_2 \sqcup 3K_6 \sqcup 4K_4$, then $\text{Spec}(\Gamma_{nc}(G)) = \{(0)^{33}, (-2)^5, (-6)^2, (-4)^3, (x_1)^1, (x_2)^1, (x_3)^1\}$, where x_1, x_2 and x_3 are roots of the equation $x^3 - 34x^2 - 312x - 576 = 0$. Since $x_1 \approx -5.08401$, $x_2 \approx -2.71078$, $x_3 \approx 41.7948$, we have

$$E(\Gamma_{nc}(G)) \approx 10 + 12 + 12 + 5.08401 + 2.71078 + 41.7948 = 83.58959.$$

We also have

$$\text{L-spec}(\Gamma_{nc}(G)) = \{(0)^1, (42)^{12}, (40)^{15}, (44)^6, (46)^{12}\} \text{ and}$$
$$\text{Q-spec}(\Gamma_{nc}(G)) = \{(44)^6, (40)^{15}, (42)^{17}, (34)^2, (38)^3, (y_1)^1, (y_2)^1, (y_3)^1\},$$

where y_1, y_2 and y_3 are roots of the equation $x^3 - 160x^2 + 7836x - 121344 = 0$. Here, $\frac{2|e(\Gamma_{nc}(G))|}{|v(\Gamma_{nc}(G))|} = \frac{960}{23}$ and $|0 - \frac{960}{23}| = \frac{960}{23}$, $|42 - \frac{960}{23}| = \frac{6}{23}$, $|40 - \frac{960}{23}| = \frac{40}{23}$, $|44 - \frac{960}{23}| = \frac{52}{23}$, $|46 - \frac{960}{23}| = \frac{98}{23}$. Therefore,

$$LE(\Gamma_{nc}(G)) = \frac{960}{23} + 12 \cdot \frac{6}{23} + 15 \cdot \frac{40}{23} + 6 \cdot \frac{52}{23} + 12 \cdot \frac{98}{23} = \frac{3120}{23}.$$

Similarly, $|44-\frac{960}{23}|=\frac{52}{23}$, $|40-\frac{960}{23}|=\frac{40}{23}$, $|42-\frac{960}{23}|=\frac{6}{23}$, $|34-\frac{960}{23}|=\frac{178}{23}$, $|38-\frac{960}{23}|=\frac{86}{23}$. Since $y_1 \approx 35.7774, y_2 \approx 40.5202$ and $y_3 \approx 83.7024$, we have $|y_1-\frac{960}{23}| \approx 137.1198, |y_2-\frac{960}{23}| \approx 28.0354, |y_3-\frac{960}{23}| \approx 965.1552$ and hence

$$\begin{aligned} LE^+(\Gamma_{nc}(G)) \approx\ & 6\cdot\frac{52}{23}+15\cdot\frac{40}{23}+17\cdot\frac{6}{23}+2\cdot\frac{178}{23}+3\cdot\frac{86}{23} \\ & + 137.1198+28.0354+965.1552=1201.0930. \end{aligned}$$

Further,

$$\begin{aligned} \text{CN-spec}(\Gamma_{nc}(G)) = \big\{ & (-44)^6, (-42)^{12}, (-40)^{20}, (-26)^3, \\ & (-4)^2, (z_1)^1, (z_2)^1, (z_3)^1 \big\}, \end{aligned}$$

where z_1, z_2 and z_3 are roots of the equation $x^3-1654x^2-86336x-921024=0$. Since $z_1 \approx 1704.96, z_2 \approx -35.9132$ and $z_3 \approx -15.042$, it follows that $E_{CN}(\Gamma_{nc}(G)) \approx 3409.9152$. We have

$$\begin{gathered} |v(\Gamma_{nc}(G))| = 46 < 83.58959 = E(\Gamma_{nc}(G)), \\ E(K_{46}) = 2(46-1) = 90 > 83.58959 = E(\Gamma_{nc}(G)) \text{ and} \\ E_{CN}(K_{46}) = 2(46-1)(46-2) = 3960 > 3409.9152 = E_{CN}(\Gamma_{nc}(G)). \end{gathered}$$

Thus, $\Gamma_{nc}(G)$ *is neither hypoenergetic, hyperenergetic nor CN-hyperenergetic*. Also, $LE(K_{46}) = 2(46-1) = 90 < \frac{3120}{23} = LE(\Gamma_{nc}(G))$ and $LE^+(K_{46}) = 2(46-1) = 90 < 1201.0930 = LE^+(\Gamma_{nc}(G))$. Therefore, $\Gamma_{nc}(G)$ *is L-hyperenergetic as well as Q-hyperenergetic*. Further,

$$\begin{aligned} E(\Gamma_{nc}(G)) = 83.58959 <& \frac{3120}{23} \\ &= LE(\Gamma_{nc}(G)) < 1201.0930 = LE^+(\Gamma_{nc}(G)). \end{aligned}$$

If $\Gamma_c(G) \cong 3K_6$, then $\text{Spec}(\Gamma_{nc}(G)) = \{(0)^{15}, (-6)^2, (12)^1\}$ and so $E(\Gamma_{nc}(G)) = 12+12 = 24$. We also have $\text{L-spec}(\Gamma_{nc}(G)) = \{(0)^1, (12)^{15}, (18)^2\}$ and $\text{Q-spec}(\Gamma_{nc}(G)) = \{(6)^2, (12)^{15}, (24)^1\}$. Here, $\frac{2|e(\Gamma_{nc}(G))|}{|v(\Gamma_{nc}(G))|} = 12$ and $|0-12| = 12, |12-12| = 0, |18-12| = 6$. Therefore,

$$LE(\Gamma_{nc}(G)) = 12+0+2\cdot 6 = 24.$$

Similarly, $|6-12| = 6, |12-12| = 0, |24-12| = 12$ and hence

$$LE^+(\Gamma_{nc}(G)) = 2\cdot 6+0+12 = 24.$$

Further, $\text{CN-spec}(\Gamma_{nc}(G)) = \{(132)^1, (24)^2, (-12)^{15}\}$ and so $E_{CN}(\Gamma_{nc}(G)) = 360$. We have

$$|v(\Gamma_{nc}(G))| = 18 < 24 = E(\Gamma_{nc}(G)),$$
$$E(K_{18}) = 2(18-1) = 34 > 24 = E(\Gamma_{nc}(G)) \text{ and}$$
$$E_{CN}(K_{18}) = 2(18-1)(18-2) = 544 > 360 = E_{CN}(\Gamma_{nc}(G)).$$

Thus, $\Gamma_{nc}(G)$ *is neither hypoenergetic, hyperenergetic nor CN-hyperenergetic*. Also, $LE(K_{18}) = 2(18-1) = 34 > 24 = LE(\Gamma_{nc}(G))$ and $LE^+(K_{18}) = 2(18-1) = 34 > 24 = LE^+(\Gamma_{nc}(G))$. Therefore, $\Gamma_{nc}(G)$ *is neither L-hyperenergetic nor Q-hyperenergetic*. Further,

$$E(\Gamma_{nc}(G)) = 24 = LE(\Gamma_{nc}(G)) = LE^+(\Gamma_{nc}(G)).$$

Hence, the result follows. □

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