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RESEARCH ARTICLE

# Minimal non-$BFC$ rings

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Abstract. We study associative rings in which every proper subring is $BFC$ (i.e., has center of finite index) and obtain a characterization of minimal non-$BFC$ unitary rings of finite characteristic.

## Introduction

Let $(R, +, \cdot)$ be an associative ring (not necessary with unity). By analogy with group theory (see e.g. [35, Chapther 14, 14.5]), a ring $R$ is called *an $FC$-ring* (or shortly $FC$) if, for any $a \in R$, the centralizer

$$C_R(a) = \{c \in R \mid c \cdot a = a \cdot c\}$$

is a subgroup of finite index in the additive group $R^+$ of $R$ [6]. In [9] such rings are called $FIC$. Commutative rings and finite rings are $FC$-rings. The concept of a Lie $FC$-ring can be introduced in the same way as for the associative case (see [5, 7]).

A ring $R$ is called *a $BFC$-ring* (or a $PE$-ring as in [9]) if every set of pairwise non-commuting elements is finite. Every $BFC$-ring is $FC$. A ring $R$ is $BFC$ if and only if $|R : Z(R)| < \infty$ (see e.g. [9, Theorem 2.1]). If $R$ is $FC$, then its adjoint group $R^\circ$ (i.e. the group with respect to the circle operation "$\circ$" defined by the rule $x \circ g = x + g + x \cdot g$) is also $FC$. The study of $FC$-groups is an important topic in group theory.

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A number of works are devoted to the study of unitary associative rings $R$ with $FC$-groups of units $U(R)$ (see e.g. [13, 17, 37] and others).

We study rings $R$ for which each proper subring is $BFC$. But first let's describe the terminology.

*Notations.* For a ring $R$, elements $x, g$ of $R$ and a non-empty subset $S$ of $R$

- $[x, g] = x \cdot g - g \cdot x$ is the additive commutator of $g$ and $x$,
- $J(R)$ the Jacobson radical of $R$,
- $F(R) = \{x \in R \mid x \text{ is of finite order in } R^+\}$ the torsion part of $R$,
- $|R : I|$ the index of an additive group $I^+$ of an ideal $I$ in $R^+$,
- $\operatorname{char} R$ the characteristic of $R$,
- $Z(R)$ the center of $R$,
- $\mathbb{P}(R)$ the prime radical of $R$ (i.e., the intersection of all prime ideals of $R$),
- $R' = [R, R]$ the additive subgroup of $R^+$ generated by all $[x, g]$,
- $C(R)$ the commutator ideal of $R$, i.e. the ideal of $R$ generated by all $[x, g]$,
- $Z_0$ the ideal of $R$ generated by all its central ideals,
- $\operatorname{lann} S = \{a \in R \mid a \cdot S = 0\}$, $\operatorname{rann} S = \{a \in R \mid S \cdot a = 0\}$ the left and right annihilators in $R$ and $\operatorname{ann} S = (\operatorname{lann} S) \cap (\operatorname{rann} S)$,
- $\langle X \rangle_{\mathrm{rg}}$ the subring of $R$ generated by $X \subseteq R$ (if $X = \varnothing$, then $\langle X \rangle_{\mathrm{rg}} = 0$),
- $x^\sigma$ is the image of $x$ with respect to a ring endomorphism $\sigma$ of $R$.

If $R$ is a skew field, then its group of units will be denoted by $R^*$. Moreover, for a prime $p$, $\mathbb{F}_{p^n}$ denotes the finite field of $p^n$ elements. Further, we define

$$FC(R) = \{x \in R \mid |R : C_R(x)| < \infty\}.$$

A field in which each nonzero element is a root of 1 is called *absolute*. If $R$ is a nonzero radical (i.e., $R$ is a group with respect to the circle multiplication "$\circ$") $FC$-ring, then the commutator ideal $C(R)$ is proper in $R$ (Proposition 1). A ring (respectively group) $X$ is said to be *minimal non-BFC* if $X$ is not $BFC$ whereas every proper subring (respectively subgroup) of $X$ is $BFC$. The study of minimal non-$BFC$ groups began in [11] (see also [4, 12, 28] and others).

We will say that a local ring $R$ contains *a coefficient ring* $S$ if $S$ is a commutative subring of $R$ such that $R = J(R)+S$ and $R/J(R) \cong S/qS$, where $q$ is a prime or $q = 0$ [16]. The presence of a coefficient ring in a local ring was investigated in [15, 16, 38].

We obtain the following.

**Theorem 1.** *Let $R$ be a unitary ring.*

(i) *If $R$ is a minimal non-$BFC$ ring, then $R$ is local.*

(ii) *If $R$ is a local ring of finite characteristic, then $R$ is a minimal non-$BFC$ ring if and only if $R$ is of one of the following types:*

  ($a$) *$R = \langle x, y\rangle_{\mathrm{rg}}$ is a 2-generated (as a ring) skew field of prime characteristic with commutative proper subrings,*

  ($b$) *$R^+$ is a $p$-group, $R = J(R)+S = C(R)+S$, where $S$ is a finite coefficient ring of $R$, $J(R)$ is commutative, $pR + J(R)^2$ is proper in $J(R)$ and central in $R$, $J(R/Z_0) = \mathbb{F}u$ with $u^2 = 0$,*

$$\mathbb{F} = \bigcup_{n=0}^{\infty} \mathbb{F}_{p^{q^n}} \tag{1}$$

*is an absolute field, $S/pS \cong \mathbb{F}_{p^{q^m}}$ is a finite field ($p$ and $q$ are primes; $m \geq 0$ is an integer), there exists a nontrivial automorphism $\sigma$ of $(S + Z_0)/Z_0$ such that $u\alpha = \alpha^\sigma u$ for each $\alpha \in (S + Z_0)/Z_0$, the multiplicative group $(S/pS)^* = \langle c + pS\rangle$ is cyclic and $c^{\frac{p^{q^m}-1}{p^{q^{m-1}}-1}} \in Z(R)$.*

We also study rings $R$ with the property that all proper subgroups of the adjoint group $R^\circ$ are $BFC$ and prove the next.

**Theorem 2.** *Let $R$ be a ring with every proper subgroup of the adjoint group $R^\circ$ to be $BFC$.*

(1) *If $R$ is nil, then $R^{\circ\prime}$ is proper in $R^\circ$.*

(2) *If $R$ is a radical ring and $R^{\circ\prime}$ is proper in $R^\circ$, then $R$ is $BFC$.*

(3) *If $R$ is a local ring, then $R$ is $BFC$.*

Finally, any unexplained terminology is standard as in [20, 35].

## 1. Preliminaries

First, we establish two lemmas.

**Lemma 1** ([33, Lemma 1]). *If $S$ is a proper subring of finite index in a ring $R$, then there exists an ideal $I$ of $R$ such that $I \subseteq S$ and $|R : I| < \infty$.*

**Lemma 2.** *Let $R$ be an $FC$-ring and $a \in R \setminus Z(R)$. Then there exists a proper ideal $I_a$ of finite index in $R$ such that $I_a \subseteq C_R(a)$ and*

$$I_a \cdot [R, a] = 0 = [R, a] \cdot I_a$$

*where $[R, a] = \{[r, a] \mid r \in R\}$. Moreover, if $R$ is radical, then $\operatorname{ann} R \neq 0$.*

*Proof.* The ideal $I_a$ exists by Lemma 1. Moreover, $rai = ria = ari$ and $ira = air = iar$ for any $r \in R$, $i \in I_a$ and consequently $[a, r]i = 0 = i[a, r]$. This means that $I_a \subseteq \operatorname{ann}[R, a]$.

Assume that $R$ is radical. Since $(R/I_a)^n = 0$ for some positive integer $n$ by [40, Theorem 1], we conclude that

$$R^n \cdot [a, R] = 0 = [a, R] \cdot R^n$$

what implies that $\operatorname{ann} R \neq 0$. □

**Lemma 3.** *A radical ring $R$ is $FC$ (respectively $BFC$) if and only if its adjoint group $R^\circ$ is an $FC$-group (respectively a $BFC$-group).*

*Proof.* It holds because $C_R(a) = C_{R^\circ}(a)$ for any $a \in R$. □

**Lemma 4.** *Let $R$ be a radical ring and $0 \neq a \in R$. If $|R : \operatorname{lann} a| < \infty$, then $\operatorname{rann} R \neq 0$.*

*Proof.* If $\operatorname{lann} a = R$, then $Ra = 0$. Therefore we assume that $\operatorname{lann} a$ is proper in $R$. Then there exists an ideal $I$ of $R$ such that $I \subseteq \operatorname{lann} a$ and $|R : I| < \infty$. Since $R^n \subseteq I$ for some positive integer $n$ in view of [40, Theorem 1] ($n \geq 2$ is the lowest possible integer with such a property), we conclude that $0 \neq R^{n-1}a \subseteq \operatorname{rann} R$. □

A ring $R$ is called *right $T$-nilpotent* if, for every sequence $\{a_i\}_{i=1}^{\infty}$ of its elements, there is a positive integer $n$ such that

$$a_n a_{n-1} \cdots a_1 = 0.$$

A ring $R$ is *$M$-nilpotent* if, for every double sequence

$$\ldots, a_{-2}, a_{-1}, a_0, a_1, a_2, \ldots$$

of its elements, there exist integers $m,\ k\ \ (k \geq 0)$ such that

$$a_m a_{m+1} \cdots a_{m+k} = 0$$

(see [36]). The set $\{a \in R \mid RaR = 0\}$ is called *a middle annihilator* of $R$. From [36] it follows that $R$ is $M$-nilpotent if and only if each its nonzero homomorphic image has nonzero middle annihilator. Each right $T$-nilpotent ring is $M$-nilpotent.

**Corollary 1.** *Let $R$ be a radical ring. If* $\operatorname{lann} a$ *is of finite index in $R$ for any $a \in R$, then $R$ is right $T$-nilpotent.*

*Proof.* In fact, every proper quotient ring of $R$ has a nonzero right annihilator by Lemma 4. Therefore, $R$ is right $T$-nilpotent by [18, Theorems 1.3]. □

The following fact is probably well known.

**Lemma 5.** *If $R$ is a nonzero $M$-nilpotent ring, then $R^2$ is proper in $R$.*

*Proof.* By contrary. Assume that $R = R^2$. Then $R^3 \neq 0$ and therefore there exists an element $r_0 \in R$ such that $0 \neq Rr_0R = R^2r_0R^2$. As a consequence, $Rr_{-1}r_0r_1R \neq 0$ for some $r_{-1}, r_1 \in R$. By the same argument we can construct a sequence $\{r_i \mid i \in \mathbb{Z}\}$ such that

$$r_{-n} \cdots r_{-2}r_{-1}r_0r_1r_2 \ldots r_n \neq 0$$

for each integer $n$, a contradiction. □

**Proposition 1.** *Let $R$ be a nonzero radical ring. If $R$ is an $FC$-ring, then its commutator ideal $C(R)$ is proper in $R$.*

*Proof.* By contrary. Assume that $C(R) = R$. Then $R^2 = R$. Inasmuch as $R/A$ is non-commutative for each proper ideal $A$ of $R$, we deduce that $\operatorname{rann}(R/A) \neq 0$ as consistent with Lemma 2 and [40, Theorem 1]. This yields that $R$ is right $T$-nilpotent by [18, Theorem 1.3]. But then $R^2$ is proper in $R$ by Lemma 5, a contradiction. □

**Corollary 2.** *Let $R$ be a radical $FC$-ring. If* $\operatorname{rann} R = 0$ (*respectively* $\operatorname{lann} R = 0$ *or $R^+$ is torsion-free*), *then $R$ is commutative.*

*Proof.* In fact, if $a \in R$ is non-central, then there exists an ideal $I_a$ of finite index in $R$ such that $[R, a] \subseteq \operatorname{ann} I_a$. Since a finite radical ring is nilpotent by [40, Theorem 1], we obtain that $R^n \subseteq I_a$ ($n$ is minimal possible). Then $\operatorname{ann} R$ is nonzero, a contradiction. □

As follows from [23, Proposition 1] (see e.g. [6, Corollary 2 and Lemma 18]), the commutator ideal $C(R)$ is of finite in $R$ if and only if $|R : Z(R)| < \infty$. If $|R : Z(R)| < \infty$, then $R$ is a $BFC$-ring and the torsion part

$$F(R^\circ) = \{x \in R \mid x \text{ is of finite order in } R^\circ\}$$

is a normal subgroup of the adjoint group $R^\circ$.

Recall that a ring is *prime* if a product of any two its nonzero ideals is nonzero.

From Lemma 2 it holds that a prime $FC$-ring is commutative or finite simple. We can even say more.

**Lemma 6.** *Let $R$ be an infinite prime ring and $I$ its nonzero ideal. If $I$ is $FC$ (as a ring), then $R$ is commutative.*

*Proof.* Assume that $R$ is non-commutative. Then $I$ is non-commutative and so there exists $a \in I \setminus Z(I)$. Then $I$ contains a proper ideal $I_a$ of finite index such that $I_a \subseteq C_I(a)$ and $[I, a] \subseteq \operatorname{ann} I_a = 0$. Since every ideal of a prime ring is a prime ring, we obtain a contradiction.

Thus $R$ is commutative. □

We note that $R^L = (R, +, [-,-])$ is a Lie ring with respect to a Lie bracket "$[-,-]$" defined by the rule $[a,b] = a \cdot b - b \cdot a$ for all $a, b \in R$ (so-called *the associated Lie ring* of $R$). If $U$ is an additive subgroup of $R$ such that $[u, r] \in U$ for any $u \in U$ and $r \in R$, then $U$ is called *a Lie ideal* of $R$. Each ideal of $R$ is its Lie ideal. Obviously that $U$ is a Lie ideal of $R$ if and only if $U$ is an ideal of the Lie ring $R^L$. For example, center $Z(R)$ is a Lie ideal of $R$.

**Lemma 7.** *If $I$ is a Lie ideal of a ring $R$, then its commutator subgroup $I'$ and the commutator ideal $C(I)$ (i.e., generated by $I'$ in $I$) are Lie ideals of $R$.*

*Proof.* Let $a, b \in I$, $r \in R$ and $c \in [I, I]$, where $c = [i, j]$ for $i, j \in I$. Then

$$[r, acb] = [r, a]cb + a[r, c]b + ac[r, b] =$$
$$= [r, a]cb - a[[i, r], j]b - a[[r, j], i]b + ac[r, b] \in C(I).$$

Hence the assertion holds. □

**Lemma 8.** *If $A$ is a finite Lie ideal of a ring $R$, then $|R : C_R(A)| < \infty$.*

*Proof.* Let $a \in A$. The group homomorphism $\psi : R^+ \ni a \mapsto [x, a] \in R^+$ has the finite image $[R, a]$ and so the kernel $\operatorname{Ker}\psi = C_R(a)$ is of finite index in $R^+$. Since $A$ is finite, the result follows. □

Considering the following lemma, we note that infinite rings with finite proper subrings were investigated in [30].

**Lemma 9.** *Let $R$ be an infinite radical ring such that $pR = 0$ for some prime $p$. If all proper left ideals of $R$ are finite, then $R^2 = 0$ and $R = u\mathbb{F}$ for some $u \in R$, where $\mathbb{F}$ is an absolute field of type (1) ($p$ and $q$ are primes).*

*Proof.* Since $Ra$ and $\operatorname{lann} a$ are left ideals of an infinite ring $R$ for each $a \in R$, we conclude that $Ra = 0$. This implies that $R^2 = 0$ and $R$ is an algebra over some field $\mathbb{F}$. Since $R$ is infinite, it is an indecomposable $\mathbb{F}$-module what yields that $R = u\mathbb{F}$ for some $u \in R$. Thus $\mathbb{F}$ is infinite.

If $\mathbb{G}$ is an infinite proper subfield of $\mathbb{F}$, then $u\mathbb{G}$ is an infinite proper ideal of $R$, a contradiction. Moreover, $\mathbb{F}_{p^m}$ is a subfield of $\mathbb{F}_{p^n}$ ($m$ and $n$ are positive integers) if and only if $m$ is a divisor of $n$. Hence $\mathbb{F}$ is of the form (1). □

By $\operatorname{IDer} R$ we denote the set of all inner derivations $\partial_x$ of $R$, where $\partial_x(y) = [x, y]$ $(x, y \in R)$.

**Lemma 10.** *Let $R$ be a ring. Then:*

($i$) *the commutator subgroup $R'$ is finite if and only if the commutator ideal $C(R)$ is finite;*

($ii$) *$R$ is $BFC$ if and only if the commutator subgroup $R'$ is finite.*

*Proof.* ($i$) ($\Leftarrow$) It holds from the fact that $R' \subseteq C(R)$.

($\Rightarrow$) Since the set $\{[x, y] \mid x, y \in R\}$ is finite, we conclude that $|\operatorname{IDer} R| < \infty$. Then $C(R)$ is finite by [6, Lemma 18(5)].

($ii$) This part it follows from [9, Lemma 2.2 and Theorem 2.1], [23, Proposition 1] and [6, Lemma 18(5)]. But we will prove it directly.

($\Leftarrow$) Inasmuch as $R'$ is finite, we conclude that

$$\operatorname{IDer} R = \{\partial_{x_1}, \dots, \partial_{x_n}\}$$

for some $x_1, \dots, x_n \in R$. Then, for every $a \in R$, there exists some $x_i \in R$ $(1 \le i \le n)$ such that $\partial_a = \partial_{x_i}$ and so $a \in x_i + Z(R)$. Hence $|R : Z(R)|$ is finite and $R$ is $BFC$.

($\Rightarrow$) If $R$ is $BFC$, then $|R : Z(R)| < \infty$ by [9, Theorem 2.1] and therefore $R = \langle Z(R), x_1, \ldots, x_n \rangle_{\mathrm{rg}}$ for some $x_1, \ldots, x_n \in R$. Then, for any $r, t \in R$, there exist $i, j$ $(1 \leq i, j \leq n)$ such that $[r, t] = [x_i, x_j]$ because $r = z_1 + x_i$, $t = z_2 + x_j$ for some $z_1, z_2 \in Z(R)$. Thus the set $R' = \{[r, t] \mid r, t \in R\}$ is finite. □

## 2. Unitary rings

Recall that a unitary ring $R$ is *local* if the quotient ring $R/J(R)$ is a skew field.

**Lemma 11.** *Let $R$ be a unitary ring with proper subrings to be $FC$ (respectively $BFC$). If $R$ is not local, then it is $FC$ (respectively $BFC$).*

*Proof.* 1) Assume that $R$ is infinite with proper subrings to be $FC$. Let $M_1$, $M_2$ be different maximal ideals of $R$. Then each $M_i$ is an $FC$-ring and the quotient ring $R/M_i$ (which is isomorphic to $M_j/(M_i \cap M_j)$) is $FC$ (here and everywhere below in the proof $i, j = 1, 2$ and $i \neq j$) and so it is commutative or finite by Lemma 6. Obviously that $M_1 \cap M_2 \subseteq FC(R)$.

We consider the possible cases.

*a*) Suppose that $R/M_1$ and $R/M_2$ are finite. If $x \in M_i \setminus M_j$, then $C_{M_i}(x) \subseteq C_R(x)$, and thus $x \in FC(R)$.

*b*) Let $|R/M_1| = |R/M_2| = \infty$ and $a \in M_i \setminus M_j$. Since $R/M_j$ is an infinite field, we have that $X_j := \langle M_j, a \rangle_{\mathrm{rg}}$ is proper in $R$. Then $C_{M_i}(a) + C_{X_j}(a)$ is of finite index in $R$ and thus $a \in FC(R)$.

*c*) Assume that $|R/M_i| = \infty$, $|R/M_j| < \infty$. Then $M_j \subseteq FC(R)$. Since $M_j$ is $BFC$, we obtain that $R$ contains an ideal $A$ of finite index such that $A \subseteq Z(M_j)$ and $A^2C(R) = 0$. If $A + \langle m \rangle_{\mathrm{rg}} \neq R$ for each $m \in M_i$, then $M_i \subseteq FC(R)$ and so $R$ is $BFC$. Therefore we assume that $A + \langle u \rangle_{\mathrm{rg}} = R$ for some $u \in M_i$. Inasmuch as $A + M_i = R$, we deduce that $A = A^k + M_i \cap A$ for any positive integer $k$. Then from $[u, a_1a_2a_3 + s] = 0$ for any $a_1, a_2, a_3 \in A$ and $s \in M_i \cap A$ it follows that $A \subseteq Z(R)$ what implies that $R$ is $BFC$.

Hence $R$ is $FC$.

2) Now we assume that $R$ is an infinite ring with proper subrings $BFC$. By the part 1), $R$ is $FC$. Assume that $R$ is non-commutative. Applying Lemma 2 we see that $R$ contains a proper ideal $B$ of finite index such that $|B : Z(B)| < \infty$. As a consequence, $R = \langle Z(B), a_1, \ldots, a_n \rangle_{\mathrm{rg}}$ for some its elements $a_1, \ldots, a_n$, where the number $n$ of generators is minimal possible. Then $n \geq 2$. Since $L_i = \langle Z(B), a_i \rangle_{\mathrm{rg}}$ is $BFC$,

$|R : L_i| < \infty$ and $|L_i : Z(L_i)| < \infty$, we conclude that $|R : Z(L_i)| < \infty$ for any $i = 1, \dots, n$. On this basis we obtain that $Z(R)$ is of finite index in $R$ and hence $R$ is $BFC$. □

Then [6, Lemma 6] can be reformulated as follows.

**Lemma 12.** *Let $R$ be a local ring. The following conditions hold.*

*($i$) If $R/J(R)$ is infinite, then $R$ is $FC$ if and only if it is commutative.*

*($ii$) If $R/J(R)$ is finite, then $R$ is $FC$ if and only if $U(R)$ is an $FC$-group.*

*Proof.* ($i$) Assume that $|R/J(R)| = \infty$.

($\Rightarrow$) The quotient ring $R/J(R)$ not contains a proper subring of finite index in view of [24, Corollary 1]. Since $C_R(a)$ is of finite index in $R$ for each $a \in R$, we conclude that $R$ is commutative.

($\Leftarrow$) Obviously.

(2) Now assume that $R/J(R)$ is finite.

($\Rightarrow$) We see that, for any $a \in R$,

$$|R : C_R(a)| < \infty \;\Leftrightarrow\; \left|J(R) : C_{J(R)}(a)\right| < \infty \;\Leftrightarrow\; \Leftrightarrow \left|1 + J(R) : C_{1+J(R)}(a)\right| < \infty$$

and, as a consequence, $U(R)$ is an $FC$-group.

($\Leftarrow$) Inasmuch as the residue field $R/J(R)$ is finite, $(R/J(R))^*$ and $U(R)/(1 + J(R))$ are isomorphic, we deduce that

$$\left|J(R) : C_{J(R)}(a)\right| < \infty \;\Leftrightarrow\; \left|U(R) : C_{U(R)}(a)\right| < \infty$$

for each $a \in R$ and so $R$ is $FC$.

□

**Example 1.** Let $A$ be a field of the form (1), $B = \mathbb{F}_{p^2}$, $a \in A$ and $b \in B$. We consider a local ring

$$R = Au + B = \{au + b \mid a \in A,\ b \in B \text{ and } u^2 = 0\}$$

with a multiplication induced by the rule

$$ua = a^p u.$$

Then

$$[au, b] = a(b^p - b)u,$$

and so the Jacobson radical $J(R) = Au$. Let $S$ be a proper subring of $R$. If $J(R) \subseteq S$, then $S = Au + \mathbb{F}_p$ and thus $S$ is commutative. So, $b^{p-1} \in Z(R)$. Assume that $J(R) \nsubseteq S$. Since $J(R) \cap S$ is proper in $J(R)$ (and thus it is finite) and $S/(J(R) \cap S)$ is a field consists of $p^2$ elements, we obtain that $S$ is finite (and consequently $BFC$). Finally, $|R : C_R(B)| = \infty$ what means that $R$ is not $BFC$, but with proper subrings to be $BFC$.

**Corollary 3.** *A minimal non-BFC (respectively non-FC) unitary ring is local.*

*Proof.* It holds in view of Lemma 11 and Example 1. □

We will need the next fact.

**Lemma 13** ([38, Theorem 2.2])**.** *Let $R$ be a local ring with the nilpotent Jacobson radical $J(R)$. If $R/J(R)$ is a field which is algebraic over $\mathbb{F}_p$, then $R$ has a coefficient ring $S$.*

The next corollary is in the some sense related to [8, Corollary 34(iv)].

**Corollary 4.** *Let $R$ be a local ring with the nilpotent Jacobson radical $J(R)$ and the finite residue field $R/J(R)$. If $J(R)$ is a principal right ideal, then $R$ is finite.*

*Proof.* Assume that $J(R)^n = 0$ for some positive integer $n$ and $J(R) = uR$ for some $u \in R$. Then $R = J(R) + S$ has a coefficient ring $S$ by Lemma 13. Since $p^iS/p^{i+1}S$ and $S/pS$ are isomorphic as additive groups for any positive integer $i$, we conclude that $S$ is finite. Then

$$\begin{aligned} R = J(R) + S = uJ(R) + uS + S = u^2J(R) + u^2S + uS + S = \cdots = \\ = u^{n-1}J(R) + u^{n-1}S + \cdots + uS + S = u^{n-1}S + \cdots + uS + S \end{aligned}$$

is finite. □

**Remark 1.** Let $\mathbb{F}$ be a field of characteristic $p > 0$. Then a nonzero element $\alpha \in \mathbb{F}$ is algebraic over the prime subfield $\mathbb{F}_p$ if and only if the field

$$\mathbb{F}_p(\alpha) = \left\{ \frac{f(\alpha)}{g(\alpha)} \mid f, g \in \mathbb{F}_p[X] \text{ and } g(\alpha) \neq 0 \right\}$$

is finite which is equivalent to that $\alpha$ is of finite order in the multiplicative group $\mathbb{F}_p(\alpha)^*$. Hence the multiplicative group $\mathbb{F}^*$ is torsion infinite if and only if $\mathbb{F}$ is an absolute field.

If $\beta \in \mathbb{F}^*$ is of infinite order, then it is transcendental over $\mathbb{F}_p$. The ring $\mathbb{F}_p[\beta]$ is isomorphic to the polynomial ring $\mathbb{F}_p[X]$, $X^{-1} \notin \mathbb{F}_p[X]$ and so $\mathbb{F}_p[\beta] \neq \mathbb{F}$.

An associative ring $R$ is *Lie nilpotent* if there exists a positive integer $n$ such that

$$[[x_1, \ldots, x_n], x_{n+1}] = 0$$

for all $x_i \in R$ $(i = 1, \ldots, n+1)$.

**Lemma 14.** *Let $R$ be a local ring with the additive $p$-group $R^+$ and let the unit group $(R/J(R))^*$ be infinite torsion. If all proper subrings of $R$ are $BFC$, then $R$ is the ones.*

*Proof.* In view of Remark 1 it follows that the multiplicative group

$$(R/J(R))^* = \bigcup_{i=1}^{\infty} \langle \overline{a}_i \rangle$$

is an union of cyclic subgroups $\langle \overline{a}_i \rangle$ such that $\langle \overline{a}_i \rangle \subseteq \langle \overline{a}_{i+1} \rangle$. Let $a_i$ be an inverse image of $\overline{a}_i$ in $R$. Then each $S_i = J(R) + \langle a_i \rangle_{\mathrm{rg}}$ is $BFC$ and so $|S_i'| < \infty$. Applying Lemmas 7 and 8 and [23, Proposition 1] we conclude that $S_i' \subseteq Z(R)$. By the same argument, if $K := J(R) + \langle a_i, a_j \rangle_{\mathrm{rg}}$ for any positive integers $i$ and $j$, then $K'$ is finite and thus it is central in $R$. Hence $[R, R] \subseteq Z(R)$ and therefore $R$ is Lie nilpotent. The unit group $U(R)$ is also nilpotent by [14, Corollary 3.4] what means that

$$U(R) = (1 + J(R)) \times (R/J(R))^*$$

is a group direct product of an unipotent $p$-group $1 + J(R)$ and a $p'$-group $(R/J(R))^*$. This implies that $[R, R] \subseteq [J(R), J(R)]$ is finite and $R$ is $BFC$. □

Unfortunately, it is not known whether it exists a (non-commutative) skew field with commutative proper subrings (see e.g. [19, 26]).

**Lemma 15.** *Let $R$ be a local ring such that the unit group $(R/J(R))^*$ is non-torsion. If $R$ is a minimal non-$BFC$ ring, then $J(R) \subseteq Z(R)$. Moreover, if $R/J(R)$ is non-commutative, then $R = \langle x, y \rangle_{\mathrm{rg}}$ is a 2-generated (as a ring) skew field of prime characteristic with commutative proper subrings.*

*Proof.* Let $b \in U(R)$ is of infinite order and $N = J(R) + \langle b \rangle_{\mathrm{rg}}$. Then $|N : J(R)| = \infty$. If we assume that $a \in N \setminus Z(N)$, then there exists an ideal $X$ of finite index in $N$ such that $[N, a] \subseteq \operatorname{ann} X$ and so $X \subseteq J(R)$, a contradiction. From this it holds that $N$ is commutative. In common with above it follows that $C_R(J(R))$ is commutative.

If $a \in R \setminus C_R(J(R))$, then $a \in U(R)$ is of finite order. Since $(\langle a \rangle_{\mathrm{rg}} + J(R))/J(R)$ is finite, we deduce that $M = \langle C_R(J(R)), a \rangle_{\mathrm{rg}}$ is proper in $R$ (and consequently it is $BFC$). Inasmuch as $J(R)$ is of infinite index in $C_R(J(R))$, we conclude that $M$ is commutative in view of Lemma 2. Hence $J(R) \subseteq Z(R)$. But then $J(R) \cdot C(R) = 0$. If $R/J(R)$ is non-commutative, then $R = C(R)$ and so $J(R)$ is zero. This implies that $R$ is a skew field by Lemma 6. If $a, b \in R$ are non-commuting, then $R = \langle a, b \rangle_{\mathrm{rg}}$ in view of Lemma 2.

Finally, $pR = 0$ for some prime $p$. □

Given the previous lemma, we note that the following is true.

**Lemma 16.** *Let $R$ be a simple ring with commutative proper subrings. If $[R', R']$ is proper in $R'$ and* $\operatorname{char} R \neq 2$, *then $R$ is commutative.*

*Proof.* Since $[R', R']$ is proper in $R'$, we deduce that $T([R', R']) = \{x \in R \mid [x, R] \subseteq [R', R']\}$ is also proper in $R$ by [20, Lemma 1.4]. Hence $[R', R']$ is abelian and $R' \subseteq Z(R)$ by [21, Lemma 1]. But then $R$ is commutative by [21, Lemma 1]. □

**Lemma 17.** *Let $R$ be a local ring such that $(R/J(R))^*$ is non-torsion abelian. If all proper subrings of $R$ are $BFC$, then $R$ is commutative.*

*Proof.* The Jacobson radical of the ring $R$ is central by Lemma 15. The ring $R$ is non-commutative and thus there are non-commuting $a, b \in R$. Let $E = \langle a, b \rangle_{\mathrm{rg}}$.

$a$) Let $a$ be of infinite order. Since $[E, a] \subseteq \operatorname{ann} I_a \subseteq C_E(a)$ for some ideal $I_a$ of finite index in $E$ by Lemma 2, we deduce that $K = (E + J(R))/J(R)$ is finite, a contradiction.

$b$) Now suppose that $a, b$ are of finite orders. Then $K$ is a finite subfield of $R/J(R)$ with the cyclic multiplicative group $K^*$ and consequently $E$ is commutative, a contradiction.

In this way $R$ is commutative. □

**Lemma 18.** *Let $R$ be a local ring with a coefficient ring $S$. If $R$ is a minimal non-$BFC$ ring, then one of the following holds:*

(i) $(R/J(R))^*$ *is non-torsion;*

(ii) $(R/J(R))^*$ *is infinite torsion,* $R^+$ *is non-torsion,*

$$C(R) \subseteq \bigcap_{n=1}^{\infty} p^n F(R) \subseteq Z(R); \tag{2}$$

(iii) $(R/J(R))^*$ *is finite and* $\operatorname{ann} R' = \operatorname{ann} C(R)$*; moreover, if* $\operatorname{char} R = p$ *is a prime, then* $J(R)$ *is a commutative ring that not contains a proper subring of finite index.*

*Proof.* Assume that $(R/J(R))^*$ is infinite torsion. Then $R/J(R)$ is commutative by [22]. Since $R$ is non-$BFC$, we conclude that $F(R)$ is proper in $J(R)$ in view of Lemma 14. If $a, b \in U(R)$, then $L = J(R) + \langle a, b\rangle_{\mathrm{rg}}$ is proper in $R$ and consequently $C_R(L') = R$ in view of Lemmas 7, 8 and [24, Corollary 1]. This implies that $[R, R] \subseteq Z(R)$ and so $R$ is Lie nilpotent. But then $C(R)$ satisfies (2).

Next assume that $R/J(R)$ is a finite field. Then it is not difficult to see that $\operatorname{ann} R' = \operatorname{ann} C(R)$. Suppose that $R$ is of prime characteristic $p$. If $I$ is an ideal of finite index in $R$ that is proper in $J(R)$, then $I + S$ (and consequently $R$) is $BFC$, a contradiction. So, $J(R)$ not contains a proper subring of finite index in view of Lemma 2. Therefore $J(R)$ is commutative. □

**Proposition 2.** *Let $R$ be a local ring of prime characteristic $p$ such that $J(R)^2 \neq J(R)$ with a coefficient ring $S$. If all proper subrings of $R$ are $BFC$, then $R$ is either $BFC$ or $R = J(R) + S$ is a group direct sum, $J(R) = C(R)$ is nilpotent, $J(R/Z_0) = \mathbb{F}u$, where $u^2 = 0$, $\mathbb{F}$ is an absolute field of prime characteristic $p$, $S \cong \mathbb{F}_{p^{q^m}}$ is a finite field ($p$ and $q$ are primes, $m \geq 0$ is an integer) such that $S^* = \langle c\rangle$ and $c^{\frac{p^{q^m}-1}{p^{q^{m-1}}-1}} \in Z(R)$. In the last case, if $a \in S_1$, where $S_1$ is a homomorphic image of $S$ in $R/Z_0$, then $ua = a^{\sigma}u$ for nontrivial automorphism $\sigma$ of $S_1$.*

*Proof.* Assume that $R$ is not $BFC$. The quotient ring $R/J(R) \cong S$ is commutative and so it is finite by Lemmas 17 and 14. Then Jacobson radical $J(R)$ is commutative by Lemma 18.

$a$) If $A$ is an infinite ideal of $R$ that is proper in $J(R)$, then $A + S$ is $BFC$ and consequently $|A + S : Z(A + S)| < \infty$. This implies that $R$ has an infinite central ideal. Then $Z_0 \neq J(R)$ because $R$ is not commutative.

Consider the quotient ring $B = R/Z_0 = J(B) + S_1$, where $S_1$ is a homomorphic image of $S$ in $B$ and $S_1 \cong S$ are isomorphic. Every ideal $H$

of $B$ that is proper in $J(B)$ is finite and consequently $H\cdot J(B)=0$. Since $J(B)^2$ is finite, we deduce that $|J(B):\operatorname{lann} a|<\infty$ for each $a\in J(B)$ and thus $J(B)=\operatorname{lann} a$. Hence $J(B)^2=0$. If $K$ is a finite ideal of $J(B)$, then $KS_1$ is a finite ideal of $B$ and so $J(B)=u\mathbb{F}$ by Lemma 9, where $\mathbb{F}$ is an absolute field and $u^2=0$. Assume that $B$ is $BFC$ (and so $B'$ is finite). Then $|B:Z(B)|<\infty$ what gives that $J(B)\subseteq Z(B)$. Then $B'\subseteq Z(B)$ and $B$ is Lie nilpotent. This gives that the unit group $U(B)$ is torsion (because $(R/J(R))^*$ is a finite $p'$-group and $1+J(R)$ is a $p$-group by [1, Lemma 2.4]) and nilpotent by [25] and [31]. Since every torsion nilpotent group is a direct product of its Sylow subgroups, we conclude that $B$ is commutative. But then $R'\subseteq Z_0$, $R$ is Lie nilpotent what forces $R$ is commutative, a contradiction. Hence $B$ is not $BFC$.

$b$) From $uS_1\subseteq \mathbb{F}u$ it follows that there exists $a^\sigma\in\mathbb{F}$ such that $ua=a^\sigma u$ for any $a\in S_1$. Then

$$(a+b)^\sigma u=u(a+b)=ua+ub=a^\sigma u+b^\sigma u=(a^\sigma+b^\sigma)u$$

and

$$(ab)^\sigma u=u(ab)=(ua)b=(a^\sigma u)b=a^\sigma(ub)=(a^\sigma b^\sigma)u$$

what means that $\sigma: S_1\ni a\mapsto a^\sigma\in\mathbb{F}$ is a ring homomorphism. Moreover, $\sigma(1)$ is nonzero and so $\sigma(1)=1$. If $a\in S_1$ is nonzero, then $a^l=1$ for some positive integer $l$ and thus $\sigma(a)\neq 0$. This yields that $\sigma$ is a field monomorphism and $\sigma(S_1)$ is isomorphic to some $\mathbb{F}_{p^{q^m}}$.

The multiplicative group $S^*=\langle c\rangle$ is cyclic of order $p^{q^m}-1$. Inasmuch as $|J(R)+E:Z(J(R)+E)|<\infty$ for each proper subfield $E$ of $S$ and $J(R)$ not contains a proper subring of finite index in view of Lemma 18, we deduce that $E$ is central. Since $S$ has unique maximal subfield of order $p^{q^{m-1}}$, we deduce that $c^{\frac{p^{q^m}-1}{p^{q^{m-1}}-1}}\in Z(R)$.

It is not difficult to see that $B'=J(B)$ and therefore $R'+Z_0=J(R)$ what gives that $C(R)+Z_0=J(R)$. Inasmuch as $R'=[R',S]$, we obtain that $C(R)+S=R$. Finally, $J(R)^2\subseteq Z_0$ and so $J(R)^2C(R)=0$. Then $C(R)^3=0$ and consequently $J(R)$ is nilpotent. □

**Lemma 19.** *If $R$ is a minimal non-BFC unitary ring and $R/J(R)$ is finite, then $J(R)^2+pR$ is proper in $J(R)$ and central in $R$, $J(R)$ is commutative and not contains a proper subring of finite index and*

- *($a$) $R^+$ is a $p$-group for some prime $p$, or*
- *($b$) $C(R)$ satisfies (2).*

*Proof.* The ring $R$ is local by Corollary 3. Then $R = J(R) + \langle a\rangle_{\rm rg}$ for some $a \in R$ and so $pR \subseteq J(R)$ for some prime $p$. Since $J(R)$ is $BFC$, we conclude that $|J(R) : Z(J(R))| < \infty$ what yields that $R$ contains an ideal $A \subseteq Z(J(R))$ of finite index. The ring $R$ is not $BFC$ and therefore $A + \langle a\rangle_{\rm rg} = R$. This means that $J(R)$ is commutative. Then $J(R)^2 \subseteq Z(R)$ in view of [27, Lemma 5.4]. Inasmuch as $pJ(R) \subseteq Z(R)$ and $R' = [J(R), \langle a\rangle_{\rm rg}]$, we deduce that

$$[pR, R] = pR' = [pJ(R), \langle a\rangle_{\rm rg}] = 0$$

and consequently $pR \subseteq Z(R)$. Thus $J(R)^2 + pR$ is proper in $J(R)$ and $J(R)$ not contains a proper subring of finite index.

Assume that the torsion part $F(R)$ is proper in $R$. Then $R/F(R)$ is commutative and $C(R) \subseteq F(R)$.

$a$) Suppose that $B = R/Z_0$ is not $BFC$. Then $J(B) = C(B)$ by Proposition 2 what forces that $J(R) = C(R) + Z_0$. If $C(R) + \langle a\rangle_{\rm rg}$ is proper in $R$, then $|R : Z(R)| < \infty$, a contradiction. Hence $C(R) + \langle a\rangle_{\rm rg} = R$. Since $J(R/C(R))$ is isomorphic to $J(R)/C(R)$ (as a ring) and it is of finite index in $R/C(R)$, we deduce that it is nilpotent by [2, Corollary 5.1]. This implies that $R^+$ is torsion and so it is a $p$-group, a contradiction.

$b$) Now let $B$ be $BFC$. Then $B$ has the center $Z(B)$ of finite index what gives that $B$ is commutative and $C(R) \subseteq Z_0$. Hence $R = J(R) + (C(R) + \langle a\rangle_{\rm rg})$ is a nilpotent Lie ring (as a sum of two nilpotent Lie ideals) by the well-known theorem of Jacobson. As a consequence, the unit group $U(R)$ is nilpotent by [14, Corollary 3.4]. The nilpotent unit group $U(R/p^nR)$ is isomorphic to a direct product of a $p$-group $1 + J(R/p^nR)$ and a $p'$-group $(R/J(R))^*$ and therefore it is abelian. Thus $C(R) \subseteq p^nR$. It is not difficult to check that $p^nR \cap F(R) = p^nF(R)$ and so $C(R)$ satisfies (2). □

*Proof of Theorem 1.* (i) It follows in view of Corollary 3.

(ii) We can assume that $\operatorname{char} R = p^k$ for some positive integer $k$ by Lemma 19.

($\Leftarrow$) Let $L$ be a proper subring of $R$. If $J(R) + L$ is proper in $R$, then it is commutative. Therefore we assume that $L$ is non-commutative and $J(R) + L = R$. If $L'$ is finite, then $L$ is $BFC$. Suppose that $L'$ is infinite. Then its image in $R/Z_0$ is infinite and, as a consequence, $R = L + Z_0$ and $R' = L'$. This implies that $C(R) = R'R + RR' = L'L + LL' \subseteq L$.

By Proposition 2, $C(R/pR) = J(R/pR)$ what yields that $R = L + pR$. Then

$$\begin{array}{lllll} p^{k-1}R & = & p^{k-1}L & & \subseteq & L, \\ p^{k-2}R & = & p^{k-1}R + p^{k-2}L = & p^{k-2}L & \subseteq & L, \\ & \vdots & & & \vdots & \\ pR & = & p^2R + pL & & \subseteq & L. \end{array}$$

This gives that $R = L$, a contradiction. Hence $R$ is a minimal non-$BFC$ ring.

($\Rightarrow$) If the multiplicative group $(R/J(R))^*$ is non-torsion, then $R$ is of type $(a)$ in view of Lemmas 15 and 17. Suppose that $(R/J(R))^*$ is torsion. Then it is finite by Lemma 14. Since $R/J(R)$ is a finite field, we conclude that $R = J(R) + \langle a\rangle_{\mathrm{rg}}$ for some $a \in U(R)$. Moreover, the ring $\langle a\rangle_{\mathrm{rg}}$ is finite and local and therefore it has a coefficient ring by Lemma 13. As a consequence, $R = J(R) + S$ and $S \cap J(R) = pS$ (and also $S$) is finite by Corollary 4. By Lemma 19, $J(R)$ is commutative and $pR + J(R)^2 \subseteq Z(R)$ is proper in $J(R)$. The rest follows from Proposition 2. □

**Remark 2.** If $R$ is a minimal non-$BFC$-ring satisfying one of the following conditions:

$(a)$ $F(R)^+$ is of bounded exponent, or

$(b)$

$$\bigcap_{n=1}^{\infty} J(R)^n = 0,$$

then $R^+$ is a $p$-group.

In fact, it holds in view of (2).

## 3. Adjoint groups with proper $BFC$-subgroups

Recall that if $R$ is an unitary ring, then

$$R^\circ \ni a \mapsto 1 + a \in U(R)$$

is a group isomorphism. Note also that in a nil ring each subring is radical. We will often use this in the arguments below without focusing. A group is called *indecomposable*, if it is not generated by a product of two proper subgroups. An indecomposable abelian group is isomorphic to a cyclic $q$-group $\mathbb{C}_{q^n}$ or to a quasicyclic $q$-group $\mathbb{C}_{q^\infty}$ ($q$ is a prime).

**Proposition 3.** *Let $R$ be a radical ring such that the derived subgroup $R^{\circ\prime}$ is proper in the adjoint group $R^\circ$. If every proper subgroup of $R^\circ$ is $BFC$, then $R^\circ$ (and consequently $R$) is $BFC$.*

*Proof.* Assume that $R$ is not $BFC$ and consequently $R^\circ$ is a minimal non-$BFC$ group. Then the quotient group $R^\circ/R^{\circ\prime}$ is a cyclic $p$-group for some prime $p$ in view of [11, Proposition 2] and the derived subgroup $R^{\circ\prime}$ is a group direct product of finitely many quasicyclic groups by [11, Propositions 4 and 3]. Hence $R^\circ$ is metabelian and so the ring $R$ is Lie metabelian by [3, Theorem A]. Then the Lie ideal $[R, R]$ is abelian and therefore the commutator ideal $C(R)$ is nil by [10, Lemma 1.7]. Since $(R/C(R))^\circ$ and $R^\circ/C(R)^\circ$ are isomorphic and abelian, we deduce that $R^{\circ\prime} \subseteq C(R)^\circ$ and $R/C(R)$ is finite in view of [11, Proposition 2]. As following from [40, Theorem 1] $R$ is nil and, moreover, $R = C(R) + \langle a\rangle_{\mathrm{rg}}$. Suppose that $W$ is an infinite proper ideal of $R$. Since $W^\circ$ is a normal subgroup of $R^\circ$ and $R^{\circ\prime}$ not contains an infinite proper normal subgroup of $R^\circ$ by [11, Theorem], we conclude that $R^{\circ\prime} \subseteq W^\circ$ and therefore $C(R) \subseteq W$. The centralizer $C_R(R')$ is a proper subring of finite index in $R$ and $C(R) \subseteq C_R(R')$ in view of [33, Lemma 1]. This gives that $C(R)$ is proper in $R$ and therefore $C(R)$ is commutative.

Let $b \in C(R)$. As far as $bC(R) = C(R)b$, we obtain that $bC(R)$ is an ideal of $R$. If it is finite, then $C(R) \subseteq \operatorname{rann} b$ and so $bC(R) = 0$. In the case when $bC(R)$ is infinite we obtain that $bC(R) = C(R)$, a contradiction. This implies that $C(R)^2 = 0$ (because $C(R)$ is nil). Hence $R$ is a nilpotent ring and $R^\circ$ is a nilpotent group by [25, 31]. But none of the groups described in [11, Theorem] is nilpotent, a contradiction.

As a matter of record $R^\circ$ (and so $R$) is $BFC$, a contradiction. Thus $R$ is $BFC$. □

**Lemma 20.** *Let $R$ be a radical ring such that the adjoint group $R^{\circ\prime} = R^\circ$ is perfect. If $R^\circ$ is a minimal non-$BFC$ group, then $R$ has a homomorphic image $B$, which is a simple commutative domain such that $Ba = B = aB$ for any nonzero $a \in B$ and either*

*(a) $B^+$ is torsion-free divisible, or*

*(b) $pB = 0$ for some prime $p$.*

*If $R$ is a nil ring with every proper subgroup of $R^\circ$ to be $BFC$, then $R^{\circ\prime}$ is proper in $R^\circ$.*

*Proof.* 1) The ring $R$ not contains a proper subring of finite index in view of [40, Theorem 1] and [33, Lemma 1]. Let $S$ be a proper ideal of $R$. Then $S$ is $BFC$ and consequently $S' \subseteq Z(R)$ by Lemma 8. This means that $S$ is Lie nilpotent and so it is (associative) nilpotent by [34, Corollary 2].

As far as $R^2 = R$, we obtain that $R$ is not $M$-nilpotent and therefore there exists some homomorphic image $K$ of $R$ such that $\operatorname{rann} K = 0 = \operatorname{lann} K$ as consistent with [36]. Without loss of generality we may assume that $R = K$.

A commutative proper right ideal $A$ of $R$ is nilpotent in view of [34, Corollary 2]. Suppose that $A$ is a non-commutative proper right ideal of $R$. Since $A$ is a radical ring by [39, Proposition 2.1 2)], it is $BFC$ in view of Lemma 3 what implies that there exists an ideal $X$ of finite index in $A$ such that $X \subseteq Z(A)$ and $X \cdot C(A) = 0$. Since $A^m \subseteq X$ for some positive integer $m$ in view of [40, Theorem 1], we deduce that $A^m \cdot C(A) = 0$. Suppose that $m$ is the smallest number with such a property. Then $0 \neq A^{m-1} \cdot C(A) \subseteq \operatorname{rann} A$ (if $m = 1$, then $C(A) \subseteq \operatorname{rann} A$). Inasmuch as $\operatorname{rann} R = 0$, we conclude that $A + RA$ is a proper ideal of $R$ what yields that $A$ is (associative) nilpotent. Hence all proper one-sided ideals of $R$ are nilpotent.

2) Assume that $R$ is not nil (then $\mathbb{P}(R) \neq R$) and so $B = R/\mathbb{P}(R)$ is without nonzero nilpotent elements (because $C(R) \subseteq \mathbb{P}(R)$). On this basis $Ba = B = aB$ for any $0 \neq a \in B$. For each nonzero $c \in B$ there is some $b \in B$ such that $c = ba$. If we assume that $ac = 0$, then $c^2 = 0$, a contradiction. This implies that $B$ is a domain. Then either $pB = 0$ for some prime $p$ or $B^+$ is divisible (and so torsion-free).

3) If $R$ is nil, then $\langle a \rangle_{\mathrm{rg}} + \langle a \rangle_{\mathrm{rg}} R$ has a nonzero left annihilator for each $a \in R$ and so is proper in $R$ what means that every element of $R$ is contained in a nilpotent ideal (and thus the adjoint group $R^\circ$ is locally nilpotent). If $R^+$ is torsion, then $R^{\circ\prime} \neq R^\circ$ by [12, p. 360, Corollary], a contradiction with assumption. In the other case $pR = R$ for any prime $p$ and so $R^+$ is divisible. Then $R/F(R)$ is a $\mathbb{Q}$-algebra. Every two elements of $R/F(R)$ generates its proper subring and so $R/F(R)$ is commutative. Hence $C(R) \subseteq F(R)$ and the result follows. □

**Remark 3.** From time to time various authors publish works devoted to the search of perfect minimal non-$FC$-groups (see e.g. [28, 32] and others). So far, no such group has been designed.

**Proposition 4.** *Let $R$ be a local ring with the proper commutator ideal $C(R)$. If all proper subgroups of the unit group $U(R)$ are $BFC$, then $U(R)$ (and so $R$) is $BFC$.*

*Proof.* Suppose that $R^\circ$ (which is isomorphic to $U(R)$) is a minimal non-$BFC$ group. If $J(R)^\circ = R^\circ$, then $J(R)^{\circ\prime} = J(R)^\circ$ by Proposition 3 and so $J(R)$ cannot contains a proper ideal of finite index in view of [40, Theorem 1]. As a consequence, a $BFC$-ring $J(R)$ is commutative, a contradiction. Hence $J(R)^\circ$ is proper in $R^\circ$ and $R^{\circ\prime}$ is proper in $R^\circ$.

If $R^\circ/R^{\circ\prime}$ is a product of two different nonzero proper subgroups, then $R^\circ = G_1 \circ G_2$ is a product of two nonzero proper normal subgroups $G_1$ and $G_2$. Since the derived subgroup $G_i'$ is finite and normal in $R^\circ$ $(i = 1, 2)$, we deduce that the quotient group $H := R^\circ/(G_1' \circ G_2')$ is a nilpotent group (as a product of two abelian normal subgroups). The derived subgroup $R^{\circ\prime}$ (and so $H'$) is infinite what gives that $H$ is not $BFC$. Then $H$ is a torsion group of one of types from [11, Theorem]. Since none of these types is a nilpotent group, we get a contradiction. This implies that $R^\circ/R^{\circ\prime}$ is an indecomposable group. If $R^\circ/R^{\circ\prime} \cong \mathbb{C}_{q^\infty}$ is quasicyclic, then $R^\circ$ is $BFC$ by [11, Corollary 2.3], a contradiction. Hence $R^\circ/R^{\circ\prime}$ is finite and so $R/J(R)$ is a finite field. Therefore $J(R)$ is commutative and then $pR \subseteq J(R)$ for some prime $p$.

Let $S$ be a proper subring of $R$. From $S \subseteq J(R)$ it follows that $S$ is $BFC$. If $J(R)$ is properly contained in $S$, then $S$ is a local ring and $S^\circ$ is a proper subgroup (and so it is $BFC$) of $R^\circ$. Then $S$ is a $BFC$-ring by Lemma 12. Thus $R$ is a minimal non-$BFC$ ring and $J(R)^2 \subseteq Z_0$ by Proposition 2. As a consequence, $C(R) \subseteq Z_0$ and so $C(R)^2 = 0$. But then $R^{\circ\prime} \subseteq Z(R)$ and we obtain a contradiction with [11, Propositions 3 and 4]. Hence $R^\circ$ (and so $R$) is $BFC$. □

*Proof of Theorem 2.* (1) It follows from Lemma 20.

(2) This part is proven in Proposition 3.

(3) It is proved in Proposition 4. □

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