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RESEARCH ARTICLE

Certain invariants of generic matrix algebras

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Abstract. Let K be a field of characteristic zero, W be the associative unital algebra generated by two generic traceless matrices X, Y. We also handle the Lie subalgebra L of the algebra W consisting of its Lie elements. Consider the subgroup $G = \langle e_{21} - e_{12} \rangle$ of the special linear group $SL_2(K)$ of order 4. In this study, we give free generators of the algebras W^G and L^G of invariants of the group G as a $C(W)^G$-module.

1. Introduction

Let X and Y be two algebraically independent traceless matrices of size 2×2 with entries from polynomial algebra with six generators. We denote by W the unital associative algebra over a field K of characteristic zero generated by $\{X, Y\}$, and by L its Lie subalgebra generated by Lie elements; i.e., elements that can be expressed as a linear combinations of the form $z_1 z_2 - z_2 z_1$, $z_1, z_2 \in W$ and the elements of the form $\alpha X + \beta Y$, $\alpha, \beta \in K$. It is well known (see [1]) the center $C(W)$ of the algebra W is generated by algebraically independent elements

$$t = 2X^2, \quad u = 2Y^2, \quad v = XY + YX.$$

It was showed in [1] that W is a free $C(W) = K[t, u, v]$-module generated by

$$I, \quad X, \quad Y, \quad [X, Y] = XY - YX,$$

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where $I = 1_W$ is the identity matrix of size 2×2. In the paper [2] (Lemma 1) by Drensky and the second named author, the free generators of the commutator ideal L' of the Lie algebra L as a $K[t,u,v]$-module was given as

$$Xv - Yt, \quad Xu - Yv, \quad [X,Y].$$

Now consider the group G of order four generated by the matrix $M = e_{21} - e_{12}$, that is a subgroup of the special linear group of size 2×2. Then

$$G = \langle M \rangle = \{I, M, M^2, M^3\} = \{I, M, -I, -M\}.$$

We consider the group action of G on W as follows.

$$M \cdot X = Y, \quad M \cdot Y = -X$$

and thus

$$M \cdot t = u, \quad M \cdot u = t, \quad M \cdot v = -v, \quad M \cdot [X,Y] = [X,Y].$$

Similarly we obtain the other actions as below.

$$M^2 \cdot X = -X, \quad M^2 \cdot Y = -Y, \quad M^2 \cdot t = t,$$
$$M^2 \cdot u = u, \quad M^2 \cdot v = v, \quad M^2 \cdot [X,Y] = [X,Y],$$
$$M^3 \cdot X = -Y, \quad M^3 \cdot Y = X, \quad M^3 \cdot t = u,$$
$$M^3 \cdot u = t, \quad M^3 \cdot v = -v, \quad M^3 \cdot [X,Y] = [X,Y].$$

We give the following lemma that will be used in the proofs of main results.

Lemma 1. *Let t, u, v be as above. Then*

$$K[t,u,v]^G = K[t+u, tu, v^2, (u-t)v].$$

Proof. Let $p(t,u,v) \in K[t,u,v]^G$. Then by $p = M \cdot p = M^2 \cdot p = M^3 \cdot p$ we have that

$$p(t,u,v) = p(u,t,-v).$$

Firstly, verifying that $t+u, tu, v^2, (u-t)v \in K[t,u,v]^G$ is straightforward. Hence, it is sufficient to show that

$$K[t+u, tu, v^2, (u-t)v] = \{p \in K[t,u,v] \mid p(t,u,v) = p(u,t,-v)\}$$

for the rest of the proof. Clearly, the first set is a subset of the latter. Now, let $p(t,u,v)$ be a polynomial, taken from the set on the right hand side, of the form

$$\sum_{0 \le a,b,c} \varepsilon_{abc} t^a u^b v^c$$

for some $\varepsilon_{abc} \in K$. Then

$$p(u,t,-v) = \sum_{0\le a,b,c} \varepsilon_{abc} u^a t^b (-v)^c = \sum_{0\le a,b}^{c \text{ even}} \varepsilon_{abc} u^a t^b v^c - \sum_{0\le a,b}^{c \text{ odd}} \varepsilon_{abc} u^a t^b v^c.$$

The equality $p(t,u,v) = p(u,t,-v)$ implies

$$\sum_{0\le a,b}^{c \text{ even}} \varepsilon_{abc} t^a u^b v^c = \sum_{0\le a,b}^{c \text{ even}} \varepsilon_{abc} u^a t^b v^c$$

and

$$\sum_{0\le a,b}^{c \text{ odd}} \varepsilon_{abc} t^a u^b v^c = -\sum_{0\le a,b}^{c \text{ odd}} \varepsilon_{abc} u^a t^b v^c$$

We rewrite $\sum_{0\le a,b}^{c \text{ even}} \varepsilon_{abc} t^a u^b v^c$ as

$$\sum_{a} \varepsilon_{aac} t^a u^a v^{2c'} + \sum_{a<b} \varepsilon_{abc} t^a u^b v^{2c'} + \sum_{a<b} \varepsilon_{bac} t^b u^a v^{2c'}$$

and $\sum_{0\le a,b}^{c \text{ even}} \varepsilon_{abc} u^a t^b v^c$ as

$$\sum_{a} \varepsilon_{aac} u^a t^a v^{2c'} + \sum_{a<b} \varepsilon_{abc} u^a t^b v^{2c'} + \sum_{a<b} \varepsilon_{bac} u^b t^a v^{2c'}$$

assuming that $c = 2c'$. These expressions yield that

$$\sum_{a<b} \varepsilon_{abc} t^a u^b v^{2c'} = \sum_{a<b} \varepsilon_{bac} u^b t^a v^{2c'}$$

and

$$\sum_{a<b} \varepsilon_{bac} t^b u^a v^{2c'} = \sum_{a<b} \varepsilon_{abc} u^a t^b v^{2c'}$$

Therefore

$$\sum_{a<b} (\varepsilon_{abc} - \varepsilon_{bac}) t^a u^b v^{2c'} = 0 = \sum_{a<b} (\varepsilon_{abc} - \varepsilon_{bac}) t^b u^a v^{2c'}$$

Consequently, $\varepsilon_{abc} = \varepsilon_{bac}$ for all a, b in the sum consisting of even powers of v. This implies that

$$\sum_{0\le a,b}^{c \text{ even}} \varepsilon_{abc} t^a u^b v^c = \sum_{a\le b} \varepsilon_{abc} (t^a u^b + t^b u^a)(v^2)^{c'}$$

that is in the algebra generated by $t+u, tu, v^2$, since

$$t^a u^b + t^b u^a \in K[t,u]^{S_2} = K[t+u, tu]$$

are symmetric polynomials in t and u.

Similar computations for the sum consisting of odd powers of v shows that

$$\varepsilon_{aac} = 0\ , \quad \varepsilon_{abc} = -\varepsilon_{bac}\ ,$$

and assuming that $c = 2c' + 1$, we have

$$\begin{aligned}
\sum_{0\leq a,b}^{c \text{ odd}} \varepsilon_{abc} t^a u^b v^c &= \sum_{a<b} \varepsilon_{abc}(t^a u^b - t^b u^a) v^{2c'+1} \\
&= \sum_{a<b} \varepsilon_{abc}(tu)^a (u^{b-a} - t^{b-a}) v^{2c'+1} \\
&= \sum_{a<b} \varepsilon_{abc}(tu)^a \Delta_{ab}(u-t) v (v^2)^{c'}.
\end{aligned}$$

where

$$\Delta_{ab} = u^{b-a-1} + u^{b-a-2}t + \cdots + ut^{b-a-2} + t^{b-a-1} \in K[t,u]^{S_2}$$

which completes the proof. □

2. Main Results

In this section, we provide generators of

$$W^G = \{w \in W \mid w = M \cdot w = M^2 \cdot w = M^3 \cdot w\}$$

and of

$$L^G = \{l \in L \mid l = M \cdot l = M^2 \cdot l = M^3 \cdot l\}$$

as $K[t,u,v]^G = K[t+u, tu, v^2, (u-t)v]$-modules.

Theorem 1. *W^G is freely generated by elements*

$$I, \quad [X, Y]$$

as a $K[t+u, tu, v^2, (u-t)v]$-module.

Proof. Initially, proving that an element in the $K[t+u, tu, v^2, (u-t)v]$-module generated by I and $[X, Y]$ is G-invariant is straightforward.

Now let $w \in W^G \subset W$ be of the form

$$w = Ip_1(t,u,v) + Xp_2(t,u,v) + Yp_3(t,u,v) + [X,Y]p_4(t,u,v)$$

for some $p_i(t,u,v) \in K[t,u,v]$, $i = 1,2,3,4$. Then

$$w = M \cdot w = M^2 \cdot w = M^3 \cdot w$$

gives the followings.

$$\begin{aligned} & Ip_1(t,u,v) + Xp_2(t,u,v) + Yp_3(t,u,v) + [X,Y]p_4(t,u,v) \\ =& Ip_1(u,t,-v) + Yp_2(u,t,-v) - Xp_3(u,t,-v) + [X,Y]p_4(u,t,-v) \\ =& Ip_1(t,u,v) - Xp_2(t,u,v) - Yp_3(t,u,v) + [X,Y]p_4(t,u,v) \\ =& Ip_1(u,t,-v) - Yp_2(u,t,-v) + Xp_3(u,t,-v) + [X,Y]p_4(u,t,-v). \end{aligned}$$

Working in the free $K[t,u,v]$-module W, by comparing the multipliers of generators $I, X, Y, [X,Y]$, we get that

$$\begin{aligned} & p_1(t,u,v) = p_1(u,t,-v), \quad p_4(t,u,v) = p_4(u,t,-v), \\ & p_2(t,u,v) = -p_3(u,t,-v) = -p_2(t,u,v) = p_3(u,t,-v), \\ & p_3(t,u,v) = p_2(u,t,-v) = -p_3(t,u,v) = -p_2(u,t,-v). \end{aligned}$$

This implies that $p_2(t,u,v) = p_3(t,u,v) = 0$, and

$$p_1(t,u,v), p_4(t,u,v) \in K[t+u, tu, v^2, (u-t)v]$$

by Lemma 1. The freeness of the generators I and $[X,Y]$ is a direct consequence of the module structure of W, that completes the proof. □

Theorem 2. *L^G is generated by the element $[X,Y]$ as a $K[t+u,\ tu,\ v^2,$ $(u-t)v]$-module.*

Proof. First of all, we have to show that there are no nonzero linear elements of L^G. Let $l = \alpha X + \beta Y \in L^G$. Then

$$\alpha X + \beta Y = \alpha Y - \beta X = -\alpha X - \beta Y = -\alpha Y + \beta X$$

that implies $\alpha = \beta = 0$. Now let

$$l = (Xv - Yt)p_1(t,u,v) + (Xu - Yv)p_2(t,u,v) + [X,Y]p_3(t,u,v) \in L'$$

be an element in $(L')^G$. Then we have

$$\begin{aligned}
&(Xv - Yt)p_1(t,u,v) + (Xu - Yv)p_2(t,u,v) + [X,Y]p_3(t,u,v)\\
=&(-Yv + Xu)p_1(u,t,-v) + (Yt - Xv)p_2(u,t,-v) + [X,Y]p_3(u,t,-v)\\
=&-(Xv - Yt)p_1(t,u,v) - (Xu - Yv)p_2(t,u,v) + [X,Y]p_3(t,u,v)\\
=&(Yv - Xu)p_1(u,t,-v) + (-Yt + Xv)p_2(u,t,-v) + [X,Y]p_3(u,t,-v)
\end{aligned}$$

that yields

$$\begin{aligned}
p_1(t,u,v) &= -p_2(u,t,-v) = -p_1(t,u,v) = p_2(u,t,-v)\\
p_2(t,u,v) &= p_1(u,t,-v) = -p_2(t,u,v) = -p_1(u,t,-v)\\
p_3(t,u,v) &= p_3(u,t,-v)
\end{aligned}$$

comparing the multipliers of free generators $(Xv-Yt), (Xu-Yv), [X,Y]$ of $K[t,u,v]$-module L'. Hence, $p_1(t,u,v) = p_2(t,u,v) = 0$, and $p_3(t,u,v)$ satisfies the condition of the statement by Lemma 1. Finally checking that an element stated in the theorem is G-invariant is straightforward. □

3. Conclusion

In this study, we examine the algebras W^G and L^G of invariant of a specific group G with its action inherited from the general linear group of size 2×2. In the further studies, one may consider the algebra of invariants of other important subgroups.

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