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RESEARCH ARTICLE

Centralizers of Jacobian derivations

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Dedicated to the 75th birthday of Professor V. M. Bondarenko

Abstract. Let $\mathbb{K}$ be an algebraically closed field of characteristic zero, $\mathbb{K}[x,y]$ the polynomial ring in variables x, y and let $W_2(\mathbb{K})$ be the Lie algebra of all $\mathbb{K}$-derivations on $\mathbb{K}[x,y]$. A derivation $D \in W_2(\mathbb{K})$ is called a Jacobian derivation if there exists $f \in \mathbb{K}[x,y]$ such that $D(h) = \det J(f,h)$ for any $h \in \mathbb{K}[x,y]$ (here $J(f,h)$ is the Jacobian matrix for f and h). Such a derivation is denoted by D_f. The kernel of D_f in $\mathbb{K}[x,y]$ is a subalgebra $\mathbb{K}[p]$ where $p = p(x,y)$ is a polynomial of smallest degree such that $f(x,y) = \varphi(p(x,y)$ for some $\varphi(t) \in \mathbb{K}[t]$. Let $C = C_{W_2(\mathbb{K})}(D_f)$ be the centralizer of D_f in $W_2(\mathbb{K})$. We prove that C is the free $\mathbb{K}[p]$-module of rank 1 or 2 over $\mathbb{K}[p]$ and point out a criterion of being a module of rank 2. These results are used to obtain a class of integrable autonomous systems of differential equations.

1. Introduction

Let $\mathbb{K}$ be an algebraically closed field of characteristic zero, $\mathbb{K}[x,y]$ the polynomial ring in variables x, y and $R = \mathbb{K}(x,y)$ the field of rational functions. Recall that a $\mathbb{K}$-linear map $D : \mathbb{K}[x,y] \longrightarrow \mathbb{K}[x,y]$ is called a $\mathbb{K}$-derivation (or a derivation if $\mathbb{K}$ is fixed) if $D(fg) = D(f)g + fD(g)$ for any $f,g \in \mathbb{K}[x,y]$. All the $\mathbb{K}$-derivations on $\mathbb{K}[x,y]$ form a Lie algebra over $\mathbb{K}$ (denoted by $W_2(\mathbb{K})$) with respect to the operation $[D_1,D_2] = D_1D_2 - D_2D_1$. Every element $D \in W_2(\mathbb{K})$ can be uniquely written in

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the form $D = f(x,y)\partial_x + g(x,y)\partial_y$, where $\partial_x := \frac{\partial}{\partial x}, \partial_y := \frac{\partial}{\partial y}$ are partial derivatives on $\mathbb{K}[x,y]$. The latter means that $W_2(\mathbb{K})$ is a free module of rank 2 over $\mathbb{K}[x,y]$ and $\{\partial_x, \partial_y\}$ is a free basis of this module. The Lie algebra $W_2(\mathbb{K})$ is, from the geometrical point of view, the Lie algebra of all polynomial vector fields on $\mathbb{K}^2$ and was studied intensively from many points of view (see, for example, [5], [4], [10]).

Let $f \in \mathbb{K}[x,y]$. The polynomial f defines a derivation $D_f \in W_2(\mathbb{K})$ by the rule: $D_f(h) = \det J(f,h)$ for any $h \in \mathbb{K}[x,y]$ (here $J(f,h)$ is the Jacobian matrix for f and h). The derivation D_f is called the Jacobian derivation associated with the polynomial f. The kernel $\operatorname{Ker} D_f$ in $\mathbb{K}[x,y]$ is an integrally closed subalgebra of $\mathbb{K}[x,y]$ and $f \in \operatorname{Ker} D_f$. By [7], $\operatorname{Ker} D_f = \mathbb{K}[p]$, where p is a generative closed polynomial for f.

We study the structure of the centralizer $C_{W_2(\mathbb{K})}(D_f)$. This centralizer is of interest because from viewpoint of theory of ODE with any derivation $D = f(x,y)\partial_x + g(x,y)\partial_y$ one can associate an autonomous system of ordinary differential equations

$$\frac{dx}{dt} = f(x,y), \ \frac{dy}{dt} = g(x,y)$$

and elements from $C_{W_2(\mathbb{K})}(D)$ give information about solutions of this system.

We give a criterion for a Jacobian derivation D_f to have the centralizer of rank 2 over $\operatorname{Ker} D_f$ (Theorem 1). We also prove that $C_{W_2(\mathbb{K})}(D_f)$ is a free module over the subalgebra $\mathbb{K}[p]$ of rank 1 or 2 (Theorem 2). We point out an example of integrable system of differential equations associated with a Jacobian derivation of special type.

We use standard notations. If $T = P\partial_x + Q\partial_y$ then the divergence $\operatorname{div} T$ is defined as for a vector field with components P, Q: $\operatorname{div} T = P'_x + Q'_y$. If $T = P\partial_x + Q\partial_y$ is divergence-free (i.e., $\operatorname{div} T = 0$), then $T = D_f$ for a polynomial f that is a "potential" for the vector field determined by T. A polynomial $f \in \mathbb{K}[x,y]$ is called a closed polynomial if the subalgebra $\mathbb{K}[f]$ is integrally closed in the polynomial algebra $\mathbb{K}[x,y]$. For any polynomial $f \in \mathbb{K}[x,y]$ there exists a closed polynomial $p(x,y)$ such that $f = \varphi(p)$ for some polynomial $\varphi \in \mathbb{K}[x,y]$. This polynomial $p(x,y)$ will be called a generative closed polynomial for $f(x,y)$. If L is a subalgebra of the Lie algebra $W_2(\mathbb{K})$ then $\dim_R RL$ will be called the rank of L and denoted by $\operatorname{rk}_{\mathbb{K}[x,y]} L$ or simply by $\operatorname{rk} L$.

2. A criterion for centralizers to be of rank 2

Some properties of derivations on polynomial rings are collected in the next lemma.

Lemma 1. (1) *Let* $D_1, D_2 \in W_2(\mathbb{K})$ *and* $f, g \in \mathbb{K}[x,y]$. *Then*

$$[fD_1, gD_2] = fD_1(g)D_2 - gD_2(f)D_1 + fg[D_1, D_2].$$

(2) *If* $f \in \mathbb{K}[x,y]$ *and* p *is a generative closed polynomial for* f, *then* $\operatorname{Ker} D_f = \mathbb{K}[p]$.

(3) *If* $T \in W_2(\mathbb{K})$ *and* $\operatorname{div} T = 0$, *then* $T = D_g$ *for some polynomial* $g \in \mathbb{K}[x,y]$.

Proof. (1) Direct calculation. (2) See, for example, [9]. (3) See, for example, [8]. □

Lemma 2. *Let* $T \in W_2(\mathbb{K})$ *and* $T(f) = \lambda f$ *for some polynomials* $f, \lambda \in \mathbb{K}[x,y]$. *Then* $[T, D_f] = D_{\lambda f} - (\operatorname{div} T) D_f$.

Proof. Let us write down the derivation T in the form $T = P\partial_x + Q\partial_y$ for some polynomials $P, Q \in \mathbb{K}[x,y]$. Then the condition $T(f) = \lambda f$ can be written in the form

$$Pf'_x + Qf'_y = \lambda f \tag{1}$$

Let us differentiate the equality (1) on x and then on y. We obtain

$$P'_x f'_x + Pf''_{x^2} + Q'_x f'_y + Qf''_{yx} = \lambda'_x f + \lambda f'_x, \tag{2}$$

$$P'_y f'_x + Pf''_{xy} + Q'_y f'_y + Qf''_{y^2} = \lambda'_y f + \lambda f'_y. \tag{3}$$

Further, write down the product of derivations T and D_f in terms of their components:

$$[T, D_f] = (P'_x f'_y - P'_y f'_x - Pf''_{yx} - Qf''_{y^2})\partial_x + (Pf''_{x^2} + Qf''_{xy} + f'_y Q'_x - f'_x Q'_y)\partial_y. \tag{4}$$

Let us denote for convenience $\alpha = -P'_y f'_x - Pf''_{yx} - Qf''_{y^2}$ and $\beta = Pf''_{x^2} + Qf''_{xy} + f'_y Q'_x$. Then using (3) and (2) we see that

$$\alpha = Q' f'_y - \lambda'_y f - \lambda f'_y, \ \beta = \lambda'_x f + \lambda f'_x - P'_x f'_x. \tag{5}$$

The equality (4) can be rewritten in the form

$$[T, D_f] = (P'_x f'_y + \alpha)\partial_x + (\beta - f'_x Q'_y)\partial_y.$$

Inserting in the last equality instead α and β their expressions from (5) we see that

$$[T, D_f] = (P'_x f'_y - \lambda'_y f - \lambda f'_y + Q'_y f'_y)\partial_x + (\lambda'_x f + \lambda f'_x - P'_x f'_x - f'_x Q'_y)\partial_y.$$

After rearranging the summands in the right part of this equality we get

$$[T, D_f] = ((\operatorname{div} T) f'_y - (\lambda f)'_y)\partial_x + ((\lambda f)'_x - (\operatorname{div} T) f'_x)\partial_y$$

The latter means that

$$[T, D_f] = (\operatorname{div} T)(f'_y \partial_x - f'_x \partial_y) + D_{\lambda f} = D_{\lambda f} - (\operatorname{div} T) D_f.$$

The proof is complete. □

Remark 1. The direct calculation shows that $D_{\lambda f} = \lambda D_f + f D_\lambda$. Therefore

$$[T, D_f] = \lambda D_f + f D_\lambda - (\operatorname{div} T) D_f = f D_\lambda + (\operatorname{div} T - \lambda) D_f.$$

Lemma 3. *Let $T \in W_2(\mathbb{K}), f \in \mathbb{K}[x, y]$ be such that $[T, D_f] = 0$. If $T(f) = c$ for some $c \in \mathbb{K}$, then $T = D_g$ for some polynomial $g \in \mathbb{K}[x, y]$.*

Proof. Let us write down the derivation T in the form $T = P\partial_x + Q\partial_y$ for some $P, Q \in \mathbb{K}[x, y]$. Then $P f'_x + Q f'_y = c$ by conditions of the lemma. Differentiating this equality first on x and then on y, we obtain the next equalities

$$P'_x f'_x + P f''_{x^2} + Q'_x f'_y + Q f''_{yx} = 0, \tag{6}$$

$$P'_y f'_x + P f''_{xy} + Q'_y f'_y + Q f''_{y^2} = 0. \tag{7}$$

We see from (6) that

$$P f''_{x^2} + Q'_x f'_y = -P'_x f\_x - Q f''_{xy}$$

and anagously from (7)

$$P'_y f'_x + P_x f''_{xy} = -Q'_y f'_y - Q f''_{y^2}.$$

Therefore it follows from (4) that

$$[T, D_f] = (-Q'_y f'_x - P'_x f'_x)\partial_y + (P'_x f'_y + Q'_y f'_y)\partial_x =$$

$$= f'_y (P'_x + Q'_y)\partial_x - f'_x (P'_x + Q'_y)\partial_y = \operatorname{div} T \cdot (-D_f).$$

Since $[T, D_f] = 0$ by conditions of the lemma, we see from the equality $[T, D_f] = -(\operatorname{div} T) D_f$, that $\operatorname{div} T = 0$. It follows from Lemma 1 that $T = D_g$ for some polynomial $g \in \mathbb{K}[x, y]$. □

Corollary 1. *(1) Let $T \in C_{W_2(\mathbb{K})}(D_f)$. If $T(f) = 0$, then $T = \varphi D_f$ for some rational function $\varphi \in \mathbb{K}(x, y)$ such that $\varphi \in \operatorname{Ker} D_f$. (2) If $T(f) = c$ for some $c \in \mathbb{K}^*$, then $T = D_g$ for some $g \in \mathbb{K}[x, y]$ such that f, g form a Jacobian pair, i.e., $D_g(f) = -D_f(g) = c$.*

Proof. (1) Take any polynomial $h \in \mathbb{K}[x, y]$ such that f, h are algebraically independent over $\mathbb{K}$ and put $g_1 = D_f(h), g_2 = T(h)$. Then $g_1 \neq 0$ because in other case $\operatorname{Ker} D_f$ would be of transcendence degree 2 in $\mathbb{K}(x, y)$ which is impossible. Note that $(g_2 D_f - g_1 T)(h) = 0$ and $(g_2 D_f - g_1 T)(f) = 0$. Since f, h form a transcendence basis of the field $\mathbb{K}(x, y)$, the next equality holds: that $g_2 D_f - g_1 T = 0$. Therefore $T = (g_2/g_1)D_f$. It follows from the equality $[T, D_f] = 0$ that $D_f(g_2/g_1) = 0$, that is $g_2/g_1 \in \operatorname{Ker} D_f$. Denoting $\varphi = g_2/g_1$ we get the proof of part (1) of the corollary.

(2) Since $T(f) = c$ we have by Lemma 1 that $T = D_g$ for some polynomial $g \in \mathbb{K}[x, y]$. But then $T(f) = D_g(f) = \det J(g, f) = c \in \mathbb{K}^*$. The latter means that the polynomials f, g form a Jacobian pair. □

Theorem 1. *Let $f \in \mathbb{K}[x, y], f = \theta(p)$ for a generative closed polynomial $p \in \mathbb{K}[x, y]$ with $\deg \theta \geq 1$. A derivation $T \in W_2(\mathbb{K})$ commutes with D_f if and only if $T(p) = \psi(p)$ for some polynomial $\psi(t) \in \mathbb{K}[t]$ and $\theta''(p)\psi(p) = \theta'(p)(\operatorname{div} T - \psi'(p))$.*

Proof. Let $[T, D_f] = 0$. By Lemma 1, $\operatorname{Ker} D_f = \mathbb{K}[p]$, therefore $T(\mathbb{K}[p]) \subseteq \mathbb{K}[p]$. Then $T(p) = \psi(p)$ for some polynomial $\psi(t) \in \mathbb{K}[t]$. Let us prove the equality

$$\theta''(p)\psi(p) = \theta'(p)(\operatorname{div} T - \psi'(p)). \tag{8}$$

First, let $\deg \psi(t) \geq 1$. Write $\psi(t) = a_0(t - \lambda_1) \ldots (t - \lambda_k)$, where $k \geq 1$ and $\lambda_i \in \mathbb{K}$ (recall that $\mathbb{K}$ is algebraically closed). Therefore $T(p) = a_0(p - \lambda_1) \ldots (p - \lambda_k)$. This equality can be written in the form

$$T(p - \lambda_1) = a_0(p - \lambda_1) \ldots (p - \lambda_k)$$

and taking $p - \lambda_1$ instead of p we can write the last equality as $T(p) = a_0 p(p - \mu_2) \ldots (p - \mu_k)$ for some $\mu_k \in \mathbb{K}$ (note that the polynomial $p - \lambda_1$ is also closed and $\mathbb{K}[p] = \mathbb{K}[p - \lambda_1]$). The last equality can be written in the form

$$T(p) = \psi(p) = p\mu(p) \text{ for } \mu(p) = a_0 p(p - \mu_2) \ldots (p - \mu_k).$$

By Lemma 2, we have

$$[T, D_p] = D_{\psi(p)} - (\operatorname{div} T)D_p = (\psi'(p) - \operatorname{div} T)D_p. \tag{9}$$

But, on the other hand, it follows from the equality

$$[T, D_f] = [T, \theta'(p)D_p] = 0$$

that $T(\theta'(p))D_p = -\theta'(p)[T, D_p]$ and therefore

$$[T, D_p] = -\frac{T(\theta'(p)}{\theta'(p)} D_p \tag{10}$$

(note that $\theta'(p) \neq 0$ because $f = \theta(p)$ and $f \neq const$).

It follows from (9) and (10) that

$$-\frac{T(\theta'(p)}{\theta'(p)} = \psi'(p) - \operatorname{div} T.$$

Therefore it holds the desired equality $\theta''(p)\psi(p) = \theta'(p)(\operatorname{div} T - \psi'(p))$ because $T(\theta'(p)) = \theta''(p)\psi(p)$.

Now let $\deg \psi(t) < 1$. The latter means that $\psi(t) = c \in \mathbb{K}$.

By Lemma 3, $T = D_g$ for some polynomial $g \in \mathbb{K}[x, y]$ and therefore $\operatorname{div} T = 0$. If $c = 0$, that is $\psi(t) \equiv 0$, then obviously (8) holds. Let $c \neq 0$. Then $D_g(p) = c$ and the polynomials p, g form a Jacobian pair. Let us show that $\deg \theta = 1$ in this case. Indeed, in other case

$$[T, D_f] = [D_g, D_f] = [D_g, \theta'(p)D_p] = \theta''(p) \cdot c \cdot D_p + \theta'(p)[D_g, D_p].$$

By Lemma 1, $[D_g, D_p] = D_c = 0$ (recall that $D_g(p) = c$) and therefore

$$[T, D_f] = \theta''(p) \cdot c \cdot D_p = 0.$$

The latter contradicts our choice of the derivation T because $c \neq 0$, and $\theta'' \neq 0$ by our assumption. Therefore $\deg \theta(t) = 1$. Taking into account the relations $\theta''(p) = 0$, $\operatorname{div} T = 0$, $\psi'(p) = 0$ we see that the equality (8) holds.

Let now $T(p) = \psi(p), f = \theta(p)$ for some closed polynomial p, and let the equality (8) hold. Let us show that $[T, D_f] = 0$, i.e. $[T, \theta'(p)D_p] = 0$. The last equality is equivalent to the equality

$$T(\theta'(p))D_p = -\theta'(p)[T, D_p]. \tag{11}$$

First, consider the case $\deg \psi(t) \geq 1$. Then as above one can assume without loss of generality that $\psi(t) = t\lambda(t)$ for some polynomial $\lambda(t) \in \mathbb{K}[t]$. By Lemma 2, we get

$$[T, D_p] = (\psi'(p) - \operatorname{div} T)D_p.$$

Using (11) one can easily show that the latter equality is equivalent to the equality (8):

$$\theta''(p)\psi(p)D_p = \theta'(p)(\operatorname{div} T - \psi'(p))D_p,$$

which holds by our assumptions. So, we have $[T, D_f] = 0$ in the case $\deg \psi(t) \geq 1$.

Consider the case $\deg \psi(t) < 1$, i.e., $\psi(t) = c$ for some $c \in \mathbb{K}$. If $\psi(t) \equiv 0$, then $\operatorname{div} T = 0$ by the equality (8). Therefore (by Lemma 1) $T = D_g$ for some polynomial $g \in \mathbb{K}[x, y]$ and $T(p) = 0 = D_g(p)$. Thus $D_p(g) = 0$, i.e., $g \in \operatorname{Ker} D_p$. It follows from the equality $\operatorname{Ker} D_p = \mathbb{K}[p]$ that $g = \mu(p)$ for some polynomial $\mu(t) \in \mathbb{K}[t]$. But then $T = D_g = \mu'(p)D_p$. Taking into account the equality $D_f = D_{\theta(p)} = \theta'(p)D_p$ we get

$$[T, D_f] = [\mu'(p)D_p, \theta'(p)D_p] = 0$$

because $\mu'(p), \theta'(p) \in \operatorname{Ker} D_p$. Let now $\psi(t) = c, c \in \mathbb{K}^*$. Then $T(p) = c$ and from the conditions of the theorem we have $\theta''(p) \cdot c = \theta'(p) \operatorname{div} T$. The latter equality implies that $\operatorname{div} T = 0$ because $\operatorname{div} T$ is a polynomial and $\deg \theta''(p) < \deg \theta'(p)$. But then $T = D_g$ for some polynomial $g(t) \in \mathbb{K}[t]$. It follows from the conditions of the theorem that $\theta''(p) = 0$ and hence $\theta(p) = \alpha p + \beta$ for some $\alpha, \beta \in \mathbb{K}, \alpha \neq 0$. Without loss of generality one can assume that $f = \theta(p) = p$. We have $T(p) = c$ and $T = D_g$. Then $D_p(g) = -c$ and therefore the polynomials p, g form a Jacobian pair. The latter means that

$$[T, D_f] = [D_g, D_f] = D_{[p.g]} = D_c = 0$$

that is T and D_f commute. The proof of the theorem is complete. □

Corollary 2. *Let $f \in \mathbb{K}[x, y]$ be a closed (in particular, irreducible) polynomial. A derivation $T \in W_2(\mathbb{K})$ commutes with D_f if and only if $T(f) = \psi(f)$ for some polynomial $\psi(t) \in \mathbb{K}[t]$ and $\operatorname{div} T = \psi'(f)$.*

Proof. Since f is closed we can take without loss of generality thet $\theta(t) = t$. Then $\theta''(t) = 0$, and one can easily show that (8) is equivalent to the equality $\operatorname{div} T = \psi'(f)$. □

In [12], a class of Jacobian derivations was studied that was induced by weakly semisimple polynomials $f \in \mathbb{K}[x, y]$ (a polynomial f is called weakly semisimple if the corresponding Jacobian derivation D_f has an eigenfunction $g \in \mathbb{K}[x, y]$ with nonzero eigenvalue $\lambda \in \mathbb{K}$, i.e. if $D_f(g) =$

λg). In [3], such polynomials were described in some cases and some examples were pointed out. Using some results from [3] one can construct Jacobian derivations whose centralizers are of rank 2 over their the ring of constants.

Example 1. Let $f(x,y), g(x,y) \in \mathbb{K}[x,y]$ be nonzero polynomials such that $D_f(g) = g$. Then

$$[fD_g - gD_f, D_g] = -D_g(f)D_g - g[D_f, D_g] = gD_g - gD_g = 0,$$

(here we use the equality $[D_f, D_g] = D_h$, where $h = D_f(g)$). The latter means that the Jacobian derivation D_g has the centralizer in $W_2(\mathbb{K})$ of rank 2 over its ring of constants. This centralizer contains two linearly independent (over $\mathbb{K}[x,y]$) derivations D_g and $fD_g - gD_f$. Let us choose, for example, the polynomials f and g of the form:

$$f(x,y) = x(x-1)y, \quad g(x,y) = x^3(x-1)y^2.$$

One can easily check that $D_f(g) = g$. So, the derivation $D_g = -2yx^3(x-1)\partial_x + (4x^3 - 3x^2)y^2\partial_y$ has the centralizer in $W_2(\mathbb{K})$ of rank 2 over $\mathbb{K}[p]$ and the corresponding system of differential equations

$$\frac{dx}{dt} = -2yx^3(x-1), \ \frac{dy}{dt} = (4x^3 - 3x^2)y^2$$

is integrable (see, for example, [6]).

3. On structure of centralizers of Jacobian derivations

Theorem 2. *Let $f \in \mathbb{K}[x,y]$ be a nonconstant polynomial and D_f the corresponding Jacobian derivation. Let p be a generative closed polynomial for f. Then the centralizer $C_{W_2(\mathbb{K})}(D_f)$ is a free module of rank 1 or 2 over the subring $\mathbb{K}[p]$ of $\mathbb{K}[x,y]$.*

Proof. Since p is a generative closed polynomial for f we have $f = \theta(p)$ for some polynomial $\theta(t) \in \mathbb{K}[t]$. Obviously $D_f = \theta'(p)D_p$ and, by Lemma 1, $\ker D_f = \ker D_p$. Let us denote for brevity $C = C_{W_2(\mathbb{K})}(D_f)$. Obviously C is a module over the subring $\mathbb{K}[p]$ of $\mathbb{K}[x,y]$. Denote $C_1 = \mathbb{K}[x,y]C$, it is obvious that C_1 is a $\mathbb{K}[x,y]$-module and $\mathrm{rk}_{\mathbb{K}[x,y]}C_1 \leq 2$ (recall that $W_2(\mathbb{K})$ is a free $\mathbb{K}[x,y]$-module of rank 2).

First, let $\mathrm{rk}_{\mathbb{K}[x,y]}C_1 = 1$. Let us show that in this case $\mathrm{rk}_{\mathbb{K}[p]}C = 1$ and D_p is a free generator of the module C. Take any $T \in C, T \neq$

0. By our assumptions on C_1, there exist polynomials $g, h \in \mathbb{K}[x, y]$ such that $gD_p + hT = 0$, and at least one of the polynomials g, h is nonzero. Since $D_p \neq 0$ we have $h \neq 0$ and therefore $T = -(g/h)D_p$. But then $D_f(g/h) = 0$. The latter means that the rational function g/h belongs to $\ker_R D_f$, where $\ker_R D_f$ is the kernel of the extension of D_f on $R = \mathbb{K}(x, y)$. Since $\ker_R D_f = \mathbb{K}(p)$ (see, for example [9]) we have $g/h = a(p)/b(p)$ for some polynomials $a(t), b(t) \in \mathbb{K}[t]$ that can be chosen to be coprime. Thus, we obtain the equality $a(p)D_p + b(p)T = 0$ and $a(p), b(p)$ are coprime. From the latter equality we see that $b(p)$ divides D_p, i.e. $D_p = b(p)D_1$ for some derivation $D_1 \in W_2(\mathbb{K})$. But then $b(p) = const$ because $D_p = -p'_y\partial_x + p'_x\partial_y$ and $b(p)$ does not divide p'_y, p'_x if $\deg b(t) > 0$. Thus, we have $T = (-a(p)/b(p))D_p$ and D_p is a free generator for the centralizer $C = C_{W_2(\mathbb{K})}(D_f)$ as a $\mathbb{K}[p]$-module.

Let now $\mathrm{rk}_{\mathbb{K}[x,y]} C_1 = 2$. Choose any $T \in C$ such that T and D_p are linearly independent over $\mathbb{K}[x, y]$. It follows from the equality $[T, D_p] = 0$ that $T(\ker(D_f)) \subseteq \ker(D_f)$, so $T(\mathbb{K}[p]) \subseteq \mathbb{K}[p]$. Therefore $T(p) = \mu(p)$ for some polynomial $\mu(t) \in \mathbb{K}[t]$. Choose among all such $T \in C$ a derivation T_0 such that $\deg\mu_0(t)$ is minimum, where $\mu_0(t)$ is the corresponding polynomial for T_0. One can easily show that for any $T \in C$ its polynomial $\mu(t)$ is divisible by $\mu_0(t)$. Really, let

$$\mu(t) = q(t)\mu_0(t) + r(t)$$

with $\deg r(t) < \deg\mu_0(t)$. Then $T - q(p)T_0 \in C$ and $(T - q(p)T_0)(p) = r(p)$. By our choice of T_0 we have $r(t) = 0$. So, every $T \in C$ can be written in the form $T = q(p)T_0 + T_1$, where $T_1 = T - q(p)T_0$ satisfies the equality $T_1(p) = 0$. Since $[T_1, D_p] = 0$ we can show using Corollary (1) that $T_1 = \delta(p)D_p$ for some polynomial $\delta(t) \in \mathbb{K}[t]$. Therefore $T = q(p)T_0 + \delta(p)D_p$. The latter means that the derivations T_0 and D_p are free generators of the $\mathbb{K}[p]$-module C. The proof is complete. □

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