© Algebra and Discrete Mathematics
Volume **36** (2023). Number 1, pp. 109–128
DOI:10.12958/adm2173

RESEARCH ARTICLE

Structure of relatively free n-tuple semigroups

A. V. Zhuchok

Dedicated to Professor V. M. Bondarenko
on the occasion of his 75th birthday

Abstract. An n-tuple semigroup is an algebra defined on a set with n binary associative operations. This notion was considered by Koreshkov in the context of the theory of n-tuple algebras of associative type. The $n > 1$ pairwise interassociative semigroups give rise to an n-tuple semigroup. This paper is a survey of recent developments in the study of free objects in the variety of n-tuple semigroups. We present the constructions of the free n-tuple semigroup, the free commutative n-tuple semigroup, the free rectangular n-tuple semigroup, the free left (right) k-nilpotent n-tuple semigroup, the free k-nilpotent n-tuple semigroup, and the free weakly k-nilpotent n-tuple semigroup. Some of these results can be applied to constructing relatively free cubical trialgebras and doppelalgebras.

1. Introduction

As usual, $\mathbb{N}$ denotes the set of all positive integers. For $n \in \mathbb{N}$ denote the set $\{1, 2, \ldots, n\}$ by $\overline{n}$. Let us recall that an *n-tuple semigroup* is a nonempty set G equipped with n binary operations $\boxed{1}$, $\boxed{2}, \ldots, \boxed{n}$, satisfying the axioms

$$\left(x \boxed{r}\ y\right) \boxed{s}\ z = x \boxed{r}\ \left(y \boxed{s}\ z\right)$$

2020 Mathematics Subject Classification: *08B20, 20M10, 20M50, 17A30, 17D99.*

Key words and phrases: *n-tuple semigroup, free n-tuple semigroup, relatively free n-tuple semigroup, semigroup.*

for all $x, y, z \in G$ and $r, s \in \overline{n}$. The term "n-tuple semigroup" appears in the article of Koreshkov [12] as a base of the concept of an n-tuple algebra of associative type, while $n > 1$ pairwise interassociative semigroups give rise to an n-tuple semigroup. The theory of interassociative semigroups was actively studied (see, e.g., [2,3,7,8,10]). Semigroups and doppelsemigroups are partial cases of n-tuple semigroups, $n \in \mathbb{N}$. Conversely, if we have an n-tuple semigroup in which $n = 1$ ($n = 2$), then it is a semigroup (doppelsemigroup). Thus, semigroups and doppelsemigroups can be characterized as n-tuple semigroups. The study of doppelsemigroups was initiated by the author in [28] and then it was continued in [5, 6, 25, 30, 33, 35, 36, 41, 44]. Note that doppelalgebras introduced by Richter [17] are linear analogs of doppelsemigroups. These objects have been characterized in [1] as algebras over some operads. Doppelalgebras were also considered in [15] under the name "cubical dialgebras". At the same time, cubical trialgebras introduced by Loday and Ronco [15] are linear analogs of 3-tuple semigroups. For $n = 2$ ($n = 3$), the variety of n-tuple semigroups contains commutative dimonoids (commutative trioids). We remind that dimonoids and trioids were introduced in [14] and [15], respectively, and studied, e.g., in [11, 23, 24, 26, 31, 32, 34, 43]. The algebras, which retained quite a few properties of 2-tuple semigroups, turned out to be bisemigroups [22] and restrictive bisemigroups [21]. One observes that restrictive bisemigroups enable one to effectively study differentiable manifolds [20]. The n-tuple semigroups can be also defined via associative pairs [18].

In [27], the author began the in-depth study of the properties of n-tuple semigroups. It is well-known that every n-tuple semigroup is a homomorphic image of some relatively free n-tuple semigroup. Therefore, a natural step is to consider relatively free n-tuple semigroups. The theory of n-tuple semigroups has developed significantly in the direction of studying free objects in the variety of n-tuple semigroups. The corresponding topic turned out to be very fruitful and rich in interesting results; see the papers [27, 29, 37, 38, 40, 42] devoted to absolutely and relatively free n-tuple semigroups.

In this survey, we systematize the main results that belong to the variety theory of n-tuple semigroups. We focus on the results clarifying the structure of free objects in the variety of n-tuple semigroups. Our purpose is to show which free algebras have already been constructed, and this will allow us to see which free algebras should be constructed further. In section 2, we give examples of n-tuple semigroups. Then, in section 3, we show connections between n-tuple semigroups and some another algebraic structures. Section 4 establishes independence of axioms of n-tuple semigroups. Finally, in sections 5–10, we present the results on the structure of free objects in the variety of n-tuple semigroups. More precisely, we give explicit structure theorems for the free n-tuple semigroup, the free commutative n-tuple semigroup, the free rectangular n-tuple semigroup, the free left (right) k-nilpotent n-tuple semigroup, the free k-nilpotent n-tuple semigroup, and the free weakly k-nilpotent n-tuple semigroup. The indicated results develop the variety theory of algebraic structures and some of them can be useful to constructing relatively free cubical trialgebras and relatively free doppelalgebras.

2. Examples of n-tuple semigroups

We first give examples of n-tuple semigroups.

(a) Let X be an alphabet, and let $F[X]$ be the free semigroup on X. Fix $k, n \in \mathbb{N}$, $k \geq n$. We denote the union of k different copies of $X \times X$ by $(X \times X)_k$. For every pair $(x_1, x_2) \in X \times X$, denote by $(x_1, x_2)_i$ with $1 \leq i \leq k$ the i-th copy of (x_1, x_2), and let

$$D_{(x_1,x_2)} = \{(x_1, x_2)_i \in (X \times X)_k \mid i \in \overline{k}\}.$$

For all $h = (x_1, x_2)_i \in (X \times X)_k$, where $x_1, x_2 \in X$ and $1 \leq i \leq k$, assume that $[h] = x_1x_2 \in F[X]$. For every $i \in \overline{n}$ let

$$\alpha_{\boxed{i}} : X \times X \to (X \times X)_k \cup \{a_1a_2 \in F[X] \mid a_1, a_2 \in X\},$$

$$(x_1, x_2) \mapsto (x_1, x_2)\alpha_{\boxed{i}}$$

be an arbitrary map such that

$$(x_1, x_2)\alpha_{\boxed{i}} = x_1x_2 \in F[X] \quad \text{or} \quad (x_1, x_2)\alpha_{\boxed{i}} \in D_{(x_1,x_2)}.$$

Define n binary operations $\boxed{1}, \boxed{2}, \ldots, \boxed{n}$ on $F[X] \cup (X \times X)_k$ by

$$a_1 \ldots a_m * b_1 \ldots b_s = a_1 \ldots a_m b_1 \ldots b_s, \quad w * h = w[h],$$
$$h * w = [h]w, \quad h * f = [h][f], \quad x_1 * x_2 = (x_1, x_2)\alpha_*$$

for all $a_1 \ldots a_m,\ b_1 \ldots b_s \in F[X]$ such that $ms > 1$, $w \in F[X]$, $h, f \in (X \times X)_k$, $x_1, x_2 \in X$ and $* \in \left\{\boxed{1}, \boxed{2}, \ldots, \boxed{n}\right\}$. The algebra

$$\left(F[X] \cup (X \times X)_k, \boxed{1}, \boxed{2}, \ldots, \boxed{n}\right)$$

is denoted by $X^\sharp(k;n)$.

Proposition 1 ([40], Proposition 3.3)**.** *For any $a,b,c \in X^\sharp(k;n)$ and $i,j,r,d \in \overline{n}$,*

$$\left(a \boxed{i} b\right) \boxed{j} c = a \boxed{r} \left(b \boxed{d} c\right).$$

Corollary 1 ([40], Corollary 3.4)**.** *$X^\sharp(k;n)$ is an n-tuple semigroup.*

(b) An n-tuple semigroup is called an *n-band* [37] if each its operation is idempotent. It is clear that each band can be regarded as an n-band.

Let $n \in \mathbb{N}$ and $G = \{a,b\} \cup \{c_i : i \in \overline{n}\} \cup \{d_i : i \in \overline{n}\}$, and in addition, let the sets $\{a,b\}$, $\{c_i : i \in \overline{n}\}$ and $\{d_i : i \in \overline{n}\}$ be mutually disjoint. We define the operations $\boxed{j}$, $j \in \overline{n}$ on the set G setting

$$\begin{aligned} a \boxed{j} a &= a, & b \boxed{j} b &= b, \\ a \boxed{j} b &= c_j, & b \boxed{j} a &= d_j, \\ a \boxed{j} c_k &= c_k, & b \boxed{j} d_k &= d_k, \\ a \boxed{j} d_k &= c_j, & b \boxed{j} c_k &= d_j, \\ c_k \boxed{j} x &= c_k, & d_k \boxed{j} x &= d_k \end{aligned}$$

for all $x \in G$ and $j,k \in \overline{n}$.

Proposition 2 ([37], Proposition 1)**.** *$\left(G, \boxed{1}, \boxed{2}, \ldots, \boxed{n}\right)$ is an n-band. Moreover, $\left(G, \boxed{i}\right) \cong \left(G, \boxed{j}\right)$ for any $i,j \in \overline{n}$.*

(c) Let T be an arbitrary n-tuple semigroup with operations $\boxed{1}, \boxed{2}, \ldots, \boxed{n}$, and let $a_1, a_2, \ldots, a_n \in T$. Define new operations $\boxed{1}_{a_1}, \boxed{2}_{a_2}, \ldots, \boxed{n}_{a_n}$ on T by

$$x \boxed{i}_{a_i} y = x \boxed{i} a_i \boxed{i} y$$

for all $x,y \in T$ and $i \in \overline{n}$.

Proposition 3 ([27], Proposition 3)**.** *$\left(T, \boxed{1}_{a_1}, \boxed{2}_{a_2}, \ldots, \boxed{n}_{a_n}\right)$ is an n-tuple semigroup.*

The n-tuple semigroup $\left(T, \boxed{1}_{a_1}, \boxed{2}_{a_2}, \ldots, \boxed{n}_{a_n}\right)$ is called a *variant* of T, or, alternatively, a *sandwich n-tuple semigroup* of the algebra T with respect to sandwich elements $a_1, a_2, \ldots, a_n$, or an *n-tuple semigroup with deformed multiplications.* The operations $\boxed{1}_{a_1}, \boxed{2}_{a_2}, \ldots, \boxed{n}_{a_n}$ are called *sandwich operations.*

Other examples of n-tuple semigroups can be found in [27, 40].

3. Connections of n-tuple semigroups

In this section, we establish connections between n-tuple semigroups and some another algebraic structures.

(a) Obviously, every semigroup is an 1-tuple semigroup.

(b) Every semigroup $(S, \cdot)$ can be considered as an n-tuple semigroup T if assume that $T = (S, \underbrace{\cdot, \cdot, \ldots, \cdot}_{n})$.

(c) Let B be an arbitrary nonempty set, and let $(\star, \circ)$ be an ordered pair of binary operations defined on B. Following Schein [21], the pair $(\star, \circ)$ is *associative* if $(x \star y) \circ z = x \star (y \circ z)$ for any $x, y, z \in B$. Associative pairs of operations were considered in [18]. Thus, in each n-tuple semigroup $\left(G, \boxed{1}, \boxed{2}, \ldots, \boxed{n}\right)$, any operations $\boxed{i}$ and $\boxed{j}$, where $i, j \in \overline{n}$, form an associative pair $\left(\boxed{i}, \boxed{j}\right)$. And conversely, binary operations $\boxed{1}$, $\boxed{2}, \ldots, \boxed{n}$ defined on a set G give rise to an n-tuple semigroup if any operations $\boxed{i}$ and $\boxed{j}$, $i, j \in \overline{n}$, form an associative pair $\left(\boxed{i}, \boxed{j}\right)$.

(d) Recall that a *duplex* [16] is a nonempty set D equipped with two binary associative operations $\dashv$ and $\vdash$. Free duplexes were constructed in [16]. Duplexes with an associative pair of operations were considered in the paper [16] in which a free object of rank 1 in the corresponding variety was constructed. A *dimonoid* is a duplex $(D, \dashv, \vdash)$ with an associative pair $(\vdash, \dashv)$ in which the axioms

$$(x \dashv y) \dashv z = x \dashv (y \vdash z), \qquad (x \dashv y) \vdash z = x \vdash (y \vdash z)$$

hold. Dimonoids were introduced by Loday [14] during constructing the universal enveloping algebra for a Leibniz algebra. A dimonoid is called *commutative* [32] if both its operations are commutative.

Proposition 4 ([27], Proposition 2)**.** *Every commutative dimonoid is a 2-tuple semigroup.*

Connections between the n-tuple semigroup $X^{\sharp}(k; n)$ from section 2, item (a) and dimonoids were characterized in [40].

(e) Recall that a *trioid* is a dimonoid $(D, \dashv, \vdash)$ equipped with a binary associative operation $\perp$ satisfying the axioms

$$(x \dashv y) \dashv z = x \dashv (y \perp z), \qquad (x \perp y) \dashv z = x \perp (y \dashv z),$$
$$(x \dashv y) \perp z = x \perp (y \vdash z), \qquad (x \vdash y) \perp z = x \vdash (y \perp z),$$
$$(x \perp y) \vdash z = x \vdash (y \vdash z).$$

Trioids were introduced by Loday and Ronco [15] in the context of algebraic topology. A trioid is called *commutative* [24] if its three operations are commutative.

Proposition 5 ([27], Proposition 1)**.** *Every commutative trioid is a 3-tuple semigroup.*

(f) One can associate 3-tuple semigroups and 2-tuple semigroups to cubical trialgebras and cubical dialgebras, respectively.

Following Loday and Ronco [15], a *cubical trialgebra* is a vector space A equipped with three binary operations: $\dashv$ called *left*, $\vdash$ called *right* and $\perp$ called *middle*, satisfying the following nine axioms:

$$(x \circ_1 y) \circ_2 z = x \circ_1 (y \circ_2 z) \tag{3.1}$$

where $\circ_1$ and $\circ_2$ are either $\dashv$ or $\vdash$ or $\perp$. The free cubical trialgebra on one generator was constructed in [15]. We get the definition of a cubical dialgebra from the definition of a cubical trialgebra by restricting ourself to the first two operations. More precisely, a *cubical dialgebra* [15] is a vector space A equipped with two binary operations: $\dashv$ called *left* and $\vdash$ called *right*, satisfying (3.1) for any $\circ_1, \circ_2 \in \{\dashv, \vdash\}$. This structure under the name "doppelalgebra" has been considered earlier by Richter [17].

It is obvious that every cubical trialgebra (cubical dialgebra) is just a linear analog of a 3-tuple semigroup (2-tuple semigroup). Thus, the results obtained for 3-tuple semigroups (2-tuple semigroups) can be applied to cubical trialgebras (cubical dialgebras).

(g) Consider connections between 2-tuple semigroups and interasociativities of a semigroup.

Let $(D, \vdash)$ be an arbitrary semigroup. Consider a semigroup $(D, \dashv)$ defined on the same set. Recall that $(D, \dashv)$ is an *interassociativity* of $(D, \vdash)$ [3] if $(\dashv, \vdash)$ and $(\vdash, \dashv)$ are associative pairs of operations on D. Descriptions of all interassociativities of the free commutative semigroup, the bicyclic semigroup and a monogenic semigroup were presented in [7, 8, 10]. Some methods of constructing interasociativities of a semigroup were developed in [2]. Thus, in any 2-tuple semigroup $(D, \dashv, \vdash)$, a semigroup $(D, \dashv)$ is an interassociativity of a semigroup $(D, \vdash)$, and conversely, if a semigroup $(D, \dashv)$ is an interassociativity of a semigroup $(D, \vdash)$, then $(D, \dashv, \vdash)$ is a 2-tuple semigroup. Hence, $n > 1$ pairwise interassociative semigroups give rise to an n-tuple semigroup.

A 2-tuple semigroup also has the name "doppelsemigroup" [28].

(h) Show connections between 2-tuple semigroups and restrictive bisemigroups.

Let B be an arbitrary nonempty set, and let $\dashv, \vdash$ be binary associative operations on B. A *restrictive bisemigroup* (see, e.g., [21, 22]) is an ordered triple $(B, \dashv, \vdash)$ with idempotent operations $\dashv$ and $\vdash$ forming an associative pair $(\dashv, \vdash)$ and satisfying the axioms

$$x \dashv y \dashv z = y \dashv x \dashv z, \quad x \vdash y \vdash z = x \vdash z \vdash y.$$

Restrictive bisemigroups have applications in the theory of binary relations. Thus, every 2-tuple semigroup $\left(G, \boxed{1}, \boxed{2}\right)$ is a restrictive bisemigroup if $\left(G, \boxed{1}\right)$ is a right normal band, and $\left(G, \boxed{2}\right)$ is a left normal band. And conversely, every restrictive bisemigroup $(B, \dashv, \vdash)$ with an associative pair $(\vdash, \dashv)$ is a 2-tuple semigroup.

4. Independence of axioms of n-tuple semigroups

This section is devoted to the question on independence of axioms of n-tuple semigroups [27].

A system of axioms $\sum$ is *independent* if any axiom α from $\sum$ cannot be deduced from the system of axioms $\sum \setminus \{\alpha\}$.

Theorem 1 ([27], Theorem 1)**.** *The system of all axioms of n-tuple semigroups ($n > 1$) is independent.*

5. Free n-tuple semigroups

In this section, we construct the free n-tuple semigroup of an arbitrary rank and consider singly generated free n-tuple semigroups [27] separately. We will use notions and notations of section 2.

The class of all n-tuple semigroups forms a variety. An n-tuple semigroup which is free in the variety of n-tuple semigroups is called a *free n-tuple semigroup.*

A *subdirect product of two algebras* A_1 and A_2 is a subalgebra U of the direct product $A_1 \times A_2$ such that the projection maps $U \to A_1$ and $U \to A_2$ are surjections.

Theorem 2 ([27], Theorem 2)**.** *Every free n-tuple semigroup is a subdirect product of a free semigroup on some set and a variant of some n-tuple semigroup defined on a free monoid of rank n.*

Now we give an explicit construction of the free n-tuple semigroup.

Let X be an arbitrary nonempty set, and let ω be an arbitrary word over the alphabet X. The length of ω is denoted by l_ω. By definition, the length of the empty word is equal to 0. Fix $n \in \mathbb{N}$ and let $Y = \{y_1, y_2, \dots, y_n\}$ be an arbitrary set consisting of n elements. Let further $F[X]$ be the free semigroup on X and $F^\theta[Y]$ the free monoid on Y with the empty word θ. Define n binary operations $\boxed{1}, \boxed{2}, \dots, \boxed{n}$ on

$$XY_n = \{(w, u) \in F[X] \times F^\theta[Y] \,|\, l_w - l_u = 1\}$$

by

$$(w_1, u_1) \boxed{i} (w_2, u_2) = (w_1 w_2, u_1 \cdot_{y_i} u_2) \tag{5.1}$$

for all $(w_1, u_1), (w_2, u_2) \in XY_n$ and $i \in \overline{n}$, where $\cdot_{y_i}$ is the sandwich-operation on $F^\theta[Y]$. The algebra $\left(XY_n, \boxed{1}, \boxed{2}, \dots, \boxed{n}\right)$ is denoted by $F_nTS(X)$. By the proof of Theorem 2, $F_nTS(X)$ is the free n-tuple semigroup.

Theorem 2 implies a corollary which describes free n-tuple semigroups of rank 1.

Corollary 2 ([27], Corollary 1)**.** $(F^\theta[Y], \cdot_{y_1}, \dots, \cdot_{y_n})$ *is the free n-tuple semigroup of rank 1.*

In [27], it was proved that the semigroups of the free n-tuple semigroup are isomorphic and its automorphism group is isomorphic to the symmetric group. In the latter paper, it was also shown how a free semigroup can be obtained from the free n-tuple semigroup by a suitable factorization. The least commutative congruence and the least rectangular congruence on the free n-tuple semigroup were presented in [37] and [29], respectively. The problems of the description of the least k-nilpotent congruence and the least weakly k-nilpotent congruence on the free n-tuple semigroup were solved in [40] and [38], respectively. The least left (right) k-nilpotent congruence on the free n-tuple semigroup and on the free semigroup was characterized in [42].

6. Free commutative n-tuple semigroups

In this section, we construct the free commutative n-tuple semigroup of an arbitrary rank and consider singly generated free commutative n-tuple semigroups [37] separately. We will use notions and notations of sections 2 and 5.

An n-tuple semigroup is called *commutative* if all its operations are commutative. The class of all commutative n-tuple semigroups forms a subvariety of the variety of n-tuple semigroups. An n-tuple semigroup which is free in the variety of commutative n-tuple semigroups is called a *free commutative n-tuple semigroup.*

In the construction of $F_nTS(X)$ instead of the free semigroup on X, take the free commutative semigroup $F^*[X]$ on X, and instead of the free monoid on Y, take the free commutative monoid $F_*^\theta[Y]$ on Y with the empty word θ. In this case, denote by $FC_nS(X)$ the algebra $\Big(XY_n, \boxed{1}, \boxed{2}, \ldots, \boxed{n}\Big)$ with operations defined by (5.1).

Theorem 3 ([37], Theorem 4)**.** *$FC_nS(X)$ is the free commutative n-tuple semigroup.*

Corollary 3 ([37], Corollary 3)**.** *$(F_*^\theta[Y], \cdot_{y_1}, \cdot_{y_2}, \ldots, \cdot_{y_n})$ is the free commutative n-tuple semigroup of rank 1.*

In [37], it was stated that the semigroups of the free commutative n-tuple semigroup are isomorphic and its automorphism group is isomorphic to the symmetric group. In addition, the authors constructed a congruence on $FC_nS(X)$ and used it to get a free commutative semigroup from the free commutative n-tuple semigroup. They also found some conditions under which the operations of an arbitrary (commutative) n-tuple semigroup coincide.

7. Free rectangular n-tuple semigroups

In this section, we construct the free rectangular n-tuple semigroup of an arbitrary rank and consider separately free rectangular n-tuple semigroups of rank 1 [29]. We also calculate the cardinality of the free rectangular n-tuple semigroup for the finite case.

A semigroup S is called *rectangular* [32] if $xyz = xz$ foll all $x, y, z \in S$. In [4], the lattice of subvarieties of the variety defined by the identity $xyz = xz$ was indicated. This variety is the union of the variety of left zero semigroups, the variety of right zero semigroups and the variety of zero semigroups, and the lattice of its subvarieties is an 8-element Boolean algebra.

For n-tuple semigroups, it is natural to consider an analog of a rectangular semigroup.

An n-tuple semigroup $\Big(G, \boxed{1}, \boxed{2}, \ldots, \boxed{n}\Big)$ is called *rectangular* if semigroups $\Big(G, \boxed{1}\Big)$, $\Big(G, \boxed{2}\Big)$, …, $\Big(G, \boxed{n}\Big)$ are rectangular. The class of all rectangular n-tuple semigroups forms a subvariety of the variety of

n-tuple semigroups. An n-tuple semigroup which is free in the variety of rectangular n-tuple semigroups is called a *free rectangular n-tuple semigroup.*

Let X be an arbitrary nonempty set, $n \in \mathbb{N}$, and let $Y = \{y_1, \ldots, y_n\}$. Define n binary operations $\boxed{1}, \boxed{2}, \ldots, \boxed{n}$ on $X \cup (Y \times X \times X \times Y)$ by

$$(a_1, b_1, c_1, d_1) \boxed{i} (a_2, b_2, c_2, d_2) = (a_1, b_1, c_2, d_2),$$

$$x \boxed{i} (a_1, b_1, c_1, d_1) = (y_i, x, c_1, d_1), \quad (a_1, b_1, c_1, d_1) \boxed{i} x = (a_1, b_1, x, y_i),$$

$$x \boxed{i} y = (y_i, x, y, y_i)$$

for all $(a_1, b_1, c_1, d_1), (a_2, b_2, c_2, d_2) \in Y \times X \times X \times Y$, $x, y \in X$ and $i \in \overline{n}$. The obtained algebra is denoted by $FR_nS(X)$.

Theorem 4 ([29], Theorem 1)**.** *$FR_nS(X)$ is the free rectangular n-tuple semigroup.*

Corollary 4 ([29], Corollary 1)**.** *The free rectangular n-tuple semigroup $FR_nS(X)$ generated by a finite set X is finite. Specifically, if $|X| = k$, then $|FR_nS(X)| = k(1 + n^2k)$.*

Now we construct an n-tuple semigroup which is isomorphic to the free rectangular n-tuple semigroup of rank 1.

Let e be an arbitrary symbol. Define n binary operations $\boxed{1}, \boxed{2}, \ldots,$ $\boxed{n}$ on $(Y \times Y) \cup \{e\}$ by

$$(a_1, d_1) \boxed{i} (a_2, d_2) = (a_1, d_2), \quad e \boxed{i} (a_1, d_1) = (y_i, d_1),$$

$$(a_1, d_1) \boxed{i} e = (a_1, y_i), \quad e \boxed{i} e = (y_i, y_i)$$

for all $(a_1, d_1), (a_2, d_2) \in Y \times Y$ and $i \in \overline{n}$. The algebra

$$\Big((Y \times Y) \cup \{e\}, \boxed{1}, \boxed{2}, \ldots, \boxed{n}\Big)$$

is denoted by FR_nS_1.

Theorem 4 implies the following statement which describes singly generated free rectangular n-tuple semigroups.

Corollary 5 ([29], Corollary 2)**.** *If $|X| = 1$, then $FR_nS_1 \cong FR_nS(X)$.*

It was established in [29] that the operations of a rectangular n-tuple semigroup coincide if and only if its semigroups are pairwise $\mathcal{P}$-related semigroups. It is also known that the semigroups of the free rectangular n-tuple semigroup are isomorphic and its automorphism group is isomorphic to the symmetric group [29].

8. Free left (right) k-nilpotent n-tuple semigroups

In this section, we construct the free left k-nilpotent n-tuple semigroup of rank 1 and show that the free left k-nilpotent n-tuple semigroup of rank $m > 1$ is a subdirect product of the free left k-nilpotent semigroup with m generators and the free left k-nilpotent n-tuple semigroup of rank 1 [42]. We also calculate the cardinality of the free left k-nilpotent n-tuple semigroup for the finite case. We will use notions and notations of section 5.

Following Schein [19], a semigroup S is called a *left (right) nilpotent semigroup of rank* p if the product of any p elements from this semigroup gives a left (right) zero. Right nilpotent semigroups appear in automata theory, namely, such semigroups are semigroups of self-adaptive automata (see [9,13]). The class of all left nilpotent semigroups of rank p is characterized by the identity $g_1g_2\ldots g_pg_{p+1} = g_1g_2\ldots g_p$. The least such p is called the *left nilpotency index* of a semigroup S [42]. Following [42], for $k \in \mathbb{N}$ a left nilpotent semigroup of left nilpotency index $\leq k$ is said to be a left k-nilpotent semigroup. Right k-nilpotent semigroups are defined dually [42]. The class of all left (right) k-nilpotent semigroups forms a subvariety of the variety of semigroups. A semigroup which is free in the variety of left (right) k-nilpotent semigroups is called a *free left (right) k-nilpotent semigroup.*

For n-tuple semigroups, it is natural to consider an analog of a left (right) nilpotent semigroup of rank p.

An n-tuple semigroup $\left(G, \boxed{1}, \boxed{2}, \ldots, \boxed{n}\right)$ is called *left nilpotent* if for some $m \in \mathbb{N}$, every $x_1, \ldots, x_m$, $x_{m+1} \in G$, and all $i \in \overline{n}$ the following identities hold:

$$(x_1 *_1 \ldots *_{m-1} x_m) \boxed{i}\ x_{m+1} = x_1 *_1 \ldots *_{m-1} x_m,$$

where $*_1, \ldots, *_{m-1} \in \left\{\boxed{1}, \boxed{2}, \ldots, \boxed{n}\right\}$. The least such m is called the *left nilpotency index* of $\left(G, \boxed{1}, \boxed{2}, \ldots, \boxed{n}\right)$. For $k \in \mathbb{N}$ a left nilpotent n-tuple semigroup of left nilpotency index $\leq k$ is said to be a *left k-nilpotent n-tuple semigroup.* Right k-nilpotent n-tuple semigroups are defined dually. The class of all left (right) k-nilpotent n-tuple semigroups forms a subvariety of the variety of n-tuple semigroups. An n-tuple semigroup which is free in the variety of left (right) k-nilpotent n-tuple semigroups is called a *free left (right) k-nilpotent n-tuple semigroup.*

Let $w \in F[X]$. Fix $k, n \in \mathbb{N}$. Following [39], if $l_w \geq k$, let $\overset{k}{\overrightarrow{w}}$ denote the initial subword with the length k of w, and if $l_w < k$, let $\overrightarrow{w}^{k} = w$. We will also regard that $\overrightarrow{u}^{0} = \theta$ for all $u \in F^\theta[Y]$. Let us assume that

$$Y^{(k)} = \{u \in F^\theta[Y] \mid l_u + 1 \leq k\}$$

and define n binary operations $\boxed{1}, \boxed{2}, \ldots, \boxed{n}$ on $Y^{(k)}$ by

$$u_1 \boxed{i}\, u_2 = \overrightarrow{u_1 y_i u_2}^{k-1}$$

for all $u_1, u_2 \in Y^{(k)}$ and $i \in \overline{n}$. The algebra obtained in this way is denoted by $Y_n^{(k)}$.

Theorem 5 ([42], Theorem 3.1). $Y_n^{(k)}$ *is the free left k-nilpotent n-tuple semigroup of rank* 1.

Further, we present the free left k-nilpotent semigroup.

Let $U_k = \{w \in F[X] \,|\, l_w \leq k\}$. A binary operation $\cdot$ is defined on U_k by the rule

$$w_1 \cdot w_2 = \overrightarrow{w_1 w_2}^{k}$$

for all $w_1, w_2 \in U_k$. With respect to this operation U_k is a semigroup generated by X. It is denoted by $FLNS_k(X)$.

Lemma 1 ([42], Lemma 3.2). $FLNS_k(X)$ *is the free left k-nilpotent semigroup.*

Now we are ready to construct the free left k-nilpotent n-tuple semigroup of an arbitrary rank. For this define n binary operations $\boxed{1}, \boxed{2}, \ldots, \boxed{n}$ on

$$W_k = \left\{(w, u) \in FLNS_k(X) \times Y_n^{(k)} \ \mid l_w - l_u = 1\right\}$$

by

$$(w_1, u_1) \boxed{i}\, (w_2, u_2) = \left(\overrightarrow{w_1 w_2}^{k}, \overrightarrow{u_1 y_i u_2}^{k-1}\right)$$

for all $(w_1, u_1), (w_2, u_2) \in W_k$ and $i \in \overline{n}$. The obtained algebra is denoted by $F_{(l)}^{k,n}NS(X)$.

Theorem 6 ([42], Theorem 3.3). $F_{(l)}^{k,n}NS(X)$ *is the free left k-nilpotent n-tuple semigroup.*

Corollary 6 ([42], Corollary 3.4). *The free left k-nilpotent n-tuple semigroup $F_{(l)}^{k,n}NS(X)$ generated by a finite set $X \times \{\theta\}$ is finite. Specifically, $|F_{(l)}^{k,n}NS(X)| = \sum_{i=1}^{k} n^{i-1} \cdot |X|^i$.*

Corollary 7 ([42], Corollary 3.5). *Every free left k-nilpotent n-tuple semigroup of rank $m > 1$ is a subdirect product of the free left k-nilpotent semigroup with m generators and the free left k-nilpotent n-tuple semigroup of rank 1.*

Corollary 8 ([42], Corollary 3.6). *$F_{(l)}^{k,1}NS(X)$ is the free left k-nilpotent semigroup.*

Theorems 5 and 6 imply the following statement.

Corollary 9 ([42], Corollary 3.7). *If $|X| = 1$, then $Y_n^{(k)} \cong F_{(l)}^{k,n}NS(X)$.*

Remark 1 ([42], Remark 3.12). *In order to construct free right k-nilpotent n-tuple semigroups we use the duality principle.*

In [42], it was established that the semigroups of the free left (right) k-nilpotent n-tuple semigroup are isomorphic and its automorphism group is isomorphic to the symmetric group.

9. Free k-nilpotent n-tuple semigroups

In this section, we construct the free k-nilpotent n-tuple semigroup of an arbitrary rank and consider separately free k-nilpotent n-tuple semigroups of rank 1 [40]. Moreover, we calculate the cardinality of the free k-nilpotent n-tuple semigroup for the finite case [40]. We will use notations of section 5.

An element 0 of an n-tuple semigroup $\left(G, \boxed{1}, \boxed{2}, \ldots, \boxed{n}\right)$ is called *zero* if $x{*}0 = 0 = 0{*}x$ for all $x \in G$ and $* \in \left\{\boxed{1}, \boxed{2}, \ldots, \boxed{n}\right\}$. An n-tuple semigroup $\left(G, \boxed{1}, \boxed{2}, \ldots, \boxed{n}\right)$ with zero 0 is called *nilpotent* if for some $m \in \mathbb{N}$ and any $x_i \in G$ with $1 \leq i \leq m+1$, and $*_j \in \left\{\boxed{1}, \boxed{2}, \ldots, \boxed{n}\right\}$ with $1 \leq j \leq m$,

$$x_1 *_1 x_2 *_2 \ldots *_m x_{m+1} = 0.$$

The least such m is called the *nilpotency index* of $\left(G, \boxed{1}, \boxed{2}, \ldots, \boxed{n}\right)$. For $k \in \mathbb{N}$ a nilpotent n-tuple semigroup of nilpotency index $\leq k$ is said to be *k-nilpotent*. The class of all k-nilpotent n-tuple semigroups forms a subvariety of the variety of n-tuple semigroups. An n-tuple semigroup which is free in the variety of k-nilpotent n-tuple semigroups is called a *free k-nilpotent n-tuple semigroup.*

Fix $k \in \mathbb{N}$. Define n binary operations $\boxed{1}, \boxed{2}, \ldots, \boxed{n}$ on

$$XY_{[k]} = \{(w, u) \in F[X] \times F^{\theta}[Y] \,|\, l_w - l_u = 1,\, l_w \leq k\} \cup \{0\} \quad \text{by}$$

$$(w_1, u_1) \boxed{i} (w_2, u_2) = \begin{cases} (w_1w_2, u_1y_iu_2), & l_{w_1w_2} \leq k, \\ 0, & l_{w_1w_2} > k, \end{cases}$$

$$(w_1, u_1) \boxed{i} 0 = 0 \boxed{i} (w_1, u_1) = 0 \boxed{i} 0 = 0$$

for all (w_1, u_1), $(w_2, u_2) \in XY_{[k]} \backslash \{0\}$ and $i \in \overline{n}$. The algebra obtained in this way is denoted by $FN_n^kS(X)$.

Theorem 7 ([40], Theorem 4.1)**.** *$FN_n^kS(X)$ is the free k-nilpotent n-tuple semigroup.*

Corollary 10 ([40], Corollary 4.2)**.** *The free k-nilpotent n-tuple semigroup $FN_n^kS(X)$ generated by a finite set $X \times \{\theta\}$ is finite. Specifically, $|FN_n^kS(X)| = \sum_{i=1}^{k} n^{i-1}|X|^i + 1$.*

Now we construct an n-tuple semigroup which is isomorphic to the free k-nilpotent n-tuple semigroup of rank 1.

Assume that $Y_k = \{u \in F^{\theta}[Y] \mid l_u + 1 \leq k\} \cup \{0\}$. Define n binary operations $\boxed{1}, \boxed{2}, \ldots, \boxed{n}$ on Y_k by

$$u_1 \boxed{i} u_2 = \begin{cases} u_1y_iu_2, & l_{u_1u_2} + 2 \leq k, \\ 0, & l_{u_1u_2} + 2 > k, \end{cases}$$

$$u_1 \boxed{i} 0 = 0 \boxed{i} u_1 = 0 \boxed{i} 0 = 0$$

for all $u_1, u_2 \in Y_k \backslash \{0\}$ and $i \in \overline{n}$. The obtained algebra is denoted by Y_n^k.

Corollary 11 ([40], Corollary 4.3)**.** *If $|X| = 1$, then $Y_n^k \cong FN_n^kS(X)$.*

According to [40], the semigroups of the free k-nilpotent n-tuple semigroup are isomorphic and its automorphism group is isomorphic to the symmetric group.

10. Free weakly k-nilpotent n-tuple semigroups

In this section, we construct the free weakly k-nilpotent n-tuple semigroup of an arbitrary rank and consider separately free weakly k-nilpotent n-tuple semigroups of rank 1 [38]. In addition, we calculate the cardinality of the free weakly k-nilpotent n-tuple semigroup for the finite case [38]. We will use notations of section 5.

A semigroup S is called *nilpotent* if $S^{n+1} = 0$ for some $n \in \mathbb{N}$. The least such n is called the nilpotency index of S. For $k \in \mathbb{N}$ a nilpotent semigroup of nilpotency index $\leq k$ is called *k-nilpotent.* An n-tuple semigroup $\left(G, \boxed{1}, \boxed{2}, \ldots, \boxed{n}\right)$ with zero is called *weakly nilpotent* if $\left(G, \boxed{1}\right)$, $\left(G, \boxed{2}\right), \ldots, \left(G, \boxed{n}\right)$ are nilpotent semigroups. A weakly nilpotent n-tuple semigroup $\left(G, \boxed{1}, \boxed{2}, \ldots, \boxed{n}\right)$ is called *weakly k-nilpotent* if $\left(G, \boxed{1}\right)$, $\left(G, \boxed{2}\right)$, $\ldots$, $\left(G, \boxed{n}\right)$ are k-nilpotent semigroups. It is not difficult to check that the variety of k-nilpotent n-tuple semigroups introduced in [40] (see also section 9) is a subvariety of the variety of weakly k-nilpotent n-tuple semigroups. An n-tuple semigroup which is free in the variety of weakly k-nilpotent n-tuple semigroups is called a *free weakly k-nilpotent n-tuple semigroup.*

For $x \in Y$ and all $u \in F^{\theta}[Y]$, the number of occurrences of the element x in u is denoted by $d_x(u)$. Obviously, $d_x(\theta) = 0$. Fix $k \in \mathbb{N}$ and define n binary operations $\boxed{1}, \boxed{2}, \ldots, \boxed{n}$ on

$$\begin{aligned}\Omega_k = \{(w, u) \in F[X] \times F^{\theta}[Y] \mid l_w - l_u = 1, \\ d_x(u) + 1 \leq k \text{ for all } x \in Y\} \cup \{0\}\end{aligned}$$

by

$$\begin{aligned}&(w_1, u_1) \boxed{i} (w_2, u_2) \\ &\qquad = \begin{cases} (w_1 w_2, u_1 y_i u_2), & d_x(u_1 y_i u_2) + 1 \leq k \text{ for all } x \in Y, \\ 0, & \text{otherwise,} \end{cases} \\ &(w_1, u_1) \boxed{i} 0 = 0 \boxed{i} (w_1, u_1) = 0 \boxed{i} 0 = 0\end{aligned}$$

for all (w_1, u_1), $(w_2, u_2) \in \Omega_k \backslash \{0\}$ and $i \in \overline{n}$. The algebra obtained in this way is denoted by $FNS_n^k(X)$.

Theorem 8 ([38], Theorem 1)**.** *$FNS_n^k(X)$ is the free weakly k-nilpotent n-tuple semigroup.*

In order to calculate the cardinality of Ω_k, let $a_0 = \alpha_0 = \alpha_{-1} = 0$, and let $\alpha_p = \sum_{i=1}^{p} a_i$ for $p \in \overline{n}$ and $\alpha = (a_1, \ldots, a_n) \in \{0, \ldots, k-1\}^n$.

Corollary 12 ([38], Corollary 1)**.** *The free weakly k-nilpotent n-tuple semigroup $FNS_n^k(X)$ generated by a finite set $X \times \{\theta\}$ is finite. Specifically, if $|X| = m$, then*

$$|\Omega_k| = 1 + \sum_{\substack{\alpha = (a_1, \ldots, a_n) \in \\ \{0, \ldots, k-1\}^n}} \left(\prod_{l=0}^{n-1} \begin{pmatrix} \alpha_n - \alpha_{l-1} \\ a_l \end{pmatrix} m^{(1+\alpha_n)} \right).$$

Now we construct an n-tuple semigroup which is isomorphic to the free weakly k-nilpotent n-tuple semigroup of rank 1.

For $k \in \mathbb{N}$ let us assume that

$$Y(k) = \{u \in F^\theta[Y] \mid d_x(u) + 1 \leq k \quad \text{for all} \quad x \in Y\} \cup \{0\}.$$

Define n binary operations $\boxed{1}, \boxed{2}, \ldots, \boxed{n}$ on $Y(k)$ by

$$u_1 \boxed{i}\, u_2 = \begin{cases} u_1 y_i u_2, & d_x(u_1 y_i u_2) + 1 \leq k \quad \text{for all} \quad x \in Y, \\ 0, & \text{otherwise}, \end{cases}$$

$$u_1 \boxed{i}\, 0 = 0 \boxed{i}\, u_1 = 0 \boxed{i}\, 0 = 0$$

for all $u_1, u_2 \in Y(k) \setminus \{0\}$ and $i \in \overline{n}$. The obtained algebra is denoted by $Y(n, k)$.

Corollary 13 ([38], Corollary 2)**.** *If $|X| = 1$, then $Y(n, k) \cong FNS_n^k(X)$.*

In [38], some properties of the free weakly k-nilpotent n-tuple semigroup were described. More precisely, the authors characterized all maximal n-tuple subsemigroups of the free weakly k-nilpotent n-tuple semigroup and gave a criterion for an isomorphism of endomorphism semigroups of free weakly k-nilpotent n-tuple semigroups. They also considered separately the question of regularity in the endomorphism semigroup of $FNS_n^k(X)$. Moreover, in [38], it was established that the semigroups of the free weakly k-nilpotent n-tuple semigroup are isomorphic and its automorphism group is isomorphic to the symmetric group.

Remark 2. *The free product of arbitrary n-tuple semigroups was constructed in [37]. In the latter paper, the authors introduced the notion of n-band of n-tuple semigroups and used it to describe the structure of free products of n-tuple semigroups. They also characterized one congruence on the free product of n-tuple semigroups in order to get a free product of semigroups from the free product of n-tuple semigroups.*

Note that the main results of sections 5–10 can be applied to constructing the corresponding relatively free cubical trialgebras and relatively free doppelalgebras.

Acknowledgments

The present paper was written during the research stay of the author at the Institute of Mathematics of the University of Potsdam (Germany) within the framework of the Philipp Schwartz Initiative program of the Alexander von Humboldt Foundation. The author would like to thank Dr. Jörg Koppitz for inviting to the Institute of Mathematics to work on this project and the University of Potsdam for hospitality. The author is also grateful to Claudia Rößling and Carolin Switala from the Welcome Center Potsdam for planning and implementing his stay in Potsdam and comprehensive information, advice and continuous support. In addition, the author would like to express his sincere gratitude to Apatsara Sareeto for the help with some organizational questions.

References

[1] M. Aguiar, M. Livernet, *The associative operad and the weak order on the symmetric groups*, J. Homotopy Relat. Struct., **2**, no. 1, 57–84 (2007).

[2] S.J. Boyd, M. Gould, A.W. Nelson, *Interassociativity of semigroups*, In: Proceedings of the Tennessee Topology Conference, Nashville, TN, USA, 1996. Singapore: World Scientific, 33–51 (1997).

[3] M. Drouzy, *La structuration des ensembles de semigroupes d'ordre 2, 3 et 4 par la relation d'interassociativité*, Manuscript (1986).

[4] T. Evans, *The lattice of semigroup varieties*, Semigroup Forum, **2**, 1–43 (1971). doi: 10.1007/BF02572269.

[5] V. Gavrylkiv, *Note on cyclic doppelsemigroups*, Algebra Discrete Math., **34**, no. 1, 15–21 (2022). doi: 10.12958/adm1991.

[6] V. Gavrylkiv, D. Rendziak, *Interassociativity and three-element doppelsemigroups*, Algebra Discrete Math., **28**, no. 2, 224–247 (2019).

[7] B.N. Givens, K. Linton, A. Rosin, L. Dishman, *Interassociates of the free commutative semigroup on n generators*, Semigroup Forum **74**, 370–378 (2007). doi: 10.1007/s00233-006-0655-9.

[8] B.N. Givens, A. Rosin, K. Linton, *Interassociates of the bicyclic semigroup*, Semigroup Forum **94**, 104–122 (2017). doi: 10.1007/s00233-016-9794-9.

[9] V.M. Glushkov, *The abstract theory of automata*, Uspekhi Mat. Nauk **16**:5 (101), 3–62 (1961) (in Russian).

[10] M. Gould, K.A. Linton, A.W. Nelson, *Interassociates of monogenic semigroups*, Semigroup Forum **68**, 186–201 (2004). doi: 10.1007/s00233-002-0028-y.

[11] J. Huang, Y. Bai, Y. Chen et al., *Constructions of free dibands and tribands*, Semigroup Forum **104**, 647–666 (2022). doi: 10.1007/s00233-022-10270-w.

[12] N.A. Koreshkov, *n-Tuple algebras of associative type*, Russ. Math. (Izvestiya VUZ. Matematika) **52**, no. 12, 28–35 (2008). doi: 10.3103/S1066369X08120050.

[13] V.I. Levenshtein, *Self-adaptive automata for decoding messages*, Dokl. Akad. Nauk. SSSR **141**, N 6, 1320–1323 (1961) (in Russian).

[14] J.-L. Loday, *Dialgebras*, In: Dialgebras and related operads: Lect. Notes Math. Springer-Verlag, Berlin. **1763**, 7–66 (2001).

[15] J.-L. Loday, M.O. Ronco, *Trialgebras and families of polytopes*, Contemp. Math. **346**, 369–398 (2004). doi: 10.1090/conm/346/06296.

[16] T. Pirashvili, *Sets with two associative operations*, Centr. Eur. J. Math. **2**, 169–183 (2003). doi: 10.2478/BF02476006.

[17] B. Richter, *Dialgebren, Doppelalgebren und ihre Homologie*, Diplomarbeit, Universitat Bonn. (1997). Available at http://www.math.uni-hamburg.de/home/richter/publications.html.

[18] A. Sade, *Groupoïdes en relation associative et semigroupes mutuellement associatifs*, Ann. Soc. Sci. Bruxelles Ser. I **75**, 52–57 (1961).

[19] B.M. Schein, *One-sided nilpotent semigroups*, Uspekhi Mat. Nauk **19**:1 (115), 187–189 (1964) (in Russian).

[20] B.M. Schein, *Relation algebras and function semigroups*, Semigroup Forum **1** (1), 1–62 (1970). doi: 10.1007/BF02573019.

[21] B.M. Schein, *Restrictive bisemigroups*, Izv. Vyssh. Uchebn. Zaved. Mat. **1** (44), 168–179 (1965) (in Russian).

[22] B.M. Schein, *Restrictive semigroups and bisemigroups*, Technical Report (Univ. of Arkansas, Fayetteville, AR, 1989).

[23] J.D.H. Smith, *Directional algebras*, Houston J. Math. **42**, no. 1, 1–22 (2016).

[24] A.V. Zhuchok, *Free commutative trioids*, Semigroup Forum **98**, no. 2, 355–368 (2019). doi: 10.1007/s00233-019-09995-y.

[25] A.V. Zhuchok, *Free left n-dinilpotent doppelsemigroups*, Commun. Algebra **45**, no. 11, 4960–4970 (2017). doi: 10.1080/00927872.2017.1287274.

[26] A.V. Zhuchok, *Free n-nilpotent dimonoids*, Algebra Discrete Math. **16**, no. 2, 299–310 (2013).

[27] A.V. Zhuchok, *Free n-tuple semigroups*, Math. Notes **103**, no. 5, 737–744 (2018). doi: 10.1134/S0001434618050061.

[28] A.V. Zhuchok, *Free products of doppelsemigroups*, Algebra Univers. **77**, no. 3, 361–374 (2017). doi: 10.1007/s00012-017-0431-6.

[29] A.V. Zhuchok, *Free rectangular n-tuple semigroups*, Chebyshevskii Sb. **20**, no. 3, 261–271 (2019). doi: 10.22405/2226-8383-2019-20-3-261-271.

[30] A.V. Zhuchok, *Relatively free doppelsemigroups*, Monograph series Lectures in Pure and Applied Mathematics. Germany, Potsdam: Potsdam University Press. **5**, 86 p. (2018).

[31] A.V. Zhuchok, *Semilatties of subdimonoids*, Asian-Eur. J. Math. **4**, no. 2, 359–371 (2011). doi: 10.1142/S1793557111000290.

[32] A.V. Zhuchok, *Structure of relatively free dimonoids*, Commun. Algebra **45**, no. 4, 1639–1656 (2017). doi: 10.1080/00927872.2016.1222404.

[33] A.V. Zhuchok, *Structure of free strong doppelsemigroups*, Commun. Algebra **46**, no. 8, 3262–3279 (2018). doi: 10.1080/00927872.2017.1407422.

[34] A.V. Zhuchok, *Trioids*, Asian-Eur. J. Math. **8**, no. 4, 1550089 (23 p.) (2015). doi: 10.1142/S1793557115500898.

[35] A.V. Zhuchok, M. Demko, *Free n-dinilpotent doppelsemigroups*, Algebra Discrete Math. **22**, no. 2, 304–316 (2016).

[36] A.V. Zhuchok, K. Knauer, *Abelian doppelsemigroups*, Algebra Discrete Math. **26**, no. 2, 290–304 (2018).

[37] A.V. Zhuchok, J. Koppitz, *Free products of n-tuple semigroups*, Ukrainian Math. J. **70**, no. 11, 1710–1726 (2019). doi: 10.1007/s11253-019-01601-2.

[38] A.V. Zhuchok, J. Koppitz, *Free weakly k-nilpotent n-tuple semigroups*, (2023). Submitted.

[39] A.V. Zhuchok, Y.A. Kryklia, *Free left n-trinilpotent trioids*, Commun. Algebra **49**, no. 2, 467–481 (2021). doi: 10.1080/00927872.2020.1802472.

[40] A.V. Zhuchok, Yul.V. Zhuchok, *Free k-nilpotent n-tuple semigroups*, Commun. Algebra **51**, no. 9, 3972–3980 (2023). doi: 10.1080/00927872.2023.2195000.

[41] A.V. Zhuchok, Yul.V. Zhuchok, J. Koppitz, *Free rectangular doppelsemigroups*, J. Algebra Appl. **19**, no. 11, 2050205 (2020). doi: 10.1142/S0219498820502059.

[42] A.V. Zhuchok, Yul.V. Zhuchok, O.O. Odintsova, *Free left k-nilpotent n-tuple semigroups*, Bul. Acad. Stiinte Repub. Mold. Mat. **94**, no. 3, 29–38 (2020).

[43] A.V. Zhuchok, Yul.V. Zhuchok, Y.V. Zhuchok, *Certain congruences on free trioids*, Commun. Algebra **47**, no. 12, 5471–5481 (2019). doi: 10.1080/00927872.2019.1631322.

[44] Y.V. Zhuchok, J. Koppitz, *Representations of ordered doppelsemigroups by binary relations*, Algebra Discrete Math. **27**, no. 1, 144–154 (2019).

Contact information

Anatolii V. Zhuchok Department of Algebra and System Analysis, Luhansk Taras Shevchenko National University, Koval Street, 3, Poltava, 36014, Ukraine; Institute of Mathematics, University of Potsdam, Karl-Liebknecht-Strasse 24-25, Potsdam, D-14476, Germany
E-Mail: zhuchok.av@gmail.com
URL:

Received by the editors: 15.09.2023.